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Short Note

The smallest radius of a ball containing the support of a compactly supported potential

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Abstract: Let $D \subset \mathbb{R}^3$ be a bounded domain. $q \in C(D)$ be a real-valued compactly supported potential, $A(\beta, \alpha, k)$ be its scattering amplitude, $k > 0$ be fixed, without loss of generality we assume $k = 1$, β be the unit vector in the direction of scattered field, α be the unit vector in the direction of the incident field. Assume that the boundary of D is a smooth surface S. Assume that $D \subset Q_a := \{x : |x| \leq a\}$, and $a > 0$ is the minimal number such that $q(x) = 0$ for $|x| > a$. Formula is derived for a in terms of the scattering amplitude.

Keywords: scattering theory, potential scattering, compactly supported potentials, the smallest radius of the ball containing the support of the potential

MSC: 35Q40, 81U05.

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1. Introduction

Let $D \subset \mathbb{R}^3$ be a bounded domain. $q \in C(D)$ be a real-valued compactly supported potential, S is a smooth boundary of D, $A(\beta, \alpha, k)$ be its scattering amplitude, $k > 0$ be fixed, we assume $k = 1$ without loss of generality, β be the unit vector in the direction of scattered field, α be the unit vector in the direction of the incident field. Assume that $D \subset Q_a := \{x : |x| \leq a\}$, $a > 0$ is the minimal number such that $q(x) = 0$ for $|x| > a$, γ is the unit vector such that $a\gamma \in S$, and for any small $\epsilon > 0$ one has $\int_{D_\eta} |q(x)|^2 dx > 0$, where D_η is the part of D in a neighborhood of the point $a\gamma$ in the region $a - \epsilon < |x| < a$. This makes $c_2 > 0$ in formula (6) below for an arbitrary small $\eta > 0$.

The known formula for the scattering amplitude is (see, for example, [1,2]):

$$A(\beta, \alpha, k) = -\frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} q(x) u(x, \alpha, k) dx, \tag{1}$$

where $u(x, \alpha, k)$ is the scattering solution (see, for example, [1], pp.256–259, [2], pp.359–439). Since $k > 0$ is fixed, we omit the dependence on k in what follows. The scattering amplitude corresponding to a compactly supported potential $q \in L^2(D)$ is an analytic function of β and α on the complex variety $M := \{z \in C^3, z \cdot z = 1\}, z \cdot z := \sum_{j=1}^{3} z_j^2$., see [1].

Our goal is to derive a formula for the a in terms of the scattering amplitude.

Let us formulate some facts. A proof of the first fact can be found in [1].; the second fact is well known.

A proof of the basic result of this research note, Theorem 1, is given in §2.

Fact 1. There exists $\nu(\alpha, \theta) \in L^2(S^2)$ such that

$$-4\pi \int_{S^2} A(\beta, \alpha) \nu(\alpha, \theta) d\alpha = \int_D e^{-i(\beta - \theta) \cdot x} q(x) dx (1 + O(\frac{1}{|\theta|})), \qquad |\theta| \to \infty, \tag{2}$$

where

$$\theta \in M := \{z \in C^3, z \cdot z = 1\}, \qquad z \cdot z = \sum_{j=1}^{3} z_j^2.$$

Estimate (2) is proved in [1], pp. 260–261. A numerical procedure for calculating $\nu(\alpha, \theta)$ is given in [1], pp. 265-266.

If u, v are real-valued vectors in $\mathbb{R}^3$, and $z = u + iv$, then $z \cdot z = 1$ if and only if

$$(u, u) - (v, v) = 1, \qquad (u, v) + (v, u) = 0,$$

where $(u, v) := \sum_{j=1}^{3} u_j v_j$, $|v|^2 = (v, v)$.

It follows from (2) that if $\beta - \theta = \xi$, $\beta, \theta \in M$, and $|\theta| \to \infty$, then

$$\lim_{|\theta|\to\infty} [-4\pi \int_{S^2} A(\beta, \alpha)\nu(\alpha, \theta) d\alpha] = Q(\xi), \tag{3}$$

where $Q(\xi) := \int_D q(x) e^{-i\xi\cdot x} dx$ is the Fourier transform of q.

