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An enriched iterative scheme for common fixed points of enriched nearly asymptotically nonexpansive mappings in Banach spaces

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Abstract: In this paper, we introduce and analyze a new two-step iterative scheme for approximating common fixed points of two enriched nearly asymptotically nonexpansive self-mappings defined on a closed convex subset of a uniformly convex Banach space. The proposed iteration combines averaged mappings with a carefully selected convex parameter to achieve an optimal contraction rate. We first establish that for enriched contractions, the new scheme converges with a rate factor equal to the contraction constant κ_t, which is strictly faster than the factor $k + \alpha_n(1-k)$ obtained for the existing Yao—Chen iteration and for the classical Mann iteration. Under more general enriched nearly asymptotically nonexpansive conditions, we prove asymptotic regularity, provided the sequence parameters lie in a closed interval within $(0,1)$ and certain conditions hold on the nearly asymptotically nonexpansive sequences. Using this regularity, we establish the demiclosedness principle for $I-\gamma$ at zero and, under Opial's condition, Fréchet differentiability of the norm, or the Kadec—Klee property of the dual space, we prove weak convergence of the iterates to a common fixed point. Furthermore, under condition (A'), we obtain strong convergence. A concrete example illustrates the applicability of the considered mapping class. The results extend and unify several recent findings in fixed point theory, particularly in the context of enriched and nearly asymptotically nonexpansive mappings.

Keywords: enriched nearly, asymptotically nonexpansive mappings, Common fixed points, Iterative scheme, Banach spaces, Condition (A')

MSC: 46M10, 46H70.

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1. Introduction

The approximation of fixed points for nonlinear operators using iterative methods is a cornerstone of nonlinear functional analysis, with far-reaching applications in optimization, variational inequalities, equilibrium problems, and the numerical treatment of differential and integral equations [1]. The theoretical foundations of this field owe much to Opial [2], who in 1967 introduced a geometric condition—now known as Opial's property—that proved essential for proving weak convergence of iterates generated by nonexpansive mappings. This was followed by important developments such as Bruck's work on almost orbits [3], which offered deeper insights into the asymptotic behavior of iterations. Around the same time, the Mann and Ishikawa iterations emerged as fundamental tools, and a landmark study by Tan and Xu [4] in 1993 provided a rigorous analysis of the Ishikawa scheme for nonexpansive mappings. Schu [5] later extended these ideas to the broader class of asymptotically nonexpansive mappings, while Takahashi and Kim [6] further enriched the landscape with approximation methods tailored to Banach spaces.

In the early 2000s, researchers refined these ideas by considering more practical and complex settings. Osilike and Aniagbosor [7], along with Takahashi and Tsukiyama [8], explored convergence in the presence of computational errors and compact domains. Kaczor [9] contributed by investigating almost orbits for asymptotically nonexpansive semigroups, adding new depth to the asymptotic theory. Although nonexpansive mappings, defined by the condition $\|Tx - Ty\| \leq \|x - y\|$, have long served as a standard model, their rigidity limits applicability. A notable relaxation came with the concept of asymptotically nonexpansive mappings, introduced by Goebel and Kirk, where the Lipschitz constant depends on the iteration step and tends to one. This class satisfies $\|T^n x - T^n y\| \leq k_n \|x - y\|$ with $k_n \to 1$, effectively bridging the gap between contractive and Lipschitzian behavior.

An even more flexible framework emerged with the introduction of nearly asymptotically nonexpansive mappings. Sahu [10] laid important groundwork in 2005 by studying demicontinuous nearly Lipschitzian mappings. This concept was later formalized by Agarwal, O'Regan, and Sahu [11], who defined such mappings through the inequality

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n),$$

where $\{a_n\}$ is a nonnegative sequence converging to zero. This class generalizes asymptotically nonexpansive mappings (recovered when $a_n = 0$) and has attracted growing interest [12,13] due to its flexibility in applications. Around the same time, Berinde's monograph [1] offered a unified and comprehensive treatment of iterative approximation methods, while Fukhar-ud-din and Khan [14] contributed convergence results with errors for asymptotically quasi-nonexpansive mappings. Yao and Chen [15] examined a modified Mann iteration for approximating common fixed points of two asymptotically nonexpansive mappings, though their work did not address convergence rates or compare performance with more efficient schemes like Picard iteration.

The following years saw continued progress. In 2008, Sahu and Ismat [12] further explored weak and strong convergence for nearly asymptotically nonexpansive mappings, and Kahn and Abbas [13] extended these ideas to common fixed point problems in 2011. More recently, Berinde and collaborators [16–19] introduced the concept of enriched contractions and enriched nonexpansive mappings, showing how classical iterative methods can be adapted to this broader class. This line of work has inspired recent studies on higher-order schemes, such as Kirk-type iterations under weak enriched contractions, as investigated by Zhou, Saleem, and Abbas [20] and Nisar, Hammad, and Elmursi [21]. In parallel, Alam and Rohen [22] proposed a modified iterative method for nearly asymptotically nonexpansive mappings, reflecting the ongoing vitality of this research direction.

Despite these advances, several open questions remain—particularly concerning optimal convergence rates and the development of iterative schemes that are both efficient and broadly applicable. The present study is situated within this evolving landscape. By building on foundational contributions—from Opial and Tan–Xu to Sahu, Agarwal et al., and the recent work on enriched mappings—we aim to address these gaps and advance the theory of fixed point approximation for generalized nonexpansive operators.

In this paper, we introduce a two-step iterative scheme that achieves an optimal contraction rate for enriched contractions, yielding a convergence factor κ_t that is strictly superior to the existing Yao–Chen iteration and the classical Mann iteration. Also, we extend the analysis to the broader class of enriched nearly asymptotically nonexpansive mappings, establishing asymptotic regularity under suitable parameter conditions and proving the demiclosedness principle for $I - \gamma$ at zero. We also provide weak convergence results under Opial's condition, Fréchet differentiability of the norm, or the Kadec–Klee property of the dual space, alongside strong convergence under condition (A'), thereby offering a complete convergence framework in uniformly convex Banach spaces. Furthermore, we illustrate the applicability of the considered mapping class through a concrete example, demonstrating the practical relevance of our theoretical findings. Collectively, our results unify, extend, and improve upon several recent developments in the literature, providing a robust and versatile approach to approximating common fixed points of enriched and nearly asymptotically nonexpansive mappings.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers. Let X be a real Banach space and Y a nonempty subset of X. A mapping $\gamma : Y \to Y$ is called *asymptotically nonexpansive* if there exists a sequence $\{\pi_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} \pi_n = 1$ such that

$$\|\gamma^n r - \gamma^n s\| \leq \pi_n \|r - s\|,$$

for all $r, s \in Y$ and $n \in \mathbb{N}$. The mapping T is called *uniformly L-Lipschitzian* if for some $L > 0$, $\|\gamma^n r - \gamma^n s\| \leq L\|r - s\|$ for all $r, s \in Y$ and $n \in \mathbb{N}$. Moreover, γ is termed a *contraction* if there exists a constant $0 < k < 1$ such that

$$\|\gamma r - \gamma s\| \leq k\|r - s\|,$$

for all $r, s \in Y$. Also, γ is termed an *enriched contraction* [17] if there exists a constant $0 < k < 1$ and there exists $\lambda \in [0, \infty)$, such that

$$\|\lambda(r - s) + \gamma r - \gamma s\| \leq (\lambda + 1)k\|r - s\|,$$

for all $r, s \in Y$.

Given a sequence $\{h_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} h_n = 0$, following Agarwal et al. [11], γ is said to be *nearly asymptotically nonexpansive* if there exists a sequence $\{\pi_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} \pi_n = 1$ such that

$$\|\gamma^n r - \gamma^n s\| \leq \pi_n(\|r - s\| + h_n),$$

for all $r, s \in Y$ and $n \in \mathbb{N}$. The mapping γ is said to be *nearly uniformly L-Lipschitzian* if $\pi_n \leq L$ for all $n \in \mathbb{N}$.

Note that every asymptotically nonexpansive mapping is nearly asymptotically nonexpansive, and every nearly asymptotically nonexpansive mapping is nearly uniformly L-Lipschitzian.

We recall that the Picard and Mann iteration processes for a mapping $\gamma : Y \to Y$ are defined respectively as:

$$\begin{cases} r_1 = r \in Y, \\ r_{n+1} = \gamma r_n, \quad n \in \mathbb{N}, \end{cases} \tag{1}$$

and

$$\begin{cases} r_1 = r \in Y, \\ r_{n+1} = (1 - \alpha_n)r_n + \alpha_n \gamma r_n, \quad n \in \mathbb{N}, \end{cases} \tag{2}$$

where $\{\alpha_n\} \subset (0, 1)$.

Recently, Agarwal et al. [11] introduced the following iteration scheme:

$$\begin{cases} r_1 = r \in Y, \\ r_{n+1} = (1 - \alpha_n)\gamma^n r_n + \alpha_n \gamma^n s_n, \\ s_n = (1 - \beta_n)r_n + \beta_n \gamma^n r_n, \quad n \in \mathbb{N}, \end{cases} \tag{3}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$. They demonstrated that this scheme converges at the same rate as the Picard iteration.

On the other hand, we state without error terms the iteration scheme studied by Yao and Chen [15] for common fixed points of two mappings:

$$\begin{cases} r_1 = r \in Y, \\ r_{n+1} = \alpha_n r_n + \beta_n \gamma^n r_n + \delta_n \xi^n r_n, \quad n \in \mathbb{N}, \end{cases} \tag{4}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\alpha_n + \beta_n + \delta_n = 1$. They did not investigate the rate of convergence of this scheme.

Definition 1. The self mappings $\gamma, \xi : Y \to Y$ are called *enriched nearly asymptotically nonexpansive* if there exists a sequences $\{h_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} h_n = 0$ and $\pi_n \in [1, \infty)$ with $\pi_n \to 1$ as $n \to \infty$ and there exists $\lambda \in [0, \infty)$, such that

$$\|\lambda(r - s) + \gamma^n r - \gamma^n s\| \leq (\lambda + 1)\pi_n\|r - s\| + \pi_n h_n, \forall\, r, s \in Y, n \geq 1.$$

By setting $\lambda = \frac{1}{t} - 1$, we get $t = \frac{1}{\lambda+1}, \forall\, t \in (0,1]$ dependent on λ. It, therefore, follows that $\|(\frac{1}{t}-1)(r-s) + \gamma^n r - \gamma^n s\| \leq \frac{1}{t}\pi_n\|r-s\| + \pi_n h_n$, which on simplification yields

$$\|\gamma_t^n r - \gamma_t^n s\| \leq \pi_n\|r-s\| + t\pi_n h_n, \quad \forall\, n \geq 1,$$

where $\gamma_t^n = (1-t)I + t\gamma^n$.

Similarly, define $\xi_t = (1-t)I + t\xi$ and $\xi_t^n = (1-t)I + t\xi^n$.

The mappings γ, ξ are said to be *enriched nearly uniformly L-Lipschitzian* if $\pi_n \leq L$ for all $n \in \mathbb{N}$.

Observe that if $\lambda = 0$, we have

$$\|0(r-s) + \gamma^n r - \gamma^n s\| \leq (0+1)\pi_n\|r-s\| + \pi_n h_n, \forall\, r,s \in Y, n \geq 1,$$

which implies that

$$\|\gamma^n r - \gamma^n s\| \leq \pi_n\|r-s\| + \pi_n h_n, \forall\, r,s \in Y, n \geq 1.$$

Therefore, every nearly asymptotically nonexpansive mapping is 0-enriched nearly asymptotically nonexpansive mapping.

Remark 1. Every asymptotically nonexpansive mapping is nearly asymptotically nonexpansive, every nearly asymptotically nonexpansive is 0-enriched nearly asymptotically nonexpansive, and every enriched nearly asymptotically nonexpansive mapping is enriched nearly uniformly L-Lipschitzian.

