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Abilov's estimate for the Laguerre-Bessel transform

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Abstract: In this work, we obtain new inequalities for the Laguerre-Bessel transform in the space $L^2_\alpha(\mathbb{X})$, by using generalized continuity modulus , where $\mathbb{X} = [0,+\infty[\times[0,+\infty[$ and $\alpha \geq 0$.

Keywords: Laguerre-Bessel transform, generalized translation operator, estimate, Lipschitz class

MSC: 43A62, 42B35, 42A10, 26A16.

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1. Introduction and preliminaries

Given $\alpha \geq 0$. The harmonic analysis on $\mathbb{X} = [0,+\infty[\times[0,+\infty[$ is generated by the following partial differential operators

$$\begin{cases} \mathcal{D}_{1,\alpha} = \frac{\partial^2}{\partial t^2} + \frac{2\alpha}{t}\frac{\partial}{\partial t}, \\ \mathcal{D}_{2,\alpha} = \frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + x^2\mathcal{D}_{1,\alpha}, \end{cases}$$

where for all $(x,t) \in \mathbb{X}$. For $(\lambda, m) \in [0,+\infty[\times\mathbb{N}$, the initial value problem

$$\begin{cases} \mathcal{D}_{1,\alpha}u = -\lambda^2 u, \\ \mathcal{D}_{2,\alpha}u = -4\lambda(m + \frac{\alpha+1}{2})u, \\ u(0,0) = 1, \frac{\partial u}{\partial x}(0,0) = \frac{\partial u}{\partial t}(0,0) = 0, \end{cases} \tag{1}$$

has a unique solution $\psi_{\lambda,m}$ given by

$$\psi_{\lambda,m}(x,t) = j_{\alpha-\frac{1}{2}}(\lambda t)\mathfrak{L}^\alpha_m(\lambda x^2), \quad \forall (x,t) \in \mathbb{X}, \tag{2}$$

where $\mathfrak{L}^\alpha_m$ is the Laguerre function defined on $\mathbb{R}_+$ by

$$\mathfrak{L}^\alpha_m(x) = e^{-\frac{x}{2}}\frac{L^\alpha_m(x)}{L^\alpha_m(0)}, \tag{3}$$

L^α_m being the Laguerre polynomial of degree m and order α, and j_α is the normalized Bessel function given by

$$j_\alpha(x) = \Gamma(\alpha+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\left(\frac{x}{2}\right)^{2k}. \tag{4}$$

For $(x,t) \in \mathbb{X}$, the generalized translation operator $T^{(\alpha)}_{(x,t)}$ is defined for $\alpha = 0$ by

$$\frac{1}{4\pi}\sum_{i,j=0}^{1}\int_0^\pi f\left(\Delta_\theta(x,y), Y + (-1)^i t + (-1)^j s\right)d\theta,$$

and for $\alpha > 0$ by

$$b_\alpha\int_{[0,\pi]^3} f\left(\Delta_\theta(x,y), \Delta_\theta(x,y)\xi\right)d\mu_\alpha(\xi,\psi,\theta),$$

where $\Delta_\theta(x,y) = \sqrt{x^2+y^2+2xy\cos\theta}$, $b_\alpha = \frac{(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)}{\pi^{\frac{3}{4}}\Gamma(\alpha)}$, $Y = xy\sin\theta$ and

$$d\mu_\alpha(\xi,\psi,\theta) = (\sin\xi)^{2\alpha-1}(\sin\psi)^{2\alpha-1}(\sin\theta)^{2\alpha}d\xi d\psi d\theta.$$

We denote by:

• $\mid x,t \mid = \mid (x,t) \mid_{\mathbb{X}} = (x^4+4t^2)^{\frac{1}{4}}$ the homogeneous norm on $\mathbb{X}$.

• $\mid \lambda, m \mid = \mid (\lambda,m) \mid_{[0,+\infty[\times\mathbb{N}} = 4\lambda(m+\frac{\alpha+1}{2})$ the quasinorm on $[0,+\infty[\times\mathbb{N}$. Let denote $\mathbb{B}_r$ the ball centered 0 and of radius r, defined by,

$$\mathbb{B}_r = \{(\lambda,m) \in [0,+\infty[\times\mathbb{N}; \mid \lambda, m \mid < r\} \ \ and \ \ \mathbb{B}_r^c = ([0,+\infty[\times\mathbb{N})\backslash\mathbb{B}_r.$$