Fact 2. If $q \in L^2(D)$ is compactly supported, then its Fourier transform is an entire function of exponential type.

In §2 we prove that the the smallest number $a > 0$ such that the ball B_a contains the support of q can be calculated by the formula:

$$a = \overline{\lim}_{|v|\to\infty} \frac{||Q(u + iv)||}{|v|}. \tag{4}$$

Here $||Q(u)||^2 = (2\pi)^3 \int_{\mathbb{R}^3} |q(x)|^2 dx$, $\|Q(u + iv)\|^2 = (2\pi)^3 \int_{\mathbb{R}^3} |q(x)|^2 e^{2v\cdot x} dx$.

Theorem 1. *Assume that $q \in L^2(D)$ is real-valued. Then the smallest radius of the ball containing the support of the potential is given by* (4).

In §2 a proof of Theorem 1 is given.

2. Proof of Theorem 1

By Plansherel's formula one gets

$$||Q(u + iv)||^2 = (2\pi)^3 \int_D e^{2v\cdot x} |q(x)|^2 dx \le c^2 e^{2|v|a}, \qquad c^2 := (2\pi)^3 \int_D |q(x)|^2 dx. \tag{5}$$

Let us prove an estimate from below:

$$||Q(u + iv)||^2 \ge c_2^2 \eta e^{2|v|(a-\eta)}, \qquad c_2^2 = \int_{D_\eta} |q|^2 dx, \tag{6}$$

where $\eta > 0$ is arbitrarily small, and D_η is sufficiently small, so that $\cos(x, \gamma) > 1 - \eta_1$, where η_1 is sufficiently small, γ is a unit vector such that $a\gamma \in \partial S$, S is the boundary of D. Thus, $v \cdot x > |v||x|(1 - \eta)$. From (5)–(6) one gets:

$$c_2 e^{|v|a(1-\eta)} \le ||Q(u + iv)|| \le c e^{|v|a}. \tag{7}$$

Taking natural logarithm of (7), dividing by $|v|$ and taking $|v| \to \infty$ yields formula (4).

Theorem 1 is proved. □

The main novelty in this short note is formula (4). This formula differs from the known result: the Paley-Wiener theorem ([3], p.181.

How can one choose the function $\nu(\alpha, \theta)$ in (3)?

Let us give a method for doing this. This method is outlined on p. 266 in [1].

For $|x| > a$ the scattering solution

$$u(x, \alpha) = e^{i\alpha\cdot x} + \sum_{\ell=0}^{\infty} A_\ell(\alpha) Y_\ell(x_0) h_\ell(r),$$

where $x_0 := \frac{x}{|x|}$, $r := |x|$, $h_\ell(r)$ are spherical Hankel functions (see [1], p.262),

$$A_\ell(\alpha) = \int_{S^2} A(\beta, \alpha) \overline{Y_\ell(\beta)} d\beta,$$

and $Y_\ell(\beta)$ are the normalized sperical harmonics (see [1], pp261–263).

Definition 1.

$$\rho(x) = e^{-i\theta\cdot x}\int_{S^2} u(x,\alpha)\nu(\alpha,\theta')d\alpha - 1, \tag{8}$$

where $\theta' - \theta = \xi, \theta, \theta' \in M$, $|\theta| \to \infty$.

It is proved in [1], p. 265-266, that $||\rho||_{|x|\le a} \le c|\theta|^{-1}, \quad |\theta| \to \infty$. Therefore, $\nu(\alpha,\theta)$ can be chosen as the solution to the minimization problem: $||\rho||_{a\le|x|\le b} = min$, where minimization is taken over $\nu(\alpha,\theta) \in L^2(S^2)$, and $0 < a < b$ are two arbitrary numbers. Practically $b - a$ can be chosen of the order of 1.

The reader may generalize Theorem 1 by assuming that $q \in L^2(D)$ rather than $q \in C(D)$.

3. Conclusion

A formula is given for the radius of the smallest ball containing the support of a compactly supported potential $q(x)$. It is assumed that $q \in L^2(D)$ is a real-valued potential, and $D \subset \mathbb{R}^3$ is a bounded domain.

References

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