We now introduce the following enriched iteration scheme to compute common fixed points of two mappings:

$$\begin{cases} r_1 = r \in Y, \\ r_{n+1} = (1-\alpha_n)\gamma_t^n r_n + \alpha_n \xi_t^n s_n, \\ s_n = (1-\beta_n)r_n + \beta_n \gamma_t^n r_n, \quad n \in \mathbb{N}, \end{cases} \tag{5}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $\gamma_t^n = (1-t)I + t\gamma^n, \xi_t^n = (1-t)I + t\xi^n$.

Let $A = \{r \in X : \|r\| = 1\}$ and let X^* denote the dual space of X, that is, the space of all continuous linear functionals f on X. The space X has:

(i) *Gâteaux differentiable norm* if

$$\lim_{k\to 0} \frac{\|r+ks\| - \|r\|}{k},$$

exists for each r and s in A;

(ii) *Fréchet differentiable norm* (see e.g., [6]) if for each r in A, the above limit exists and is attained uniformly for s in A. In this case, it is also well-known that

$$\langle l, J(r)\rangle + \frac{1}{2}\|r\|^2 \leq \frac{1}{2}\|r+l\|^2 \leq \langle l, J(r)\rangle + \frac{1}{2}\|r\|^2 + a(\|l\|),$$

for all $r, l \in X$, where J is the Fréchet derivative of the functional $\frac{1}{2}\|\cdot\|^2$ at $r \in X$, $\langle\cdot,\cdot\rangle$ denotes the pairing between X and X^*, and a is an increasing function defined on $[0,\infty)$ such that $\lim_{t\downarrow 0} \frac{a(k)}{k} = 0$;

(iii) *Opial property* [2] if for any sequence $\{r_n\}$ in X, $r_n \rightharpoonup r$ implies that

$$\limsup_{n\to\infty} \|r_n - r\| < \limsup_{n\to\infty} \|r_n - s\|,$$

for all $s \in X$ with $s \neq r$;

(iv) *Kadec–Klee property* if for every sequence $\{r_n\}$ in X, $r_n \rightharpoonup r$ and $\|r_n\| \to \|r\|$ together imply $r_n \to r$ as $n \to \infty$.

Let δ denote the modulus of uniform convexity. Recall that if X is a uniformly convex Banach space, then (see e.g., [3])

$$\|gr + (1-g)s\| \leq 1 - 2g(1-g)\delta(\|r-s\|),$$

for all $g \in [0,1]$ and for all $r, s \in X$ such that $\|r\| \leq 1$, $\|s\| \leq 1$.

A mapping $\gamma : Y \to Y$ is said to be *demiclosed* at $s \in X$ if for each sequence $\{r_n\}$ in Y and each $r \in X$, $r_n \rightharpoonup r$ and $\gamma r_n \to s$ imply that $r \in Y$ and $\gamma r = s$.

We now state the following lemmas, which will be used later.

Lemma 1 ([5])**.** *Suppose that X is a uniformly convex Banach space and $0 < p \leq g_n \leq q < 1$ for all $n \in \mathbb{N}$. Let $\{r_n\}$ and $\{s_n\}$ be two sequences in X such that*

$$\limsup_{n\to\infty} \|r_n\| \leq \theta, \quad \limsup_{n\to\infty} \|s_n\| \leq \theta, \quad \text{and} \quad \lim_{n\to\infty} \|g_n r_n + (1-g_n)s_n\| = \theta,$$

hold for some $\theta \geq 0$. Then $\lim_{n\to\infty} \|r_n - s_n\| = 0$.

Lemma 2. *If $\{k_n\}$, $\{l_n\}$, and $\{h_n\}$ are sequences of nonnegative real numbers such that*

$$k_{n+1} \leq (1 + l_n)k_n + h_n, \quad \sum_{n=1}^{\infty} l_n < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} h_n < \infty,$$

then $\lim_{n\to\infty} k_n$ exists.

Lemma 3 ([11])**.** *Let X be a uniformly convex Banach space satisfying Opial's condition, and let Y be a nonempty closed convex subset of X. Let γ be a uniformly continuous, nearly asymptotically nonexpansive mapping of Y into itself. Then $I - \gamma$ is demiclosed with respect to zero.*

Lemma 4 ([9])**.** *Let X be a reflexive Banach space such that X^* has the Kadec–-Klee property. Let $\{r_n\}$ be a bounded sequence in X, and let $r^*, s^* \in W = \overline{w}(r_n)$, where W denotes the weak limit set of $\{r_n\}$. Suppose that*

$$\lim_{n\to\infty} \|gr_n + (1-g)r^* - s^*\|,$$

exists for all $g \in [0,1]$. Then $r^* = s^*$.

3. Main results

Recall that if $r_n \to c$ and $s_n \to c$, we say that $\{r_n\}$ is better than $\{s_n\}$ if $\|r_n - c\| \leq \|s_n - c\|$ for all n (see [1]).

Proposition 1. *Let Y be a nonempty closed convex subset of a normed space X. Let $\xi, \gamma : Y \to Y$ be two enriched self-contractions with a common fixed point $c \in Y$ and contraction constant $k \in (0,1)$.*

Consider two iterative schemes:

Scheme A (4)*:*

$$r_{n+1} = \alpha_n r_n + \beta_n \gamma r_n + \delta_n \xi r_n, \quad \alpha_n + \beta_n + \delta_n = 1, \quad \alpha_n, \beta_n, \delta_n \in (0,1).$$

Scheme B (5)*:*

$$r_{n+1} = (1-\alpha_n)\gamma_t r_n + \alpha_n \xi_t s_n, \quad s_n = (1-\beta_n)r_n + \beta_n \gamma_t r_n,$$

with $t = \frac{1}{1+\lambda} \in (0,1], \forall \lambda \in [0,\infty)$.

Then:

1. *Scheme A converges with rate factor $k + \alpha_n(1-k)$.*
2. *Scheme B converges with rate factor κ_t.*

3. *For any $\alpha_n \in (0,1)$ with $\alpha_n > 1-t$, we have $\kappa_t < k + \alpha_n(1-k)$, so Scheme B is asymptotically strictly faster.*

Proof. Let c be a common fixed point of ξ and γ.

We now wish to establish the contraction factors.

First, we compute the contraction factor of γ_t for any $x, y \in Y$. Observe that

$$\begin{aligned}
\|\gamma_t x - \gamma_t y\| &= \|(1-t)(x-y) + t(\gamma x - \gamma y)\| \\
&\leq (1-t)\|x-y\| + t\|\gamma x - \gamma y\| \\
&\leq (1-t)\|x-y\| + tk\|x-y\| \\
&= [1-t(1-k)]\|x-y\| \\
&= \kappa_t\|x-y\|.
\end{aligned}$$

Thus γ_t is a contraction with factor $\kappa_t := 1-t(1-k)$.

Similarly,

$$\begin{aligned}
\|\xi_t x - \xi_t y\| &= \|(1-t)(x-y) + t(\xi x - \xi y)\| \\
&\leq (1-t)\|x-y\| + t\|\xi x - \xi y\| \\
&\leq (1-t)\|x-y\| + tk\|x-y\| \\
&= [1-t(1-k)]\|x-y\| \\
&= \kappa_t\|x-y\|.
\end{aligned}$$

Hence ξ_t has the same factor κ_t.

Now, suppose

$$1-t(1-k) = k \quad \Longrightarrow \quad 1-k = t(1-k) \quad \Longrightarrow \quad t = 1.$$

Hence for $t = 1$ which also gives $\gamma_t = \gamma$, we observe that $\kappa_t = k$.

Furthermore, for (4)(Scheme A)

$$\begin{aligned}
\|r_{n+1} - c\| &= \|\alpha_n(r_n - c) + \beta_n(\gamma r_n - c) + \delta_n(\xi r_n - c)\| \\
&\leq \alpha_n\|r_n - c\| + \beta_n k\|r_n - c\| + \delta_n k\|r_n - c\| \\
&= (\alpha_n + (1-\alpha_n)k)\|r_n - c\| \\
&= (k + \alpha_n(1-k))\|r_n - c\|.
\end{aligned}$$

Thus the contraction factor is $k + \alpha_n(1-k)$.

Also, for (5)(Scheme B)

$$\begin{aligned}
\|r_{n+1} - c\| &= \|(1-\alpha_n)(\gamma_t r_n - c) + \alpha_n(\xi_t s_n - c)\| \\
&\leq (1-\alpha_n)\|\gamma_t r_n - c\| + \alpha_n\|\xi_t s_n - c\| \\
&\leq (1-\alpha_n)\kappa_t\|r_n - c\| + \alpha_n\kappa_t\|s_n - c\|.
\end{aligned}$$

But

$$\begin{aligned}
\|s_n - c\| &= \|(1-\beta_n)(r_n - c) + \beta_n(\gamma_t r_n - c)\| \\
&\leq (1-\beta_n)\|r_n - c\| + \beta_n\kappa_t\|r_n - c\| \\
&= (1-\beta_n(1-\kappa_t))\|r_n - c\| \\
&\leq \|r_n - c\|,
\end{aligned}$$

since $1-\kappa_t \geq 0$. Substituting back, we obtain

$$\begin{aligned}
\|r_{n+1} - c\| &\leq (1-\alpha_n)\kappa_t\|r_n - c\| + \alpha_n\kappa_t\|r_n - c\| \\
&= \kappa_t\|r_n - c\|.
\end{aligned}$$

Therefore, $\forall t \in (0,1]$ and for any $\alpha_n \in (0,1)$ with $\alpha_n > 1-t$, we have:

$$\kappa_t = 1 - t(1-k) < k + \alpha_n(1-k).$$

Hence,

$$\frac{\|r_{n+1}-c\|_B}{\|r_n - c\|_B} \leq 1 - t(1-k) < k + \alpha_n(1-k) \geq \frac{\|r_{n+1}-c\|_A}{\|r_n - c\|_A}.$$

Therefore, Scheme B(5) has a strictly smaller asymptotic contraction factor, meaning it converges faster than Scheme A(4).

This completes the proof. □

Clearly, our scheme has a better rate of convergence. We also note it has a contraction factor of k only when $t = 1$.

The next theorem is crucial to our subsequent results. From this point onward, $\mathcal{F}$ denotes the set of common fixed points of the mappings γ and ξ.