• $L^p_\alpha(\mathbb{X})$, $p \in [1,+\infty]$, the spaces of measurable functions on $\mathbb{X}$ such that

$$\begin{cases} \|f\|_{p,\alpha} = \left[\int_{\mathbb{X}} |f(x,t)|^p dm_\alpha(x,t)\right]^{\frac{1}{p}} < +\infty, \ \ if \ \ p \in [1,+\infty[, \\ \|f\|_{\infty,\alpha} = esssup_{(x,t)\in\mathbb{X}}|f(x,t)| < +\infty, \end{cases}$$

where dm_α the weighted Lebesgue measure on $\mathbb{X}$, given by

$$dm_\alpha(x,t) = \frac{x^{2\alpha+1}t^{2\alpha}}{\Gamma(\alpha+1)\Gamma(\alpha+\frac{1}{2})}dxdt.$$

• $L^p_{\gamma_\alpha}([0,+\infty[\times\mathbb{N})$, $p \in [1,+\infty]$, the spaces of measurable functions on $[0,+\infty[\times\mathbb{N}$ such that

$$\begin{cases} \|g\|_{\gamma_\alpha,p} = \left[\int_{[0,+\infty[\times\mathbb{N}} |g(\lambda,m)|^p d\gamma_\alpha(\lambda,m)\right]^{\frac{1}{p}} < +\infty, \text{ if } p \in [1,+\infty), \\ \|g\|_{\gamma_\alpha,\infty} = \text{esssup}_{(\lambda,m)\in[0,+\infty[\times\mathbb{N}} |g(\lambda,m)| < +\infty, \end{cases}$$

where $d\gamma_\alpha$ is the positive measure defined on $[0,+\infty[\times\mathbb{N}$ by

$$\int_{[0,+\infty[\times\mathbb{N}} g(\lambda,m)d\gamma_\alpha(\lambda,m) = \frac{1}{2^{2\alpha-1}\Gamma\left(\alpha+\frac{1}{2}\right)}\sum_{m=0}^{\infty} L^\alpha_m(0)\int_0^{+\infty} g(\lambda,m)\lambda^{3\alpha+1}d\lambda.$$

The Fourier Laguerre-Bessel transform of a function in $L^1_\alpha(\mathbb{X})$ is given by

$$\mathcal{F}_{LB}f(\lambda,m) = \int_{\mathbb{X}} f(x,t)\psi_{\lambda,m}(x,t)dm_\alpha(x,t), \ \ (\lambda,m) \in [0,+\infty[\times\mathbb{N}.$$

From [1], it is well known that Fourier Laguerre-Bessel transform can be inverted to

$$\mathcal{F}^{-1}_{LB}f(x,t) = \int_{[0,+\infty[\times\mathbb{N}} f(\lambda,m)\psi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m), \ \ (x,t) \in \mathbb{X}.$$

It is well-known (see [1–4]) that the Fourier Laguerre-Bessel transform $\mathcal{F}_{LB}$ satisfies the following properties:

- The integral transform can be extended to an isometric isomorphism $L^2_\alpha(\mathbb{X})$ to $L^2_{\gamma_\alpha}([0,+\infty[\times\mathbb{N})$ and we have the Plancherel formula

$$\|f\|_{2,\alpha} = \|\mathcal{F}_{LB}f\|_{\gamma_\alpha,2}, \ \ for \ \ f \in L^1_\alpha(\mathbb{X}) \cap L^2_\alpha(\mathbb{X}). \tag{5}$$

- We also have the inverse formula of the generalized Fourier transform:

$$f(x,t) = \int_{[0,+\infty[\times\mathbb{N}} \mathcal{F}_{LB}f(\lambda,m)\psi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m), \ \ (x,t) \in \mathbb{X},$$

provided $\mathcal{F}_{LB}f \in L^1_{\gamma_\alpha}([0,+\infty[\times\mathbb{N})$.

- For $f \in L^2_\alpha(\mathbb{X})$, $(x,t) \in \mathbb{X}$ and $(\lambda, m) \in [0, +\infty[\times\mathbb{N}$, we have

$$\mathcal{F}_{LB}(T^{(\alpha)}_{(x,t)} f)(\lambda, m) = \psi_{\lambda,m}(x,t)\mathcal{F}_{LB}(f)(\lambda, m), \tag{6}$$

and from (6), we get

$$\mathcal{F}_{LB}(T^{(\alpha)}_{(x,t)} f - f)(\lambda, m) = (\psi_{\lambda,m}(x,t) - 1)\mathcal{F}_{LB}(f)(\lambda, m). \tag{7}$$

Now we define the finite differences of order $k \in \mathbb{N}$ and step $(x,t) \in \mathbb{X}$ by

$$\Delta^k_{(x,t)} f(y,s) = (T^{(\alpha)}_{(x,t)} - I)^k f(y,s),$$

where I denotes the unit operator and $(x,t) \neq (0,0)$.

Lemma 1. [5] *For a fixed $(x,t) \neq (0,0)$, we have*

$$\mathcal{F}_{LB}(\Delta^k_{(x,t)} f)(\lambda, m) = (\psi_{\lambda,m}(x,t) - 1)^k \mathcal{F}_{LB}(f)(\lambda, m). \tag{8}$$

The k^{th} order generalized modulus of continuity of function $f \in L^2_\alpha(\mathbb{X})$ is defined as

$$\Omega_k(f, \delta) = \sup_{0<|x,t|\leq\delta} \|\Delta^k_{(x,t)} f\|_{2,\alpha}. \tag{9}$$

Let $\mathrm{W}^{r,k}_{2,\phi}(\mathcal{D}_{2,\alpha})$ denote the class of functions $f \in L^2_\alpha(\mathbb{X})$ that have generalized derivatives satisfying the estimate

$$\Omega_k\left(\mathcal{D}^r_{2,\alpha} f, \delta\right) = O\left(\phi\left(\delta^k\right)\right), \delta \longrightarrow 0,$$

i.e.,

$$\mathrm{W}^{r,k}_{2,\phi}(\mathcal{D}_{2,\alpha}) = \{f \in L^2_\alpha(\mathbb{X}) / \mathcal{D}^r_{2,\alpha} f \in L^2_\alpha(\mathbb{X}), \Omega_k\left(\mathcal{D}^r_{2,\alpha} f, \delta\right) = O\left(\phi\left(\delta^k\right)\right), \delta \to 0\}, \tag{10}$$

where $\phi(t)$ is any nonnegative function given on $[0, \infty[$. For the Laguerre-Bessel operator $\mathcal{D}_{2,\alpha}$, we have $\mathcal{D}^0_{2,\alpha} f = f, \mathcal{D}^r_{2,\alpha} f = \mathcal{D}_{2,\alpha}\left(\mathcal{D}^{r-1}_{2,\alpha} f\right), r = 1, 2, \ldots$.

The goal of the present paper is to prove an analogue of the results by Abilov et al. ([6,7]) for the Laguerre-Bessel transform. It is a special case of the following classical question in harmonic analysis: describe the relation between the regularity of a function and the rapidity of decay of its Fourier transform. More precisely, we have:

Theorem 1. *Given ϕ, r, k and $f \in \mathrm{W}^{r,k}_{2,\phi}(\mathcal{D}_{2,\alpha})$. Then there exists a constant $c_1 > 0$ such that the following inequality holds, for all $N > 0$*

$$\int_{\mathbb{B}^c_N} | \mathcal{F}_{LB} f(\lambda, m) | \, d\gamma_\alpha(\lambda, m) = O\left(N^{-2r}\left(\phi\left(c_1 N^{-k}\right)\right)^2\right),$$

as $N \longrightarrow +\infty$, where the constant in the $O-$symbol depends only on r, k, α.

In the case where $\phi(t) = t^v, v > 0$, we have:

Theorem 2. *Let $\phi(t) = t^v$. Then*

$$\sqrt{\int_{\mathbb{B}^c_N} | \mathcal{F}_{LB} f(\lambda, m) |^2 \, d\gamma_\alpha(\lambda, m)} = O(N^{-r-kv}) \text{ as } N \longrightarrow +\infty \Longleftrightarrow f \in \mathrm{W}^{r,k}_{2,\phi}(\mathcal{D}_{2,\alpha}),$$

where $r = 0, 1, ..., k = 1, 2, ..., 0 < v < 1$.

2. Auxiliary results

In the remainder of this paper, we refer to $c_1, c_2, c_3, ...,$ as positive constants which are generally different in different places and which may depend on k, r, α and other inessential parameters. To prove the main results, we need to rely on some preliminary results.