Theorem 1. *Let Y be a nonempty closed convex subset of a uniformly convex Banach space X. Let γ and ξ be two enriched nearly asymptotically nonexpansive self-mappings of Y with sequences $\{\pi_n\}$ and $\{h_n\}$ such that*

$$\sum_{n=1}^{\infty} h_n < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} (\pi_n - 1) < \infty.$$

Let $\{r_n\}$ be defined by the iteration scheme (5), where $\{\alpha_n\}$ and $\{\beta_n\}$ are contained in $[\epsilon, 1-\epsilon]$ for all $n \in \mathbb{N}$ and for some $\epsilon \in (0,1)$. If $\mathcal{F} \neq \emptyset$, then

$$\lim_{n\to\infty} \|r_n - \gamma r_n\| = 0 = \lim_{n\to\infty} \|r_n - \xi r_n\|.$$

Proof. Let $c \in \mathcal{F}$. Then

$$\begin{aligned}
\|r_{n+1} - c\| &= \|\left(1-\alpha_n\right)\gamma_t^n r_n + \alpha_n \xi_t^n s_n - c\| \\
&\leq \left(1-\alpha_n\right)\|\gamma_t^n r_n - c\| + \alpha_n\|\xi_t^n s_n - c\| \\
&= \left(1-\alpha_n\right)\|\left(1-t\right)r_n + t\gamma^n r_n - c\| + \alpha_n\|\left(1-t\right)s_n + t\xi^n s_n - c\| \\
&= \left(1-\alpha_n\right)\|\left(1-t\right)\left(r_n - c\right) + t\gamma^n r_n - c\| + \alpha_n\|\left(1-t\right)\left(s_n - c\right) + t\xi^n s_n - c\| \\
&= \left(1-\alpha_n\right)\|\left(1 - \frac{1}{\lambda+1}\right)\left(r_n - c\right) + \frac{1}{\lambda+1}\gamma^n r_n - c\| + \alpha_n\|\left(1 - \frac{1}{\lambda+1}\right)\left(s_n - c\right) + \frac{1}{\lambda+1}\xi^n s_n - c\| \\
&= \left(1-\alpha_n\right)\left[\frac{1}{\lambda+1}\|\lambda\left(r_n - c\right) + \gamma^n r_n - c\|\right] + \alpha_n\left[\frac{1}{\lambda+1}\|\lambda\left(s_n - c\right) + \xi^n s_n - c\|\right] \\
&\leq \left(1-\alpha_n\right)\left[\frac{1}{\lambda+1}\left(\left(\lambda+1\right)\pi_n\|r_n - c\| + \pi_n h_n\right)\right] + \alpha_n\left[\frac{1}{\lambda+1}\left(\left(\lambda+1\right)\pi_n\|s_n - c\| + \pi_n h_n\right)\right] \\
&= \left(1-\alpha_n\right)\left[\pi_n\|r_n - c\| + \pi_n\frac{h_n}{\lambda+1}\right] + \alpha_n\left[\pi_n\|s_n - c\| + \pi_n\frac{h_n}{\lambda+1}\right] \\
&= \left(1-\alpha_n\right)\pi_n\|r_n - c\| + \alpha_n\pi_n\|\left(1-\beta_n\right)r_n + \beta_n\gamma_t^n r_n - c\| + \pi_n\frac{h_n}{\lambda+1} \\
&= \left(1-\alpha_n\right)\pi_n\|r_n - c\| + \alpha_n\pi_n\|\left(1-\beta_n\right)\left(r_n - c\right) + \beta_n\left(\gamma_t^n r_n - c\right)\| + \pi_n\frac{h_n}{\lambda+1} \\
&\leq \left(1-\alpha_n\right)\pi_n\|r_n - c\| + \alpha_n\pi_n\left(1-\beta_n\right)\|r_n - c\| + \alpha_n\pi_n\beta_n\|\gamma_t^n r_n - c\| + \pi_n\frac{h_n}{\lambda+1} \\
&= \left(1-\alpha_n\right)\pi_n\|r_n - c\| + \alpha_n\pi_n\left(1-\beta_n\right)\|r_n - c\| + \alpha_n\pi_n\beta_n\|\left(1-t\right)r_n + t\gamma^n r_n - c\| + \pi_n\frac{h_n}{\lambda+1} \\
&= \left(1-\alpha_n\right)\pi_n\|r_n - c\| + \alpha_n\pi_n\left(1-\beta_n\right)\|r_n - c\| + \alpha_n\pi_n\beta_n\|\left(1 - \frac{1}{\lambda+1}\right)\left(r_n - c\right) \\
&\quad + \frac{1}{\lambda+1}\left(\gamma^n r_n - c\right)\| + \pi_n\frac{h_n}{\lambda+1}
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n)\,\pi_n\|r_n-c\| + \alpha_n\pi_n\,(1-\beta_n)\,\|r_n-c\| + \alpha_n\pi_n\beta_n\frac{1}{\lambda+1}\left[\|\lambda\,(r_n-c)+\gamma^n r_n-c\|\right] \\
&\quad + \pi_n\frac{h_n}{\lambda+1} \\
&\leq (1-\alpha_n)\,\pi_n\|r_n-c\| + \alpha_n\pi_n\,(1-\beta_n)\,\|r_n-c\| + \alpha_n\pi_n\beta_n\pi_n\|r_n-c\| + \alpha_n\pi_n\beta_n\pi_n\frac{h_n}{\lambda+1} \\
&\quad + \pi_n\frac{h_n}{\lambda+1} \\
&= \pi_n\left[(1-\alpha_n)+\alpha_n\,(1-\beta_n)+\alpha_n\beta_n\pi_n\right]\|r_n-c\| + \pi_n\,(1+\alpha_n\beta_n\pi_n)\,\frac{h_n}{\lambda+1} \\
&= \pi_n\left[1+(\pi_n-1)\,\alpha_n\beta_n\right]\|r_n-c\| + \pi_n\,(1+\alpha_n\beta_n\pi_n)\,\frac{h_n}{\lambda+1} \\
&\leq (1+l_n)\,\|r_n-c\| + h_n',
\end{aligned}
$$

where $l_n = (1+\alpha_n\beta_n\pi_n)(\pi_n-1)$ and $h_n' = \pi_n(1+\alpha_n\beta_n\pi_n)\frac{h_n}{\lambda+1}$. Since $\sum(\pi_n-1) < \infty$ and $\sum h_n < \infty$, we have $\sum l_n < \infty$ and $\sum h_n' < \infty$. Thus, by Lemma 2, $\lim\limits_{n\to\infty}\|r_n-c\|$ exists, and $\{r_n\}$ is bounded. Denote this limit by θ.

Now,

$$
\begin{aligned}
\|s_n-c\| &= \|\beta_n\gamma_t^n r_n + (1-\beta_n)\,r_n - c\| \\
&= \|\beta_n\,(\gamma_t^n r_n - c) + (1-\beta_n)\,(r_n-c)\| \\
&\leq \beta_n\,\|\gamma_t^n r_n - c\| + (1-\beta_n)\,\|r_n-c\| \\
&= \beta_n\,\|(1-t)\,r_n + t\gamma^n r_n - c\| + (1-\beta_n)\,\|r_n-c\| \\
&= \beta_n\left\|\left(1-\frac{1}{\lambda+1}\right)(r_n-c) + \frac{1}{\lambda+1}\gamma^n r_n - c\right\| + (1-\beta_n)\,\|r_n-c\| \\
&= \beta_n\left[\frac{1}{\lambda+1}\,\|\lambda\,(r_n-c)+\gamma^n r_n - c\|\right] + (1-\beta_n)\,\|r_n-c\| \\
&\leq \beta_n\left(\pi_n\,\|r_n-c\| + \frac{\pi_n}{\lambda+1}h_n\right) + (1-\beta_n)\,\|r_n-c\| \\
&= (1+(\pi_n-1))\,\|r_n-c\| + \beta_n\frac{\pi_n}{\lambda+1}h_n,
\end{aligned}
$$

hence

$$\limsup_{n\to\infty}\|s_n-c\| \leq \theta.$$

Also,

$$\|\gamma_t^n r_n - c\| \leq \pi_n\|r_n-c\| + \frac{\pi_n}{\lambda+1}h_n \quad \text{for all } n = 1,2,\ldots,$$

implies that

$$\limsup_{n\to\infty}\|\gamma_t^n r_n - c\| \leq \theta. \tag{6}$$

Furthermore,

$$\|\xi_t^n s_n - c\| \leq \pi_n\|s_n-c\| + \frac{\pi_n}{\lambda+1}h_n,$$

since $\limsup\limits_{n\to\infty}\|s_n-c\| \leq \theta$, we have that

$$\limsup_{n\to\infty}\|\xi_t^n s_n - c\| \leq \theta. \tag{7}$$

Moreover,

$$
\begin{aligned}
\theta &= \lim_{n\to\infty}\|r_{n+1}-c\| \\
&= \lim_{n\to\infty}\|(1-\alpha_n)(\gamma_t^n r_n - c) + \alpha_n(\xi_t^n s_n - c)\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \lim_{n\to\infty} [(1-\alpha_n)\|\gamma_t^n r_n - c\| + \alpha_n\|\xi_t^n s_n - c\|] \\
&\leq \lim_{n\to\infty} \left[(1-\alpha_n)\left(\pi_n\|r_n - c\| + \frac{\pi_n}{\lambda+1}h_n\right) + \alpha_n\left(\pi_n\|s_n - c\| + \frac{\pi_n}{\lambda+1}h_n\right)\right] \\
&= \lim_{n\to\infty} \left[\pi_n\left((1-\alpha_n)\|r_n - c\| + \alpha_n\|s_n - c\|\right) + \frac{\pi_n}{\lambda+1}h_n\right] \\
&\leq \lim_{n\to\infty} \left[\pi_n\theta + \frac{\pi_n}{\lambda+1}h_n\right] \\
&= \theta,
\end{aligned}
$$

Then,

$$
\theta = \lim_{n\to\infty} \|r_{n+1} - c\| = \lim_{n\to\infty} \|(1-\alpha_n)(\gamma_t^n r_n - c) + \alpha_n(\xi_t^n s_n - c)\|.
$$

From Lemma 1 we observe that all the conditions are satisfied in (6), (7) and the above relation, then we obtain

$$
\lim_{n\to\infty} \|\gamma_t^n r_n - \xi_t^n s_n\| = 0. \tag{8}
$$

Also,

$$
\begin{aligned}
\|r_{n+1} - c\| &= \|(1-\alpha_n)\gamma_t^n r_n + \alpha_n\xi_t^n s_n - c\| \\
&= \|(\gamma_t^n r_n - c) + \alpha_n(\xi_t^n s_n - \gamma_t^n r_n)\| \\
&\leq \|\gamma_t^n r_n - c\| + \alpha_n\|\xi_t^n s_n - \gamma_t^n r_n\|,
\end{aligned}
$$

which gives

$$
\theta \leq \liminf_{n\to\infty} \|\gamma_t^n r_n - c\|,
$$

so that with (6) yields $\lim_{n\to\infty} \|\gamma_t^n r_n - c\| = \theta$.

Furthermore,

$$
\begin{aligned}
\|\gamma_t^n r_n - c\| &\leq \|\gamma_t^n r_n - \xi_t^n s_n\| + \|\xi_t^n s_n - c\| \\
&\leq \|\gamma_t^n r_n - \xi_t^n s_n\| + \pi_n\|s_n - c\| + \frac{\pi_n}{\lambda+1}h_n,
\end{aligned}
$$

implies

$$
\theta \leq \liminf_{n\to\infty} \|s_n - c\|. \tag{9}
$$

From $\limsup_{n\to\infty} \|s_n - c\| \leq \theta$ above and (9), we get

$$
\lim_{n\to\infty} \|s_n - c\| = \theta.
$$

Now, $\|\gamma_t^n r_n - c\| \leq \pi_n\|r_n - c\| + \frac{\pi_n}{\lambda+1}h_n$ implies that

$$
\limsup_{n\to\infty} \|\gamma_t^n r_n - c\| \leq \theta.
$$

Hence, $\theta = \lim_{n\to\infty} \|s_n - c\| = \lim_{n\to\infty} \|(1-\beta_n)(r_n - c) + \beta_n(\gamma_t^n r_n - c)\|$ implies, by Lemma 1 again, we have that

$$
\lim_{n\to\infty} \|\gamma_t^n r_n - r_n\| = 0. \tag{10}
$$

Since

$$
\begin{aligned}
\|\gamma_t^n r_n - r_n\| &= \|(1-t)(r_n - r_n) + t(\gamma^n r_n - r_n)\| \\
&= t\|\gamma^n r_n - r_n\| \\
\Longrightarrow \quad \|\gamma^n r_n - r_n\| &= \frac{1}{t}\|\gamma_t^n r_n - r_n\|
\end{aligned}
$$

$$\implies \lim_{n\to\infty} \|\gamma^n r_n - r_n\| = \frac{1}{t} \lim_{n\to\infty} \|\gamma_t^n r_n - r_n\| = 0.$$

Also observe that

$$\|s_n - r_n\| = \beta_n \|\gamma_t^n r_n - r_n\|.$$

Thus, by (10),

$$\lim_{n\to\infty} \|s_n - r_n\| = 0.$$

Again, observe that

$$\begin{aligned}\|r_{n+1} - r_n\| &= \|(1-\alpha_n)\gamma_t^n r_n + \alpha_n \xi_t^n s_n - r_n\| \\ &\leq \|\gamma_t^n r_n - r_n\| + \alpha_n \|\gamma_t^n r_n - \xi_t^n s_n\| \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

so that

$$\|r_{n+1} - s_n\| \leq \|r_{n+1} - r_n\| + \|s_n - r_n\| \to 0 \quad \text{as } n \to \infty.$$

Furthermore, from

$$\|r_n - \xi_t^n s_n\| \leq \|r_n - \gamma_t^n r_n\| + \|\gamma_t^n r_n - \xi_t^n s_n\| \to 0 \quad \text{as } n \to \infty,$$

we get

$$\|r_{n+1} - \xi_t^n s_n\| \leq \|r_{n+1} - r_n\| + \|r_n - \xi_t^n s_n\|,$$

so that

$$\lim_{n\to\infty} \|r_{n+1} - \xi_t^n s_n\| = 0.$$

Again, observe that since $s_n \to c$ and ξ^n is asymptotically nonexpansive, it follows that $\lim_{n\to\infty} \|\xi^n s_n - c\| = 0$. Then,

$$\begin{aligned}\|r_{n+1} - \xi^n s_n\| &= \|r_{n+1} - \xi_t^n s_n + \xi_t^n s_n - \xi^n s_n\| \\ &\leq \|r_{n+1} - \xi_t^n s_n\| + \|\xi_t^n s_n - \xi^n s_n\| \\ &= \|r_{n+1} - \xi_t^n s_n\| + \| \left[(1-t)s_n + t\xi^n s_n\right] - \xi^n s_n\| \\ &= \|r_{n+1} - \xi_t^n s_n\| + (1-t)\|s_n - \xi^n s_n\| \\ &\leq \|r_{n+1} - \xi_t^n s_n\| + (1-t)\left[\|s_n - c\| + \|\xi^n s_n - c\|\right].\end{aligned}$$

Taking $\limsup_{n\to\infty}$ on both sides of the above inequality, we obtain

$$\limsup_{n\to\infty} \|r_{n+1} - \xi^n s_n\| \leq 0.$$

Since $\|r_{n+1} - \xi^n s_n\| \geq 0$ for all n, then

$$\lim_{n\to\infty} \|r_{n+1} - \xi^n s_n\| = 0.$$

We shall now make use of the fact that every enriched nearly asymptotically nonexpansive mapping is enriched nearly uniformly L-Lipschitzian. Thus

$$\|r_{n+1} - \gamma_t r_{n+1}\| \leq \|r_{n+1} - \gamma_t^{n+1} r_{n+1}\| + \|\gamma_t^{n+1} r_{n+1} - \gamma_t^{n+1} r_n\| + \|\gamma_t^{n+1} r_n - \gamma_t r_{n+1}\|. \tag{11}$$

Since both r_{n+1} and $\gamma_t^{n+1} r_{n+1}$ converge to the same fixed point c, we have

$$\lim_{n\to\infty} \|r_{n+1} - \gamma_t^{n+1} r_{n+1}\| = 0. \tag{12}$$

Using the nearly uniformly L-Lipschitzian property of γ_t^n, we get

$$\|\gamma_t^{n+1} r_{n+1} - \gamma_t^{n+1} r_n\| \leq L\|r_{n+1} - r_n\| + \frac{L}{\lambda+1} h_n.$$

From the convergence of the iterative scheme, we have $\lim_{n\to\infty} \|r_{n+1} - r_n\| = 0$. Also, since $\sum_{n=1}^{\infty} h_n < \infty$, it follows that $h_n \to 0$. Therefore,

$$\lim_{n\to\infty} \left(L\|r_{n+1} - r_n\| + \frac{L}{\lambda+1} h_n \right) = 0. \tag{13}$$

Again, using the nearly uniformly L-Lipschitzian property, we observe that

$$\|\gamma_t^{n+1} r_n - \gamma_t r_{n+1}\| \leq L\|\gamma_t^n r_n - r_{n+1}\| + \frac{L}{\lambda+1} h_n.$$

From the iteration scheme and (8),

$$\|\gamma_t^n r_n - r_{n+1}\| = \alpha_n \|\gamma_t^n r_n - \xi_t^n s_n\| \to 0.$$

Hence,

$$\lim_{n\to\infty} \|\gamma_t^{n+1} r_n - \gamma_t r_{n+1}\| = 0. \tag{14}$$

Now, substituting (12), (13), and (14) into (11), and taking the limit superior on both sides, we obtain

$$\limsup_{n\to\infty} \|r_{n+1} - \gamma_t r_{n+1}\| \leq 0 + 0 + 0 = 0.$$

Since $\|r_{n+1} - \gamma_t r_{n+1}\| \geq 0$ for all $n \in \mathbb{N}$, we have

$$0 \leq \liminf_{n\to\infty} \|r_{n+1} - \gamma_t r_{n+1}\| \leq \limsup_{n\to\infty} \|r_{n+1} - \gamma_t r_{n+1}\| \leq 0.$$

Therefore,

$$\lim_{n\to\infty} \|r_{n+1} - \gamma_t r_{n+1}\| = 0.$$

Equivalently, shifting the index,

$$\lim_{n\to\infty} \|r_n - \gamma_t r_n\| = 0.$$

Furthermore, observe that

$$\begin{aligned}
\|r_n - \gamma_t r_n\| &= \|r_n - ((1-t)r_n + t\gamma r_n)\| \\
&= t\|r_n - \gamma r_n\| \\
\implies \|r_n - \gamma r_n\| &= \frac{1}{t}\|r_n - \gamma_t r_n\|.
\end{aligned}$$

Taking limit as $n \to \infty$, we obtain

$$\implies \quad \lim_{n\to\infty} \|r_n - \gamma r_n\| = 0. \tag{15}$$

Furthermore,

$$\begin{aligned}
\|r_n - \xi_t^n r_n\| &\leq \|r_n - r_{n+1}\| + \|r_{n+1} - \xi_t^n s_n\| + \|\xi_t^n s_n - \xi_t^n r_n\| \\
&\leq \|r_n - r_{n+1}\| + \|r_{n+1} - \xi_t^n s_n\| + \left(L\|s_n - r_n\| + \frac{L}{\lambda+1} h_n\right) \\
&\to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

and

$$\begin{aligned}
\|r_{n+1} - \xi_t r_{n+1}\| &\leq \|r_{n+1} - \xi_t^{n+1} r_{n+1}\| + \|\xi_t^{n+1} r_{n+1} - \xi_t r_{n+1}\| \\
&\leq \|r_{n+1} - \xi_t^{n+1} r_{n+1}\| + \left(L\|\xi_t^n r_{n+1} - r_{n+1}\| + \frac{L}{\lambda+1} h_n\right)
\end{aligned}$$

$$\leq \|r_{n+1} - \xi_t^{n+1} r_{n+1}\| + \left(L\|\xi_t^n r_{n+1} - \xi_t^n s_n\| + L\|\xi_t^n s_n - r_{n+1}\| + \frac{L}{\lambda+1} h_n \right)$$
$$\leq \|r_{n+1} - \xi_t^{n+1} r_{n+1}\| + L^2 \|r_{n+1} - s_n\| + L\|\xi_t^n s_n - r_{n+1}\| + (L+1)\frac{L}{\lambda+1} h_n,$$

which yields

$$\lim_{n\to\infty} \|r_n - \xi_t r_n\| = 0.$$

Similarly, observe that

$$\|r_n - \xi_t r_n\| = \|r_n - ((1-t)r_n + t\xi r_n)\|$$
$$\implies \|r_n - \xi r_n\| = \frac{1}{t}\|r_n - \xi_t r_n\|.$$

Now, taking limit as $n \to \infty$ in the above inequality, we get

$$\implies \lim_{n\to\infty} \|r_n - \xi r_n\| = 0,$$

completing the proof. □

Lemma 5. *For any $c_1, c_2 \in \mathcal{F}$, $\lim\limits_{n\to\infty} \|\tau r_n + (1-\tau)c_1 - c_2\|$ exists for all $\tau \in [0,1]$ under the conditions of Theorem 1.*

Proof. The cases $\tau = 0$ and $\tau = 1$ are trivial: $\|\tau r_n + (1-\tau)c_1 - c_2\| = \|c_1 - c_2\|$ when $\tau = 0$, and $\|r_n - c_2\|$ when $\tau = 1$, whose limit exists by Theorem 1.

Now assume $\tau \in (0,1)$. By Theorem 1, $\lim_{n\to\infty} \|r_n - c\|$ exists for all $c \in \mathcal{F}$, and therefore $\{r_n\}$ is bounded. Thus, there exists a real number $\rho > 0$ such that $\{r_n\} \subseteq V \equiv B_\rho(0) \cap Y$, so that V is a closed convex nonempty subset of Y. Define

$$d_n(\tau) = \|\tau r_n + (1-\tau)c_1 - c_2\|,$$

for all $\tau \in [0,1]$. Then $\lim_{n\to\infty} d_n(0) = \|c_1 - c_2\|$ and $\lim_{n\to\infty} d_n(1) = \lim_{n\to\infty} \|r_n - c_2\|$ exist. Let $\tau \in (0,1)$.

Claim 1. For each $n \in \mathbb{N}$, there exists a mapping $T_n : V \to V$ such that $T_n r_n = r_{n+1}$, $T_n c = c$ for all $c \in \mathcal{F}$, and

$$\|T_n r - T_n s\| \leq \pi_n^2 \left(\|r - s\| + \Phi_n \right),$$

where $\Phi_n = (1 + \alpha_n \beta_n)\frac{h_n}{\lambda+1}$ and $\Phi_n \to 0$.

Proof of Claim 1. Define $F_n : V \to V$ by $F_n r = (1-\beta_n)r + \beta_n \gamma_t^n r$, and $T_n : V \to V$ by

$$T_n r = (1-\alpha_n)\gamma_t^n r + \alpha_n \xi_t^n (F_n r).$$

Then $T_n r_n = r_{n+1}$ and $T_n c = c$ for all $c \in \mathcal{F}$. Moreover,

$$\|F_n r - F_n s\| \leq \pi_n \|r - s\| + \beta_n \frac{\pi_n}{\lambda+1} h_n,$$

and

$$\|T_n r - T_n s\| \leq (1-\alpha_n)\pi_n \|r - s\| + \alpha_n \pi_n \|F_n r - F_n s\| + \frac{\pi_n}{\lambda+1} h_n$$
$$\leq \pi_n^2 \|r - s\| + \alpha_n \beta_n \frac{\pi_n^2}{\lambda+1} h_n + \frac{\pi_n^2}{\lambda+1} h_n$$
$$= \pi_n^2 \left(\|r - s\| + \Phi_n \right),$$

where $\Phi_n = (1 + \alpha_n \beta_n)\frac{h_n}{\lambda+1}$. Since $\sum h_n < \infty$, we have $\Phi_n \to 0$. This proves the claim. □

Claim 2. Let $R_{n,m} = T_{n+m-1} \cdots T_n$ for $m \geq 1$. Then $R_{n,m} r_n = r_{n+m}$, $R_{n,m} c = c$ for all $c \in \mathcal{F}$, and

$$\|R_{n,m} r - R_{n,m} s\| \leq \Pi_{n,m} \left(\|r - s\| + \Upsilon_{n,m} \right),$$

where

$$\Pi_{n,m} = \prod_{i=n}^{n+m-1} \pi_i^2, \quad \Upsilon_{n,m} = \sum_{i=n}^{n+m-1} \Phi_i.$$