Lemma 2. [1] *For all $(\lambda, m) \in [0, +\infty[\times\mathbb{N}$, the functions $\psi_{\lambda,m}$ is infinitely differentiable on $\mathbb{R}^2$, even with respect to each variable and we have*

$$sup_{(x,t)\in\mathbb{X}}|\psi_{\lambda,m}(x,t)| = 1. \tag{11}$$

Lemma 3. *If $f \in \mathrm{W}^{r,k}_{2,\phi}(\mathcal{D}_{2,\alpha})$, we have*

(i)
$$\mathcal{F}_{LB}(\mathcal{D}^r_{2,\alpha}f)(\lambda, m) = (-1)^r \mid \lambda, m \mid^r \mathcal{F}_{LB}(f)(\lambda, m), r \in \mathbb{N}. \tag{12}$$

(ii)
$$\|\Delta^k_{(x,t)}(\mathcal{D}^r_{2,\alpha}f)\|^2_{2,\alpha} = \int_{[0,+\infty[\times\mathbb{N}} \mid \psi_{\lambda,m}(x,t) - 1 \mid^{2k}\mid \lambda, m \mid^{2r}\mid \mathcal{F}_{LB}(f) \mid^2 d\gamma_\alpha(\lambda, m), \tag{13}$$

where $r = 0, 1, ..., k$.

Proof. *(i)* From (1), we have

$$\mathcal{F}_{LB}(\mathcal{D}_{2,\alpha}f)(\lambda, m) = - \mid \lambda, m \mid \mathcal{F}_{LB}(f)(\lambda, m).$$

The result follows easily by induction on r.

(ii) From (12), (8) and (5), we obtain

$$\|\Delta^k_{(x,t)}(\mathcal{D}^r_{2,\alpha}f)\|^2_{2,\alpha} = \int_{[0,+\infty[\times\mathbb{N}} \mid \psi_{\lambda,m}(x,t) - 1 \mid^{2k}\mid \lambda, m \mid^{2r}\mid \mathcal{F}_{LB}(f) \mid^2 d\gamma_\alpha(\lambda, m),$$

where $r = 0, 1, ..., k$. □

Lemma 4. *Let $N > 0$, the following assertions are verified:*

(i) *There exist $c_2 > 0$ such that for all $(\lambda, m) \in [0, +\infty[\times\mathbb{N}$ and $(x, t) \in \mathbb{X}$,*

$$\mid \psi_{\lambda,m}(x,t) - 1 \mid\leq c_2 \mid \lambda, m \mid\mid x, t \mid. \tag{14}$$

(ii) *There exist $c_3 > 0$ such that for all $(\lambda, m) \in \mathbb{B}^c_N$ and $(x, t) \in \mathbb{X}$,*

$$\mid \psi_{\lambda,m}(x,t) \mid\leq c_3 \left(\mid \lambda, m \mid x^2\right)^{-\frac{\alpha}{2}-\frac{1}{4}}. \tag{15}$$

Proof. *(i)* From ([8], Lemma 3.1), we deduce that for every $(\lambda, m) \in [0, +\infty[\times\mathbb{N}$ and $(x, t) \in \mathbb{X}$, we have

$$\psi_{\lambda,m}(x,t) = 1 - \frac{(\lambda t)^2}{4(\alpha + \frac{1}{2})} - \frac{\mid \lambda, m \mid x^2}{4(\alpha + 1)} + \kappa_{\alpha,m}\lambda^2 x^4 + o(\lambda^2 \mid x, t \mid^4), \tag{16}$$

with $\kappa_{\alpha,m}$ depending only on α and m.

In consequence, there exist $c_4 > 0$ and $\eta > 0$ such that for all $(x, t) \in \mathbb{X}$,

$$|\lambda, m| \mid x, t \mid^2 < \eta \Rightarrow |\psi_{\lambda,m}(x,t) - 1|^2 \leq c_4|\lambda, m|^2 \mid x, t \mid^2.$$

On the other hand, From ([9],Lemma 4.3) then

$$\lim_{|\lambda,m|\to+\infty} \varphi_{\lambda,m}(x,t) = 0,$$

where $\varphi_{\lambda,m}(x,t) = e^{i\lambda t}\mathfrak{L}_m^\alpha(\lambda x^2)$ the Laguerre Kernel, and from [10], we have the asymptotic formula for the normalized Bessel function j_α when $x \longrightarrow +\infty$:

$$j_\alpha(x) = \frac{\Gamma(\alpha+1)}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{2}{x}\right)^{\alpha+\frac{1}{2}} \cos\left(x-(2\alpha+1)\frac{\pi}{4}\right) + o\left(\frac{1}{x^{\frac{3}{2}}}\right). \tag{17}$$

Hence as

$$\psi_{\lambda,m}(x,t) = j_{\alpha-\frac{1}{2}}(\lambda t)\frac{1}{e^{i\lambda t}}\psi_{\lambda,m}(x,t),$$

then $\lim_{|\lambda,m|\to+\infty}\psi_{\lambda,m}(x,t) = 0$, we get

$$\lim_{|\lambda,m|\to+\infty}\left(\frac{|\psi_{\lambda,m}(x,t)-1|}{|\lambda,m|\mid x,t\mid}\right) = 0.$$