Proof of Claim 2. The result follows by iterating Claim 1. Indeed,

$$\begin{aligned}\|R_{n,m}r - R_{n,m}s\| &\le \left(\prod_{i=n}^{n+m-1} \pi_i^2\right)\left(\|r-s\| + \sum_{i=n}^{n+m-1} \Phi_i\right) \\ &= \Pi_{n,m}\left(\|r-s\| + \Upsilon_{n,m}\right).\end{aligned}$$

This proves the claim. □

Claim 3. For $n, m \in \mathbb{N}$, define

$$\tau_n = \tau r_n + (1-\tau)c_1,$$

and

$$\begin{aligned}\alpha_{n,m} &= \tau\|r_n - c_1\| + \Upsilon_{n,m}, \\ \beta_{n,m} &= (1-\tau)\|r_n - c_1\| + \Upsilon_{n,m}, \\ \Pi_n &= \prod_{i=n}^{\infty} \pi_i^2, \quad \Upsilon_n = \sum_{i=n}^{\infty} \Phi_i.\end{aligned}$$

Then $\Pi_n \to 1$, $\Upsilon_n \to 0$, and with

$$p_{n,m} = \frac{c_1 - R_{n,m}\tau_n}{\Pi_{n,m}\alpha_{n,m}}, \quad q_{n,m} = \frac{R_{n,m}\tau_n - R_{n,m}r_n}{\Pi_{n,m}\beta_{n,m}},$$

we have $\|p_{n,m}\| \le 1$, $\|q_{n,m}\| \le 1$, and

$$\|p_{n,m} - q_{n,m}\| = \frac{\|y_{n,m} - z_{n,m}\|}{\Delta_{n,m}},$$

where

$$\begin{aligned}y_{n,m} &= \|r_n - c_1\|(R_{n,m}\tau_n - \tau R_{n,m}r_n - (1-\tau)c_1), \\ z_{n,m} &= (c_1 + R_{n,m}r_n - 2R_{n,m}\tau_n)\Upsilon_{n,m},\end{aligned}$$

and

$$\Delta_{n,m} = \Pi_{n,m}\alpha_{n,m}\beta_{n,m}.$$

Proof of Claim 3. The bounds $\|p_{n,m}\| \le 1$ and $\|q_{n,m}\| \le 1$ follow directly from Claim 2. The identity

$$\|p_{n,m} - q_{n,m}\| = \frac{\|y_{n,m} - z_{n,m}\|}{\Delta_{n,m}},$$

follows by direct algebraic expansion of the numerators. Also, $\Pi_n \to 1$ and $\Upsilon_n \to 0$ since $\sum(\pi_n - 1) < \infty$ and $\sum h_n < \infty$. This proves the claim. □

Claim 4. For all $n, m \in \mathbb{N}$,

$$\|\tau p_{n,m} + (1-\tau)q_{n,m}\| = \frac{1}{\Delta_{n,m}}\left\|\tau(1-\tau)\|r_n - c_1\|(c_1 - r_{n+m}) + x_{n,m}\Upsilon_{n,m}\right\|,$$

where

$$x_{n,m} = \tau c_1 + (1-2\tau)R_{n,m}\tau_n - (1-\tau)R_{n,m}r_n.$$

Proof of Claim 4. This follows by direct expansion:

$$\|\tau p_{n,m} + (1-\tau)q_{n,m}\| = \left\|\frac{\beta_{n,m}\tau(c_1 - R_{n,m}\tau_n) + \alpha_{n,m}(1-\tau)(R_{n,m}\tau_n - R_{n,m}r_n)}{\Delta_{n,m}}\right\|.$$

Substituting $\alpha_{n,m} = \tau\|r_n - c_1\| + \Upsilon_{n,m}$ and $\beta_{n,m} = (1-\tau)\|r_n - c_1\| + \Upsilon_{n,m}$, and simplifying, yields the desired identity. This proves the claim. $\square$

Claim 5.

$$\lim_{m,n\to\infty} \|y_{n,m}\| = 0.$$

Proof of Claim 5: Let $\Delta = \sup_{n,m} \Delta_{n,m}$. Since X is uniformly convex and $\|p_{n,m}\| \leq 1$, $\|q_{n,m}\| \leq 1$, applying the uniform convexity inequality to $p_{n,m}$ and $q_{n,m}$ gives

$$2\Delta\delta\left(\frac{\|y_{n,m} - z_{n,m}\|}{\Delta}\right) \leq \Pi_n\left[\|r_n - c_1\|^2 + \frac{N_1\Upsilon_n}{\tau(1-\tau)}\right] - \|r_n - c_1\|\|c_1 - r_{n+m}\| + \frac{\|x_{n,m}\|\Upsilon_n}{\tau(1-\tau)},$$

for some constant $N_1 > 0$. Taking the limit as $m, n \to \infty$, and using $\Pi_n \to 1$, $\Upsilon_n \to 0$, $\|r_n - c_1\| \to \theta$, and $\|c_1 - r_{n+m}\| \to \theta$, we obtain

$$\lim_{m,n\to\infty} \|y_{n,m} - z_{n,m}\| = 0.$$

Since $\|z_{n,m}\| \leq N_2\Upsilon_{n,m}$ for some $N_2 > 0$, the triangle inequality yields

$$\|y_{n,m}\| \leq \|y_{n,m} - z_{n,m}\| + \|z_{n,m}\| \to 0.$$

This proves the claim. $\square$

Claim 6.

$$\lim_{m,n\to\infty} \|R_{n,m}\tau_n - \tau R_{n,m}r_n - (1-\tau)c_1\| = 0.$$

Proof of Claim 6. From Claim 5, we have

$$\|r_n - c_1\| \cdot \|R_{n,m}\tau_n - \tau R_{n,m}r_n - (1-\tau)c_1\| \to 0.$$

If $\theta = 0$, then $r_n \to c_1$ and the result is trivial. Assume $\theta > 0$. Then there exists $N \in \mathbb{N}$ such that $\|r_n - c_1\| \geq \theta/2 > 0$ for all $n \geq N$. Hence,

$$\|R_{n,m}\tau_n - \tau R_{n,m}r_n - (1-\tau)c_1\| \leq \frac{2}{\theta}\|y_{n,m}\| \to 0.$$

This proves the claim. $\square$

Now we complete the proof. For any $n, m \in \mathbb{N}$,

$$\begin{aligned}
d_{n+m}(\tau) &= \|\tau r_{n+m} + (1-\tau)c_1 - c_2\| \\
&= \|R_{n,m}\tau_n - c_2\| \\
&\leq \|R_{n,m}\tau_n - c_2\| + \|R_{n,m}\tau_n - \tau R_{n,m}r_n - (1-\tau)c_1\| \\
&\leq \Pi_{n,m}\left(\|\tau_n - c_2\| + \Upsilon_{n,m}\right) + \|R_{n,m}\tau_n - \tau R_{n,m}r_n - (1-\tau)c_1\| \\
&\leq \Pi_n\left(\|\tau_n - c_2\| + \Upsilon_n\right) + \|R_{n,m}\tau_n - \tau R_{n,m}r_n - (1-\tau)c_1\|.
\end{aligned}$$

Taking $\limsup_{m\to\infty}$ on both sides and using Claim 6, we get

$$\limsup_{m\to\infty} d_{n+m}(\tau) \leq \Pi_n\left(\|\tau_n - c_2\| + \Upsilon_n\right).$$

Now, taking $\limsup_{n\to\infty}$ on both sides:

$$\limsup_{n\to\infty} d_n(\tau) \leq \liminf_{n\to\infty} \Pi_n\left(\|\tau_n - c_2\| + \Upsilon_n\right).$$

Since $\Pi_n \to 1$ and $\Upsilon_n \to 0$, and $\|\tau_n - c_2\| = d_n(\tau)$, we obtain

$$\limsup_{n\to\infty} d_n(\tau) \leq \liminf_{n\to\infty} d_n(\tau).$$

Hence,

$$\limsup_{n\to\infty} d_n(\tau) = \liminf_{n\to\infty} d_n(\tau),$$

so $\lim_{n\to\infty} \|\tau r_n + (1-\tau)c_1 - c_2\|$ exists for all $\tau \in [0,1]$. This completes the proof of Lemma 5. $\square$

Lemma 6. *Assume that the conditions of Theorem 1 are satisfied. Then, for any $c_1, c_2 \in \mathcal{F}$, $\lim_{n\to\infty} \langle r_n, J(c_1 - c_2)\rangle$ exists; in particular, $\langle c - l, J(c_1 - c_2)\rangle = 0$ for all $c, l \in \overline{w}(r_n)$, where $\overline{w}(r_n)$ denotes the weak limit set of $\{r_n\}$.*

Proof. Take $r = c_1 - c_2$ with $c_1 \neq c_2$ and $h = \tau(r_n - c_1)$ in the inequality ((ii)) to obtain:

$$\begin{aligned}\frac{1}{2}\|c_1 - c_2\|^2 + \tau\langle r_n - c_1, J(c_1 - c_2)\rangle &\leq \frac{1}{2}\|\tau r_n + (1-\tau)c_1 - c_2\|^2 \\ &\leq \frac{1}{2}\|c_1 - c_2\|^2 + \tau\langle r_n - c_1, J(c_1 - c_2)\rangle + b(\tau\|r_n - c_1\|).\end{aligned}$$

Now $\sup_{n\geq 1} \|r_n - c_1\| \leq N'$ for some $N' > 0$, it follows that

$$\begin{aligned}\frac{1}{2}\|c_1 - c_2\|^2 + \tau \limsup_{n\to\infty}\langle r_n - c_1, J(c_1 - c_2)\rangle &\leq \frac{1}{2}\lim_{n\to\infty} \|\tau r_n + (1-\tau)c_1 - c_2\|^2 \\ &\leq \frac{1}{2}\|c_1 - c_2\|^2 + b(\tau N') + \tau \liminf_{n\to\infty}\langle r_n - c_1, J(c_1 - c_2)\rangle.\end{aligned}$$

This implies that,

$$\limsup_{n\to\infty}\langle r_n - c_1, J(c_1 - c_2)\rangle \leq \liminf_{n\to\infty}\langle r_n - c_1, J(c_1 - c_2)\rangle + \frac{b(\tau N')}{\tau N'}N'.$$

If $\tau \to 0$, then since $\lim_{\tau\downarrow 0} \frac{b(\tau)}{\tau} = 0$, we obtain

$$\limsup_{n\to\infty}\langle r_n - c_1, J(c_1 - c_2)\rangle \leq \liminf_{n\to\infty}\langle r_n - c_1, J(c_1 - c_2)\rangle.$$

Therefore, $\lim_{n\to\infty} \langle r_n - c_1, J(c_1 - c_2)\rangle$ exists for all $c_1, c_2 \in \mathcal{F}$; in particular, we have

$$\langle c - l, J(c_1 - c_2)\rangle = 0 \quad \forall\, c, l \in \overline{w}(r_n).$$

This completes the proof of Lemma (6). $\square$

Theorem 2. *Let X be a uniformly convex Banach space which satisfies Opial's condition, and let Y be a closed convex subset of X. Let $\gamma : Y \to Y$ be a uniformly continuous enriched nearly asymptotically nonexpansive mapping. Then $I - \gamma$ is demiclosed at 0; that is, whenever $\{r_n\}$ is a sequence in Y such that $r_n \rightharpoonup c$ and $(I-\gamma)r_n \to 0$, it follows that $(I-\gamma)c = 0$.*

Proof. Let $\{r_n\}$ be a sequence in Y such that $r_n \rightharpoonup c \in Y$ and $\|r_n - \gamma r_n\| \to 0$ as $n \to \infty$.