Hence, there exist $c_5 > 0$ and $A > 0$, such that

$$|\lambda,m| > A \Rightarrow |\psi_{\lambda,m}(x,t)-1|^2 \le c_5|\lambda,m|^2 \mid x,t\mid^2 .$$

If $\frac{\eta}{|x,t|^2} < A$. Take

$$M = \max_{\frac{\eta}{|x,t|^2}\le|\lambda,m|\le A} \frac{|\psi_{\lambda,m}(x,t)-1|^2}{|\lambda,m|^2\mid x,t\mid^2}.$$

Therefore for all $(\lambda,m) \in B^c_{\frac{\eta}{|x,t|^2}}$, we have

$$|\psi_{\lambda,m}(x,t)-1| \le c_6|\lambda,m|\mid x,t\mid,$$

where $c_6 = min(\sqrt{c_5},\sqrt{M})$. Hence we have the result where $c_2 = max(\sqrt{c_4},c_6)$.

(ii) From ([11], Page 87), we have the asymptotic formula

$$L_n^\alpha(x) \approx \frac{\Gamma(n+\alpha+1)}{n!}e^{x/2}(\kappa_m x)^{-\alpha/2}J_\alpha(2\sqrt{\kappa_m x}), \quad m\to\infty, \tag{18}$$

where $\kappa_m = m+\frac{\alpha+1}{2}$, for all $0\le x\le a$ and arbitrary finite $a>0$.

On the other hand, it was shown in [12] and also in ([13], p. 355], the following estimate

$$\sqrt{x}J_p(x) = O(1), x\ge 0, \tag{19}$$

where $J_p(x)$ is the Bessel function of the first kind. Therefore, it follows from (3), (17), (18) and (19) that $\mathfrak{L}_m^\alpha(\lambda x^2) = O((|\lambda,m|x^2)^{-\frac{\alpha}{2}-\frac{1}{4}})$, the proof is completed. □

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. For a given $f \in W^{r,k}_{2,\phi}(\mathcal{D}_{2,\alpha})$ and $N>0$, we have

$$\int_{\mathbb{B}_N^c} \mid \mathcal{F}_{LB}f(\lambda,m)\mid^2 d\gamma_\alpha(\lambda,m) \le I_1 + I_2, \tag{20}$$

where

$$I_1 = \int_{\mathbb{B}_N^c} \mid \psi_{\lambda,m}(x,t)\mid\mid \mathcal{F}_{LB}f(\lambda,m)\mid^2 d\gamma_\alpha(\lambda,m),$$

and

$$I_2 = \int_{\mathbb{B}_N^c} \mid \psi_{\lambda,m}(x,t)-1\mid\mid \mathcal{F}_{LB}f(\lambda,m)\mid^2 d\gamma_\alpha(\lambda,m).$$

From (15), we have

$$I_1 \le c_3(Nx^2)^{-\frac{\alpha}{2}-\frac{1}{4}}\int_{\mathbb{B}_N^c} \mid \mathcal{F}_{LB}f(\lambda,m)\mid^2 d\gamma_\alpha(\lambda,m).$$

Choose a constant c_7 such that the number $c_8 = 1 - c_3 c_7^{-\frac{\alpha}{2}-\frac{1}{4}}$ is positive.
Setting $\mid x, t \mid = \frac{c_7}{N}$ in the inequality (20), we have

$$c_8 \int_{\mathbb{B}_N^c} \mid \mathcal{F}_{LB} f(\lambda, m) \mid^2 d\gamma_\alpha(\lambda, m) \leq I_2. \tag{21}$$

By Hölder inequality the second term in (21) satisfies

$$\begin{aligned}
I_2 \leq & \left(\int_{\mathbb{B}_N^c} \mid \psi_{\lambda,m}(x,t) - 1 \mid^{2k} \mid \mathcal{F}_{LB} f(\lambda, m) \mid^2 d\gamma_\alpha(\lambda, m) \right)^{\frac{1}{2k}} \left(\int_{\mathbb{B}_N^c} \mid \mathcal{F}_{LB} f(\lambda, m) \mid^2 d\gamma_\alpha(\lambda, m) \right)^{1-\frac{1}{2k}} \\
\leq & \left(\int_{\mathbb{B}_N^c} |\lambda, m|^{-2r} \mid \psi_{\lambda,m}(x,t) - 1 \mid^{2k} |\lambda, m|^{2r} \mid \mathcal{F}_{LB} f(\lambda, m) \mid^2 d\gamma_\alpha(\lambda, m) \right)^{\frac{1}{2k}} \\
& \times \left(\int_{\mathbb{B}_N^c} \mid \mathcal{F}_{LB} f(\lambda, m) \mid^2 d\gamma_\alpha(\lambda, m) \right)^{1-\frac{1}{2k}} \\
\leq & N^{-\frac{r}{k}} \left(\int_{\mathbb{B}_N^c} \mid \psi_{\lambda,m}(x,t) - 1 \mid^{2k} \mid \lambda, m \mid^{2r} \mid \mathcal{F}_{LB} f(\lambda, m) \mid^2 d\gamma_\alpha(\lambda, m) \right)^{\frac{1}{2k}} \\
& \times \left(\int_{\mathbb{B}_N^c} \mid \mathcal{F}_{LB} f(\lambda, m) \mid^2 d\gamma_\alpha(\lambda, m) \right)^{1-\frac{1}{2k}}.
\end{aligned}$$

We conclude that

$$\int_{\mathbb{B}_N^c} \mid \psi_{\lambda,m}(x,t) - 1 \mid^{2k} \mid \lambda, m \mid^{2r} \mid \mathcal{F}_{LB} f(\lambda, m) \mid^2 d\gamma_\alpha(\lambda, m) \leq \|\Delta_{(x,t)}^k (\mathcal{D}_{2,\alpha}^r f)\|_{2,\alpha}^2.$$

Therefore

$$I_2 \leq N^{-\frac{r}{k}} (\|\Delta_{(x,t)}^k (\mathcal{D}_{2,\alpha}^r f)\|_{2,\alpha})^{\frac{1}{k}} \left(\int_{\mathbb{B}_N^c} \mid \mathcal{F}_{LB} f(\lambda, m) \mid^2 d\gamma_\alpha(\lambda, m) \right)^{1-\frac{1}{2k}}.$$

For $f \in W_{2,\phi}^{r,k}(\mathcal{D}_{2,\alpha})$ there exist a constant $c_9 > 0$ such that

$$\|\Delta_{(x,t)}^k (\mathcal{D}_{2,\alpha}^r f)\|_{2,\alpha}^2 \leq c_9 (\phi(\delta^k))^2 \ \ as\ \delta \to 0,$$

by virtue of (9) and (10). For $\delta = \frac{c_7}{N}$, we obtain

$$c_8 \int_{\mathbb{B}_N^c} \mid \mathcal{F}_{LB} f(\lambda, m) \mid^2 d\gamma_\alpha(\lambda, m) \leq N^{-\frac{r}{k}} \left(c_9 \phi\left((\frac{c_7}{N})^k \right) \right)^{\frac{1}{k}} \left(\int_{\mathbb{B}_N^c} \mid \mathcal{F}_{LB} f(\lambda, m) \mid^2 d\gamma_\alpha(\lambda, m) \right)^{1-\frac{1}{2k}},$$

then

$$c_8 \left(\int_{\mathbb{B}_N^c} \mid \mathcal{F}_{LB} f(\lambda, m) \mid^2 d\gamma_\alpha(\lambda, m) \right)^{\frac{1}{2k}} \leq N^{-\frac{r}{k}} \left(c_9 \phi\left((\frac{c_7}{N})^k \right) \right)^{\frac{1}{k}}.$$

Therefore

$$\int_{\mathbb{B}_N^c} \mid \mathcal{F}_{LB} f(\lambda, m) \mid^2 d\gamma_\alpha(\lambda, m) = O\left(N^{-2r} \left(\phi\left((\frac{c_7}{N})^k \right) \right)^2 \right),$$

for all $N > 0$. The theorem is proved with $c_1 = c_7^k$. $\square$

Proof of Theorem 2. We prove sufficiency by using Theorem 1, let $f \in W_{2,\phi}^{r,k}(\mathcal{D}_{2,\alpha})$ then

$$\sqrt{\int_{\mathbb{B}_N^c} \mid \mathcal{F}_{LB} f(\lambda, m) \mid^2 d\gamma_\alpha(\lambda, m)} = O(N^{-r-kv}). \tag{22}$$