From the definition of $\gamma_t = (1-t)I + t\gamma$, we have

$$\|\gamma_t r_n - r_n\| = t\|\gamma r_n - r_n\| \to 0 \quad \text{as } n \to \infty.$$

Thus $\gamma_t r_n = r_n + (\gamma_t r_n - r_n) \rightharpoonup c$.

Since γ is enriched nearly asymptotically nonexpansive, γ_t is asymptotically nonexpansive. Indeed, there exist sequences $\{\pi_n\}$ and $\{h_n\}$ with $\pi_n \to 1$, $h_n \to 0$, and

$$\|\gamma_t^n r - \gamma_t^n s\| \leq \pi_n\|r - s\| + \frac{\pi_n}{\lambda + 1}h_n \quad \forall r, s \in Y,\ n \in \mathbb{N}.$$

From Lemma 3, since X is uniformly convex and satisfies Opial's condition, and γ_t is asymptotically nonexpansive, the mapping $I - \gamma_t$ is demiclosed at 0. Therefore, from $r_n \rightharpoonup c$ and $\|\gamma_t r_n - r_n\| \to 0$, we have

$$\gamma_t c = c.$$

Therefore,

$$c = \gamma_t c = (1-t)c + t\gamma c,$$

which yields $t(\gamma c - c) = 0$. Since $t > 0$, we obtain $\gamma c = c$.

Thus $(I - \gamma)c = 0$, proving that $I - \gamma$ is demiclosed at 0. This completes the proof of Theorem 2. □

Theorem 3. *Let X be a uniformly convex Banach space, and let Y, γ, ξ, and $\{r_n\}$ be as in Theorem 1. Assume that one of the following conditions holds:*

(a) *X satisfies Opial's condition; or*
(b) *X has a Fréchet differentiable norm; or*
(c) *The dual space X^* of X satisfies the Kadec–Klee property.*

If $\mathcal{F} \neq \emptyset$, then $\{r_n\}$ converges weakly to a point of $\mathcal{F}$.

Proof. Let $c \in \mathcal{F}$. Then $\lim_{n\to\infty} \|r_n - c\|$ exists, as proved in Theorem 1. We now prove that $\{r_n\}$ has a unique weak subsequential limit in $\mathcal{F}$. Indeed, let g and g^* be weak limits of the subsequences $\{r_{n_i}\}$ and $\{r_{n_j}\}$ of $\{r_n\}$, respectively.

By Theorem 1, $\lim_{n\to\infty} \|r_n - \gamma r_n\| = 0$, and from Theorem 2, $I - \gamma$ is demiclosed with respect to zero; therefore, we obtain $\gamma g = g$. Similarly, $\xi g = g$. Similarly, we can prove that $g^* \in \mathcal{F}$.

Next, we show uniqueness. To this end, first assume condition (a) holds. If g and g^* are distinct, then by Opial's condition,

$$\begin{aligned}
\lim_{n\to\infty} \|r_n - g\| &= \lim_{n_i\to\infty} \|r_{n_i} - g\| \\
&< \lim_{n_i\to\infty} \|r_{n_i} - g^*\| \\
&= \lim_{n\to\infty} \|r_n - g^*\| \\
&= \lim_{n_j\to\infty} \|r_{n_j} - g^*\| \\
&< \lim_{n_j\to\infty} \|r_{n_j} - g\| \\
&= \lim_{n\to\infty} \|r_n - g\|,
\end{aligned}$$

which is a contradiction. Therefore, $g = g^*$.

Now assume condition (b) holds. By Lemma 6, $\langle c - l, J(c_1 - c_2)\rangle = 0, \quad \forall\, c, l \in \overline{w}(r_n)$. Therefore,

$$\|g - g^*\|^2 = \langle g - g^*, J(g - g^*)\rangle = 0,$$

implies $g = g^*$.

Finally, suppose condition (c) holds. Since $\lim_{n\to\infty} \|\tau r_n + (1-\tau)g - g^*\|$ exists for all $\tau \in [0,1]$ by Lemma 5, it then follows from Lemma 4 that $g = g^*$.

Consequently, $\{r_n\}$ converges weakly to a point of $\mathcal{F}$, and completing the proof of Theorem (3). □

Two mappings $\xi, \gamma : Y \to Y$, where Y is a subset of a normed space X, are said to satisfy the condition (A') [14] if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(k) > 0$ for all $k \in (0,\infty)$, such that either $\|r - \xi r\| \geq f(d(r,\mathcal{F}))$ or $\|r - \gamma r\| \geq f(d(r,\mathcal{F})), \quad \forall\, r \in Y$, where

$$d(r,\mathcal{F}) = \inf\{\|r - c\| : c \in \mathcal{F} = F(\xi) \cap F(\gamma)\}.$$

Theorem 4. *Let X be a real Banach space, and let Y, ξ, γ, $\mathcal{F}$, and $\{r_n\}$ be as in Theorem 1. Then $\{r_n\}$ converges strongly to a point of $\mathcal{F}$ if and only if $\lim_{n\to\infty} d(r_n, \mathcal{F}) = 0$, where*

$$d(r, \mathcal{F}) = \inf\{\|r - c\| : c \in \mathcal{F}\}.$$

Proof. Suppose $\{r_n\}$ converges strongly to some $c^* \in \mathcal{F}$. Then $\|r_n - c^*\| \to 0$ as $n \to \infty$. Since $d(r_n, \mathcal{F}) \leq \|r_n - c^*\|$ for all n, it follows immediately that $\lim_{n\to\infty} d(r_n, \mathcal{F}) = 0$.

Assume $d(r_n, \mathcal{F}) \to 0$. By Theorem 1, $\{r_n\}$ is bounded and $\lim_{n\to\infty} \|r_n - c\|$ exists for every $c \in \mathcal{F}$. Denote this limit by $\phi(c)$.

We first show that ϕ is constant on $\mathcal{F}$. Let $c_1, c_2 \in \mathcal{F}$. By Lemma 5, for each $\tau \in [0, 1]$, the limit

$$L(\tau) := \lim_{n\to\infty} \|\tau r_n + (1-\tau)c_1 - c_2\|,$$

exists. In particular, $L(0) = \|c_1 - c_2\|$ and $L(1) = \phi(c_2)$.

Since $d(r_n, \mathcal{F}) \to 0$, choose $c_n \in \mathcal{F}$ with $\|r_n - c_n\| \to 0$. Since $\{r_n\}$ is bounded, $\{c_n\}$ is bounded, so it has a weakly convergent subsequence $c_{n_k} \rightharpoonup c \in \mathcal{F}$. Then for this subsequence,

$$L(\tau) = \lim_{k\to\infty} \|\tau r_{n_k} + (1-\tau)c_1 - c_2\| = \|\tau c + (1-\tau)c_1 - c_2\|.$$

Taking $\tau = 1$ gives $\phi(c_2) = \|c - c_2\|$. Similarly, interchanging c_1 and c_2 gives $\phi(c_1) = \|c - c_1\|$.

Now, by Lemma 5, the function L is the limit of convex functions, hence convex. Also, $L(\tau) = \|\tau c + (1-\tau)c_1 - c_2\|$. Applying the same argument with c_1 and c_2 interchanged gives

$$L(\tau) = \|\tau c + (1-\tau)c_2 - c_1\|.$$

Equating the two expressions for $L(\tau)$ at $\tau = 0$ and $\tau = 1$ gives $\|c_1 - c_2\| = \|c_2 - c_1\|$ (trivial) and $\|c - c_2\| = \|c - c_1\|$. Hence $\phi(c_1) = \phi(c_2)$.

Thus ϕ is constant on $\mathcal{F}$. Let this constant value be $\theta \geq 0$.

Now, since $d(r_n, \mathcal{F}) \to 0$, for each n, choose $c_n \in \mathcal{F}$ with $\|r_n - c_n\| \to 0$. Then

$$\|r_n - c_n\| \to 0.$$

We claim that $\{c_n\}$ is Cauchy. For any $m, n \in \mathbb{N}$,

$$\|c_m - c_n\| \leq \|c_m - r_m\| + \|r_m - r_n\| + \|r_n - c_n\|.$$

It suffices to show that $\{r_n\}$ is Cauchy. For any $\varepsilon > 0$, choose N such that for all $n \geq N$, $d(r_n, \mathcal{F}) < \varepsilon/4$. Fix $c \in \mathcal{F}$. Since $\lim_{n\to\infty} \|r_n - c\| = \theta$, there exists $N' \geq N$ such that for all $m, n \geq N'$,

$$|\|r_m - c\| - \|r_n - c\|| < \varepsilon/2.$$

Then for $m, n \geq N'$,

$$\|r_m - r_n\| \leq \|r_m - c\| + \|r_n - c\| < \varepsilon.$$

Thus $\{r_n\}$ is Cauchy and hence converges strongly to some $c \in Y$. Since $d(r_n, \mathcal{F}) \to 0$, we have $c \in \mathcal{F}$.

This completes the proof of Theorem 4. □

Applying Theorem 4, we obtain a strong convergence result for the scheme (5) under condition (A') as follows.

Theorem 5. *Let X be a real Banach space, and let Y, ξ, γ, $\mathcal{F}$, and $\{r_n\}$ be as in Theorem 1. Let ξ and γ satisfy condition (A') and $\mathcal{F} \neq \emptyset$. Then $\{r_n\}$ converges strongly to a common fixed point in $\mathcal{F}$.*

Proof. As proved in Theorem 1,

$$\lim_{n\to\infty} \|r_n - \xi r_n\| = 0 = \lim_{n\to\infty} \|r_n - \gamma r_n\|. \tag{16}$$

From condition (A') and (16), we obtain either

$$\lim_{n\to\infty} f(d(r_n,\mathcal{F})) \leq \lim_{n\to\infty} \|r_n - \gamma r_n\| = 0,$$

or

$$\lim_{n\to\infty} f(d(r_n,\mathcal{F})) \leq \lim_{n\to\infty} \|r_n - \xi r_n\| = 0.$$

In both cases,

$$\lim_{n\to\infty} f(d(r_n,F)) = 0.$$

Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$ and $f(k) > 0, \quad \forall\, k \in (0,\infty)$, it follows that

$$\lim_{n\to\infty} d(r_n,\mathcal{F}) = 0.$$

Clearly, all the conditions of Theorem 4 are satisfied; hence, by its conclusion, $\{r_n\}$ converges strongly to a point of $\mathcal{F}$, completing the proof. $\square$

We present a concrete example of a mapping that is *enriched nearly asymptotically nonexpansive* which is **NOT** *nearly asymptotically nonexpansive*.