It is easy to show, that there exists a function $f \in L^2_\alpha(\mathbb{X})$ such that $\mathcal{D}^r_{2,\alpha} f \in L^2_\alpha(\mathbb{X})$, from relation (13), we obtain

$$\begin{aligned}\|\Delta^k_{(x,t)}(\mathcal{D}^r_{2,\alpha} f)\|^2_{2,\alpha} =& \int_{[0,+\infty[\times\mathbb{N}} \mid \psi_{\lambda,m}(x,t)-1\mid^{2k}\mid \lambda,m\mid^{2r}\mid \mathcal{F}_{LB}(f)\mid^2 d\gamma_\alpha(\lambda,m),\\ =& A_1 + A_2,\end{aligned}$$

where

$$A_1 = \int_{\mathbb{B}_N} \mid \psi_{\lambda,m}(x,t)-1\mid^{2k}\mid \lambda,m\mid^{2r}\mid \mathcal{F}_{LB}(f)\mid^2 d\gamma_\alpha(\lambda,m),$$

and

$$A_2 = \int_{\mathbb{B}^c_N} \mid \psi_{\lambda,m}(x,t)-1\mid^{2k}\mid \lambda,m\mid^{2r}\mid \mathcal{F}_{LB}(f)\mid^2 d\gamma_\alpha(\lambda,m),$$

and $N = E\left(\frac{1}{|x,t|}\right)$, the integer part of the number $\frac{1}{|x,t|}$. From Lemma 2, we have the estimate

$$\begin{aligned}A_2 \leq& c_{10}\int_{\mathbb{B}^c_N} \mid \lambda,m\mid^{2r}\mid \mathcal{F}_{LB}(f)\mid^2 d\gamma_\alpha(\lambda,m),\\ =& c_{10}\sum_{l=0}^{+\infty}\int_{\mathbb{B}_{N+l+1}\setminus\mathbb{B}_{N+l}} \mid \lambda,m\mid^{2r}\mid \mathcal{F}_{LB}(f)\mid^2 d\gamma_\alpha(\lambda,m),\\ \leq& c_{10}\sum_{l=0}^{+\infty}(N+l+1)^{2r}\int_{\mathbb{B}_{N+l+1}\setminus\mathbb{B}_{N+l}} \mid \mathcal{F}_{LB}(f)\mid^2 d\gamma_\alpha(\lambda,m),\\ =& c_{10}\sum_{l=0}^{+\infty} a_l\left(\mathfrak{I}_l - \mathfrak{I}_{l+1}\right),\end{aligned}$$

with $a_l = (N+l+1)^{2r}$ and $\mathfrak{I}_l = \int_{\mathbb{B}^c_{N+l}} \mid \mathcal{F}_{LB}(f)\mid^2 d\gamma_\alpha(\lambda,m)$. For all integers $M \geq 1$, the Abel transformation shows

$$\begin{aligned}\sum_{l=0}^{M} a_l(\mathfrak{I}_l - \mathfrak{I}_{l+1}) =& a_0\mathfrak{I}_0 - a_M\mathfrak{I}_{M+1} + \sum_{l=1}^{M}(a_l - a_{l-1})\mathfrak{I}_l\\ \leq& a_0\mathfrak{I}_0 + \sum_{l=1}^{M}(a_l - a_{l-1})\mathfrak{I}_l,\end{aligned}$$

because $a_M\mathfrak{I}_{M+1} \geq 0$. Moreover by the finite increments theorem, we have $a_l - a_{l-1} \leq 2r(N+l+1)^{2r-1}$. On the other hand, by (22), there exists $c_{11} > 0$ such that, for all $N > 0$

$$\mathfrak{I}_l \leq c_{11}(N+l)^{-2r-2kv}.$$

For $N \geq 1$, we have

$$\begin{aligned}\sum_{l=0}^{M} a_l(\mathfrak{I}_l - \mathfrak{I}_{l+1}) \leq& a_0\mathfrak{I}_0 + \sum_{l=1}^{M}(a_l - a_{l-1})\mathfrak{I}_l,\\ \leq& c_{11}\left(1+\frac{1}{N}\right)^{2r} N^{-2kv} + 2rc_{11}\sum_{l=1}^{M}\left(1+\frac{1}{N+l}\right)^{2r-1}(N+l)^{-1-2kv}\\ \leq& c_{11}2^{2r}N^{-2kv} + 2^{2r}rc_{11}\sum_{l=1}^{M}(N+l)^{-1-2kv}.\end{aligned}$$

Finally, by the integral comparison test, we have

$$\begin{aligned}\sum_{l=1}^{M}(N+l)^{-1-2kv} \leq& \sum_{\mu=N+1}^{+\infty}(\mu)^{-1-2kv}\\ \leq& \int_N^{+\infty} t^{-1-2kv}dt = \frac{1}{2kv}N^{-2kv}.\end{aligned}$$