Example 1. Let $E = \mathbb{R}$ with the usual norm $\|x\| = |x|$, and let $C = \left[-\frac{1}{\tau},1\right] \subset \mathbb{R}$, where $\tau \in (1,2)$ with $\tau \in \mathbb{Q}$ is fixed. Define $\gamma : C \to C$ by

$$\gamma(r) = \begin{cases} -\tau r, & r \in \left[-\frac{1}{\tau},0\right] \cap \mathbb{Q}, \\ -\frac{1}{\tau} r, & r \in \left[-\frac{1}{\tau},0\right] \setminus \mathbb{Q}, \\ -\frac{1}{\tau} r, & r \in [0,1] \cap \mathbb{Q}, \\ -\tau r, & r \in [0,1] \setminus \mathbb{Q}. \end{cases}$$

For any $r \in C$, *Case 1:* $r \in \left[-\frac{1}{\tau},0\right] \cap \mathbb{Q}$
Then $\gamma(r) = -\tau r \in [0,1] \cap \mathbb{Q}$ (since $\tau, r \in \mathbb{Q}$). So $\gamma(\gamma(r)) = \gamma(-\tau r) = -\frac{1}{\tau}(-\tau r) = r$.
Case 2: $r \in \left[-\frac{1}{\tau},0\right] \setminus \mathbb{Q}$
Then $\gamma(r) = -\frac{1}{\tau} r \in [0,1] \setminus \mathbb{Q}$. So $\gamma(\gamma(r)) = \gamma\left(-\frac{1}{\tau}r\right) = -\tau\left(-\frac{1}{\tau}r\right) = r$.
Case 3: $r \in [0,1] \cap \mathbb{Q}$
Then $\gamma(r) = -\frac{1}{\tau} r \in \left[-\frac{1}{\tau},0\right] \cap \mathbb{Q}$. So $\gamma(\gamma(r)) = \gamma\left(-\frac{1}{\tau}r\right) = -\tau\left(-\frac{1}{\tau}r\right) = r$.
Case 4: $r \in [0,1] \setminus \mathbb{Q}$
Then $\gamma(r) = -\tau r \in \left[-\frac{1}{\tau},0\right] \setminus \mathbb{Q}$. So $\gamma(\gamma(r)) = \gamma(-\tau r) = -\frac{1}{\tau}(-\tau r) = r$.
Therefore,

$$\gamma^2(r) = r \quad \text{for all } r \in C.$$

Since $\gamma^2 = \mathrm{id}$, we have

$$\gamma^3(r) = \gamma^2(\gamma(r)) = \gamma(r).$$

Using $\gamma^2 = \mathrm{id}$:

$$\gamma^4(r) = \gamma^2(\gamma^2(r)) = \gamma^2(r) = r.$$

Therefore,

$$\gamma^4(r) = r \quad \text{for all } r \in C.$$

Proceeding in this manner, we observe that for all $n \in \mathbb{N}$, since $\gamma^2 = \mathrm{id}$, we have

$$\gamma^n(r) = \begin{cases} \gamma(r), & \text{if } n \text{ is odd}, \\ r, & \text{if } n \text{ is even}. \end{cases}$$

That is

$$\gamma^n = \begin{cases} \gamma, & n \text{ odd}, \\ \mathrm{id}, & n \text{ even}. \end{cases}$$

Choose $q \in (0, \infty)$ such that $q > \tau - 1$. Define sequences $\{\pi_n\}$ and $\{h_n\}$ by

$$\pi_n = 1 \quad \text{for all } n \in \mathbb{N}, \qquad h_n = \frac{1}{2^n} \quad \text{for all } n \in \mathbb{N}.$$

Then $\pi_n \to 1$ and $\sum_{n=1}^{\infty} h_n = 1 < \infty$.
We claim that for all $r, s \in C$ and all $n \in \mathbb{N}$,

$$|q(r-s) + \gamma^n r - \gamma^n s| \leq (q+1)\pi_n |r-s| + \pi_n h_n.$$

If n is even, then $\gamma^n r = r$ and $\gamma^n s = s$, so the inequality above becomes

$$|q(r-s) + r - s| = |(q+1)(r-s)| = (q+1)|r-s| \leq (q+1)\pi_n |r-s| + \pi_n h_n,$$

which holds since $\pi_n = 1$ and $h_n \geq 0$.
If n is odd, then $\gamma^n r = \gamma r$ and $\gamma^n s = \gamma s$. For $r, s \in C$, we have that

- If $r, s \in [-\frac{1}{\tau}, 0]$, then $|\gamma r - \gamma s| = \tau |r - s|$.
- If $r, s \in [0, 1]$, then $|\gamma r - \gamma s| = \frac{1}{\tau}|r - s|$.
- If r and s lie in different intervals, say $r \in \left[-\frac{1}{\tau}, 0\right]$ and $s \in [0, 1]$, then $\gamma r = -\tau r$ and $\gamma s = -\frac{1}{\tau}s$. Hence,

$$|q(r-s) + \gamma r - \gamma s| = \left|q(r-s) - \tau r + \frac{1}{\tau}s\right| = \left|(q-\tau)r + \left(-q + \frac{1}{\tau}\right)s\right|.$$

Since $r \leq 0 \leq s$, we have $|r| + |s| = -r + s = s - r = |r - s|$. Therefore,

$$\begin{aligned} |q(r-s) + \gamma r - \gamma s| &\leq |q-\tau||r| + \left|-q + \frac{1}{\tau}\right||s| \\ &\leq \max\left\{|q-\tau|, \left|-q + \frac{1}{\tau}\right|\right\}(|r| + |s|) \\ &= \max\left\{|q-\tau|, q - \frac{1}{\tau}\right\}|r-s|. \end{aligned}$$

Now, since $q > \tau - 1$, we have $q - \tau > -1$, so $|q - \tau| < q + 1$ because $q - \tau < q$ and $q - \tau > -1$. Also, $q - \frac{1}{\tau} < q < q + 1$. Hence,

$$\max\left\{|q-\tau|, q - \frac{1}{\tau}\right\} \leq q + 1.$$

Thus,

$$|q(r-s) + \gamma r - \gamma s| \leq (q+1)|r-s| + h_n.$$

The case $r \in [0, 1]$ and $s \in \left[-\frac{1}{\tau}, 0\right]$ is symmetric and follows by interchanging r and s.
Since $q > \tau - 1$, we have $q - \tau > -1$, which ensures

$$|q(r-s) + \gamma r - \gamma s| \leq (q+1)|r-s| \leq \pi_n (q+1)|r-s| + \pi_n h_n,$$

for all cases. Hence the inequality holds for all $n \in \mathbb{N}$.

Thus, γ is an enriched nearly asymptotically nonexpansive mapping with sequences $\{\mu_n\} = \{1\}$ and $\{h_n\} = \{2^{-n}\}$.

Suppose, for contradiction, that γ is nearly asymptotically nonexpansive. Then there exists sequences $\{\mu_n\}$ and $\{h_n\}$ with $\mu_n \to 1$ and $h_n \to 0$ such that

$$|\gamma^n r - \gamma^n s| \leq \mu_n(|r-s| + h_n) \quad \forall r, s \in C,\ n \in \mathbb{N}.$$

Take $r = -\frac{1}{\tau}$ and $s = 0$. Then $r, s \in [-\frac{1}{\tau}, 0]$, so:

$$\gamma r = -\tau\left(-\frac{1}{\tau}\right) = 1, \qquad \gamma s = 0.$$

Thus for odd n,

$$|\gamma^n r - \gamma^n s| = |\gamma r - \gamma s| = |1 - 0| = 1.$$

But $|r - s| = \left|-\frac{1}{\tau} - 0\right| = \frac{1}{\tau}$. Hence,

$$1 \leq \mu_n \cdot \frac{1}{\tau} + \mu_n h_n.$$

Therefore,

$$\mu_n \geq \frac{1}{\frac{1}{\tau} + h_n} = \frac{\tau}{1 + \tau h_n}.$$

Taking $\liminf_{n\to\infty}$ gives

$$\liminf_{n\to\infty} \mu_n \geq \lim_{n\to\infty} \frac{\tau}{1 + \tau h_n} = \tau > 1,$$

which contradicts $\mu_n \to 1$ (since then $\liminf_{n\to\infty} \mu_n = 1$).

Therefore, γ is **NOT** nearly asymptotically nonexpansive.

The mapping γ is enriched nearly asymptotically nonexpansive, but it is not nearly asymptotically nonexpansive. Hence, the class of enriched nearly asymptotically nonexpansive mappings is strictly larger than the class of nearly asymptotically nonexpansive mappings (and also strictly larger than the class of asymptotically nonexpansive mappings).

We now present an example of two mappings that are *enriched nearly asymptotically nonexpansive* which are **NOT** *nearly asymptotically nonexpansive*.

Example 2. Let $X = \mathbb{R}$ with the usual norm $\|x\| = |x|$. Define two mappings as follows:

1. *Mapping γ:* Let $Y = \left[-\frac{1}{\tau}, 1\right] \subset \mathbb{R}$, where $\tau \in (1, 2)$ is fixed. Define $\gamma : Y \to Y$ by

$$\begin{cases} -\tau r, & r \in [-1/\tau, 0], \\ -\frac{1}{\tau} r, & r \in [0, 1]. \end{cases}$$

2. *Mapping ξ:* Let $X = \left[-\frac{1}{\tau}, \tau\right] \subset \mathbb{R}$, where $\tau \in (1, 2)$ is fixed. Define $\xi : X \to X$ by

$$\xi(r) = \begin{cases} -\tau^2 r, & \text{if } -\frac{1}{\tau} \leq r \leq 0, \\ -\frac{1}{\tau^2} r, & \text{if } 0 \leq r \leq \tau. \end{cases}$$

Remark 2. Both mappings have the same common fixed point:

$$\mathcal{F} = F(\gamma) \cap F(\xi) = \{0\}.$$

From Example 1 above, without the rational/irrational splitting, we also obtain

$$\gamma^n(r) = \begin{cases} r, & \text{if } n \text{ is even}, \\ \gamma(r), & \text{if } n \text{ is odd}. \end{cases}$$

For any $r \in X$:

- If $-\frac{1}{\tau} \leq r \leq 0$, then $\xi(r) = -\tau^2 r \in [0, \tau]$.
- If $0 \leq r \leq \tau$, then $\xi(r) = -\frac{1}{\tau^2} r \in \left[-\frac{1}{\tau}, 0\right]$.

Thus, for all $r \in X$:

$$\xi^2(r) = \xi(\xi(r)) = r.$$

Therefore:

$$\xi^n(r) = \begin{cases} r, & \text{if } n \text{ is even,} \\ \xi(r), & \text{if } n \text{ is odd.} \end{cases}$$

Suppose, for contradiction, that ξ is nearly asymptotically nonexpansive. Then there exist sequences $\{\pi_n\} \subset [1, \infty)$ with $\pi_n \to 1$ and $\{h_n\} \subset [0, \infty)$ with $h_n \to 0$ such that

$$|\xi^n(r) - \xi^n(s)| \leq \pi_n(|r - s| + h_n) \quad \forall r, s \in X,\ n \in \mathbb{N}.$$

Take $r = -\frac{1}{\tau}$ and $s = 0$, then

$$\xi(r) = -\tau^2\left(-\frac{1}{\tau}\right) = \tau, \quad \xi(s) = 0.$$

For odd n, we have

$$|\xi^n(r) - \xi^n(s)| = |\xi(r) - \xi(s)| = |\tau - 0| = \tau.$$

But

$$|r - s| = \frac{1}{\tau}.$$

Thus

$$\tau \leq \pi_n\left(\frac{1}{\tau} + h_n\right).$$

Therefore,

$$\pi_n \geq \frac{\tau}{\frac{1}{\tau} + h_n} = \frac{\tau^2}{1 + \tau h_n}.$$

Taking $\liminf_{n\to\infty}$ gives

$$\liminf_{n\to\infty} \pi_n \geq \lim_{n\to\infty} \frac{\tau^2}{1 + \tau h_n} = \tau^2 > 1,$$

which contradicts $\pi_n \to 1$ (since then $\liminf_{n\to\infty} \pi_n = 1$).

Therefore, ξ is **NOT** nearly asymptotically nonexpansive.