Letting $M \to +\infty$, we see that, for $r \geq 0$ and $k, v > 0$, there exists a constant c_{12} such that, for all $N \geq 1$,

$$A_2 \leq c_{12} N^{-2kv}.$$

Consequently, for all $|x,t| > 0$, we get $A_2 \leq c_{12}|x,t|^{2kv},\ as\ |x,t| \to 0$.
We estimate A_1, by relation (14).

$$\begin{aligned}
A_1 \leq & c_2 \mid x,t \mid^{2k} \int_{\mathbb{B}_N} \mid \lambda, m \mid^{2k} \mid \lambda, m \mid^{2r} \mid \mathcal{F}_{LB}(f) \mid^2 d\gamma_\alpha(\lambda, m), \\
\leq & c_2 \mid x,t \mid^{2k} \int_{\mathbb{B}_N} \mid \lambda, m \mid^{2k+2r} \mid \mathcal{F}_{LB}(f) \mid^2 d\gamma_\alpha(\lambda, m), \\
\leq & c_2 \mid x,t \mid^{2k} \sum_{l=0}^{N-1} \int_{\mathbb{B}_{l+1} \setminus \mathbb{B}_l} \mid \lambda, m \mid^{2k+2r} \mid \mathcal{F}_{LB}(f) \mid^2 d\gamma_\alpha(\lambda, m), \\
\leq & c_2 \mid x,t \mid^{2k} \sum_{l=0}^{N-1} (l+1)^{2k+2r} \int_{\mathbb{B}_{l+1} \setminus \mathbb{B}_l} \mid \mathcal{F}_{LB}(f) \mid^2 d\gamma_\alpha(\lambda, m), \\
= & c_2 \mid x,t \mid^{2k} \sum_{l=0}^{N-1} b_l (\mathcal{I}_l - \mathcal{I}_{l+1}),
\end{aligned}$$

with $b_l = (l+1)^{2k+2r}$ and $\mathcal{I}_l = \int_{\mathbb{B}_l^c} \mid \mathcal{F}_{LB}(f) \mid^2 d\gamma_\alpha(\lambda, m)$. Using a summation by parts transforms and proceeding as with A_2 and the fact that $\mathcal{I}_l \leq c_{11} l^{-2r-2kv}$ by hypothesis, we obtain

$$\begin{aligned}
A_1 \leq & c_2 \mid x,t \mid^{2k} \sum_{l=0}^{N-1} b_l (\mathcal{I}_l - \mathcal{I}_{l+1}) \\
\leq & c_2 \mid x,t \mid^{2k} \left(b_0 \mathcal{I}_0 + \sum_{l=1}^{N-1} \mathcal{I}_l (b_l - b_{l-1}) \right) \\
\leq & c_2 \mid x,t \mid^{2k} \left(\mathcal{I}_0 + c_{11}(2r+2k) \sum_{l=1}^{N-1} (l+1)^{2r+2k-1} l^{-2r-2kv} \right).
\end{aligned}$$

From the inequality $l+1 \leq 2l$, we conclude that $A_1 \leq c_2 \mid x,t \mid^{2k} \left(\mathcal{I}_0 + c_{13} \sum_{l=1}^{N-1} l^{2k-2kv-1} \right)$. As a consequence of a series comparison, we have the inequality,

$$\mu \sum_{l=1}^{N-1} l^{\mu-1} \leq N^\mu \ \ for \ \ \mu > 0 \ \ and \ \ N \geq 2.$$

If $\mu = 2k - 2kv > 0$ for $0 < v < 1$, then we obtain

$$A_1 \leq c_2 \mid x,t \mid^{2k} \left(\mathcal{I}_0 + c_{14} N^{2k-2kv} \right) \leq c_2 \mid x,t \mid^{2k} \left(\mathcal{I}_0 + c_{14} \mid x,t \mid^{2kv-2k} \right),$$

since $N \leq \frac{1}{|x,t|}$. If $|x,t|$ is sufficiently small then $\mathcal{I}_0 \leq c_{14}|x,t|^{2kv-2k}$. Then we have $A_1 \leq c_{15} \mid x,t \mid^{2kv}$.
Combining the estimates for A_1 and A_2 gives

$$\|\Delta^k_{(x,t)}(\mathcal{D}^r_{2,\alpha} f)\|_{2,\alpha} = O(\mid x,t \mid^{kv}), as \ \ |x,t| \to 0.$$

Consequently

$$\Omega_k \left(\mathcal{D}^r_{2,\alpha} f, \delta \right) = O(\delta^{kv}) = O\left(\phi\left(\delta^k \right) \right), \delta \longrightarrow 0.$$

Therefore the necessity is proved and the proof of this theorem is completed. □

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