Choose $q^* > \tau^2 - 1$. Define:

$$\pi_n = 1, \quad h_n = \frac{1}{n^2} \quad \forall n \in \mathbb{N}.$$

We need to show:

$$|q^*(r - s) + \xi^n(r) - \xi^n(s)| \leq (q^* + 1)\pi_n|r - s| + \pi_n h_n.$$

Case 1: n is even. Then $\xi^n(r) = r$, $\xi^n(s) = s$:

$$|q^*(r - s) + r - s| = (q^* + 1)|r - s| \leq (q^* + 1)|r - s| + h_n.$$

Case 2: n is odd. Then $\xi^n(r) = \xi(r)$, $\xi^n(s) = \xi(s)$:

For $r, s \in \left[-\frac{1}{\tau}, 0\right]$:

$$|q^*(r - s) + \xi(r) - \xi(s)| = |q^*(r - s) - \tau^2(r - s)| = |q^* - \tau^2||r - s|.$$

Since $q^* > \tau^2 - 1$, we have $q^* - \tau^2 > -1$, so $|q^* - \tau^2| \leq q^* + 1$. Thus:

$$|q^*(r - s) + \xi(r) - \xi(s)| \leq (q^* + 1)|r - s| \leq (q^* + 1)|r - s| + h_n.$$

For $r, s \in [0, \tau]$:

$$|q^*(r-s) + \xi(r) - \xi(s)| = \left|q^*(r-s) + \frac{1}{\tau^2}(r-s)\right| = \left(q^* + \frac{1}{\tau^2}\right)|r-s| \le (q^*+1)|r-s| + h_n.$$

For $r \in \left[-\frac{1}{\tau}, 0\right]$, $s \in [0, \tau]$, then $\xi r = -\tau^2 r$ and $\xi s = -\frac{1}{\tau^2}s$. Hence,

$$|q^*(r-s) + \xi r - \xi s| = \left|q^*(r-s) - \tau^2 r + \frac{1}{\tau^2}s\right| = \left|(q^* - \tau^2)r + \left(-q^* + \frac{1}{\tau^2}\right)s\right|.$$

Since $r \le 0 \le s$, we have $|r| + |s| = -r + s = s - r = |r-s|$. Therefore,

$$\begin{aligned} |q^*(r-s) + \xi r - \xi s| &\le |q^* - \tau^2||r| + \left|-q^* + \frac{1}{\tau^2}\right||s| \\ &\le \max\left\{|q^* - \tau^2|, \left|-q^* + \frac{1}{\tau^2}\right|\right\}(|r| + |s|) \\ &= \max\left\{|q^* - \tau^2|, q^* - \frac{1}{\tau^2}\right\}|r-s|. \end{aligned}$$

Now, since $q^* > \tau^2 - 1$, we have $q^* - \tau^2 > -1$, so $|q^* - \tau^2| < q^* + 1$ (because $q^* - \tau^2 < q^*$ and $q^* - \tau^2 > -1$). Also, $q^* - \frac{1}{\tau^2} < q^* < q^* + 1$. Hence,

$$\max\left\{|q^* - \tau^2|, q^* - \frac{1}{\tau^2}\right\} \le q^* + 1.$$

Thus,

$$|q^*(r-s) + \xi r - \xi s| \le (q^*+1)|r-s| \le (q^*+1)|r-s| + h_n.$$

The case $r \in [0, \tau]$ and $s \in \left[-\frac{1}{\tau}, 0\right]$ is symmetric and follows by interchanging r and s.
Therefore, ξ is enriched nearly asymptotically nonexpansive.

Cross-term inequality for the iteration scheme

For the iteration scheme (5), we desire to show that

$$|\lambda(r-s) + \gamma^n(r) - \xi^n(s)| \le (\lambda+1)\pi_n|r-s| + \pi_n h_n.$$

Let $\lambda = \max q, q^*$. We now show that for all $r \in Y$, $s \in X$ and $n \in \mathbb{N}$.
Case 1: n is even
Then $\gamma^n(r) = r$ and $\xi^n(s) = s$ such that

$$|\lambda(r-s) + \gamma^n(r) - \xi^n(s)| = |\lambda(r-s) + r - s| = |(\lambda+1)(r-s)| = (\lambda+1)|r-s|.$$

Thus,

$$|\lambda(r-s) + \gamma^n(r) - \xi^n(s)| \le (\lambda+1)|r-s| + h_n.$$

Case 2: n is odd
Then $\gamma^n(r) = \gamma(r)$ and $\xi^n(s) = \xi(s)$ which gives

$$|\lambda(r-s) + \gamma^n(r) - \xi^n(s)| = |\lambda(r-s) + \gamma(r) - \xi(s)|.$$

We now consider several subcases.
Subcase 2a: $r, s \in \left[-\frac{1}{\tau}, 0\right]$
Then $\gamma(r) = -\tau r$ and $\xi(s) = -\tau^2 s$

$$\begin{aligned} |\lambda(r-s) + \gamma(r) - \xi(s)| &= |\lambda(r-s) - \tau r + \tau^2 s| \\ &= |(\lambda - \tau)r + (-\lambda + \tau^2)s|. \end{aligned}$$

From $\lambda > \tau^2 - 1$, we have

$$|(\lambda-\tau)r + (-\lambda+\tau^2)s| = |(\lambda-\tau)r - (\lambda-\tau^2)s|.$$

The expression is linear in r and s, so its maximum over the rectangle $[-1/\tau, 0] \times [-1/\tau, 0]$ occurs at corners. Checking corners, we observe that if

- $(r,s) = (0,0)$: value $= 0$, $|r-s| = 0$.
- $(r,s) = (-1/\tau, 0)$: value $= |\lambda-\tau|/\tau \leq (\lambda+1)/\tau$, and $|r-s| = 1/\tau$.
- $(r,s) = (0, -1/\tau)$: value $= |\lambda-\tau^2|/\tau \leq (\lambda+1)/\tau$, and $|r-s| = 1/\tau$.
- $(r,s) = (-1/\tau, -1/\tau)$: value $= 0$, $|r-s| = 0$.

Thus, in all cases,

$$|(\lambda-\tau)r - (\lambda-\tau^2)s| \leq (\lambda+1)|r-s|.$$

Hence,

$$|\lambda(r-s) + \gamma(r) - \xi(s)| \leq (\lambda+1)|r-s| \leq (\lambda+1)|r-s| + h_n.$$

Subcase 2b: $r \in [0,1]$ and $s \in [0,\tau]$
Then $\gamma(r) = \frac{1}{\tau}r$ and $\xi(s) = \frac{1}{\tau^2}s$. Hence,

$$\begin{aligned}|\lambda(r-s) + \gamma(r) - \xi(s)| &= \left|\lambda(r-s) + \frac{1}{\tau}r - \frac{1}{\tau^2}s\right| \\ &= \left|\left(\lambda + \frac{1}{\tau}\right)r - \left(\lambda + \frac{1}{\tau^2}\right)s\right|.\end{aligned}$$

Define $f(r,s) = \left(\lambda + \frac{1}{\tau}\right)r - \left(\lambda + \frac{1}{\tau^2}\right)s$. Since f is linear in r and s, its maximum over the rectangle $[0,1] \times [0,\tau]$ occurs at corners. Checking corners, we have

- $(r,s) = (0,0)$: $|f| = 0$, $|r-s| = 0$.
- $(r,s) = (1,0)$: $|f| = \lambda + \frac{1}{\tau} \leq \lambda + 1 = (\lambda+1)|r-s|$.
- $(r,s) = (0,\tau)$: $|f| = \lambda\tau + \frac{1}{\tau} \leq (\lambda+1)\tau = (\lambda+1)|r-s|$, since $\frac{1}{\tau} \leq \tau$.
- $(r,s) = (1,\tau)$: $|f| = \lambda(\tau-1) \leq (\lambda+1)(\tau-1) = (\lambda+1)|r-s|$.

Thus, for all $r \in [0,1]$, $s \in [0,\tau]$,

$$|f(r,s)| \leq (\lambda+1)|r-s|.$$

Therefore,

$$|\lambda(r-s) + \gamma(r) - \xi(s)| \leq (\lambda+1)|r-s| \leq (\lambda+1)|r-s| + h_n.$$

Subcase 2c: $r \in [0,1]$, $s \in \left[-\frac{1}{\tau}, 0\right]$
Then $\gamma(r) = \frac{1}{\tau}r$ and $\xi(s) = -\tau^2 s$. We have

$$|\lambda(r-s) + \gamma(r) - \xi(s)| = |\lambda(r-s) + \frac{1}{\tau}r + \tau^2 s| = |(\lambda + \frac{1}{\tau})r + (\tau^2 - \lambda)s|.$$

Since $r \geq 0$ and $s \leq 0$, we have $|r| + |s| = r - s = |r-s|$. Also, from $\lambda > \tau^2 - 1$ and $\tau \in (1,2)$, we get

$$\left|\lambda + \frac{1}{\tau}\right| = \lambda + \frac{1}{\tau} \leq \lambda + 1, \qquad |\tau^2 - \lambda| \leq \lambda + 1.$$

Therefore, by the triangle inequality,

$$\begin{aligned}|(\lambda + \frac{1}{\tau})r + (\tau^2-\lambda)s| &\leq |\lambda + \frac{1}{\tau}||r| + |\tau^2 - \lambda||s| \\ &\leq (\lambda+1)(|r| + |s|) \\ &= (\lambda+1)|r-s|.\end{aligned}$$

Thus,

$$|\lambda(r-s)+\gamma(r)-\xi(s)| \le (\lambda+1)|r-s| \le (\lambda+1)|r-s|+h_n,$$

Subcase 2d: $r \in \left[-\frac{1}{\tau}, 0\right]$, $s \in [0,\tau]$

Then $\gamma(r) = -\tau r$ and $\xi(s) = \frac{1}{\tau^2}s$. We have

$$|\lambda(r-s)+\gamma(r)-\xi(s)| = |(\lambda-\tau)r - (\lambda+\frac{1}{\tau^2})s|.$$

Since $r \le 0$ and $s \ge 0$, we have $|r|+|s| = -r+s = s-r = |r-s|$. Also, from $\lambda > \tau^2-1$ and $\tau \in (1,2)$, we get

$$|\lambda-\tau| \le \lambda+1, \qquad \left|\lambda+\frac{1}{\tau^2}\right| = \lambda+\frac{1}{\tau^2} \le \lambda+1.$$

Therefore, by the triangle inequality,

$$\begin{aligned} |(\lambda-\tau)r-(\lambda+\frac{1}{\tau^2})s| &\le |\lambda-\tau||r| + \left|\lambda+\frac{1}{\tau^2}\right||s| \\ &\le (\lambda+1)(|r|+|s|) \\ &= (\lambda+1)|r-s|. \end{aligned}$$

Thus,

$$|\lambda(r-s)+\gamma(r)-\xi(s)| \le (\lambda+1)|r-s| \le (\lambda+1)|r-s|+h_n,$$

Therefore, for all $r \in Y$ and $s \in X$ (or $r, s \in X$, since $Y \subset X$) and all $n \in \mathbb{N}$,

$$|\lambda(r-s)+\gamma^n(r)-\xi^n(s)| \le (\lambda+1)\pi_n|r-s|+\pi_n h_n,$$

where $\pi_n = 1$ and $h_n = \frac{1}{n^2}$.

This establishes the cross-term inequality for the iteration scheme. Combined with the earlier proofs that γ is enriched nearly asymptotically nonexpansive (from Example 1) and ξ is enriched nearly asymptotically nonexpansive, we conclude that both mappings belong to the class of enriched nearly asymptotically nonexpansive mappings.

4. Conclusion

In this paper, we introduced and analyzed a new two-step iterative scheme for approximating common fixed points of two enriched nearly asymptotically nonexpansive mappings in uniformly convex Banach spaces. For enriched contractions, the proposed method achieves an optimal contraction rate factor of k, which is strictly faster than the factor $k+\alpha_n(1-k)$ obtained for the existing Yao–Chen iteration. Under suitable summability conditions, we established asymptotic regularity, namely $\lim_{n\to\infty}\|r_n-\gamma r_n\| = \lim_{n\to\infty}\|r_n-\xi r_n\| = 0$. Using the demiclosedness principle together with Opial's condition, Fréchet differentiability of the norm, or the Kadec–Klee property of the dual space, weak convergence to a common fixed point was proved. Moreover, under condition (A'), strong convergence was obtained. Concrete examples demonstrated the applicability of the considered mapping class, and the results unify and extend several recent developments in fixed point theory.

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