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Barycentric Expansions of Reciprocal Polynomial Series and Arithmetic Consequences

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Abstract: We study reciprocal series $\sum_{n\geq 0} 1/\prod_{j=1}^{d}(n+\alpha_j)$ with strictly positive shifts α_j. Centering the shifts at their arithmetic mean μ gives, under the condition $\eta := \max_j |\alpha_j - \mu| < \mu$, an exact expansion in Hurwitz zeta values $\zeta(d+k,\mu)$ whose coefficients are the complete homogeneous symmetric polynomials of the centered shifts. We give two explicit remainder bounds and show that the barycenter is the unique center that removes the linear term. In the cubic case the coefficients are governed by the dispersion V and the asymmetry E through $h_k = \frac{V}{2}h_{k-2} + Eh_{k-3}$. A sign-controlled double expansion shows that, when $E \leq 0$, all terms of the expansion are nonnegative, so every truncation is a certified lower bound; when $E > 0$ the variance-corrected lower bound fails in general. For polynomials with distinct rational roots, none in $\{0,1,2,\ldots\}$, the series is a rational number plus a linear form Λ in digamma values at canonical arguments in $(0,1]$. By Baker's theorem the series is either rational (exactly when $\Lambda = 0$) or transcendental. We decide which case occurs when the roots lie in at most two classes modulo 1, in every degree, and give a sufficient sign criterion in general; the case of quadratic polynomials with distinct rational roots is classified completely.

Keywords: reciprocal polynomial series, Hurwitz zeta function, digamma function, complete homogeneous symmetric polynomials, barycentric expansion, symmetric invariants, rational roots, transcendence

MSC: Primary 33B15; Secondary 11M35, 05E05, 11J86.

1. Introduction

Reciprocal polynomial series provide a simple meeting point of elementary analysis, special functions, and arithmetic. For $d \geq 2$ and $\alpha_1, \ldots, \alpha_d > 0$, define

$$\mathcal{S}_d(\alpha_1,\ldots,\alpha_d) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha_1)\cdots(n+\alpha_d)}. \tag{1}$$

The series converges absolutely because its summand is $O(n^{-d})$.

Partial fractions express (1) through the digamma function, and divided differences of the digamma function together with the Hermite–Genocchi formula give integral representations. The present paper organizes the same object differently: instead of separating the linear factors, we center them at their arithmetic mean

$$\mu = \frac{\alpha_1 + \cdots + \alpha_d}{d}, \qquad y_j = \alpha_j - \mu, \qquad \sum_{j=1}^{d} y_j = 0, \tag{2}$$

and expand around this common center. With $u = n + \mu$,

$$\prod_{j=1}^{d}(n+\alpha_j) = u^d \prod_{j=1}^{d}\Big(1 + \frac{y_j}{u}\Big), \tag{3}$$

and the reciprocal product is governed by the generating function of the complete homogeneous symmetric polynomials. This produces a convergent Hurwitz-zeta expansion whenever

$$\eta := \max_j |y_j| < \mu. \tag{4}$$

The expansion (Theorem 1) is elementary, and the coefficient recurrence (Theorem 2) is the classical relation between complete and elementary symmetric functions. We regard the following consequences as the contribution of the paper.

(i) The barycenter is the *unique* admissible center removing the linear term (Proposition 1); after centering, the whole coefficient sequence depends only on $e_2, \dots, e_d$.
(ii) Two explicit, certified remainder bounds are provided: a general one depending only on η/μ (Theorem 3) and a separation-sensitive one (Proposition 2), which in the examples of §10 is sharper by several orders of magnitude.
(iii) In degree three, a sign-controlled double expansion (Proposition 3): when $E \le 0$ all terms are nonnegative and every truncation is a lower bound (Corollary 1), whereas for $E > 0$ the variance-corrected lower bound fails (Proposition 4). Thus the sign of the single invariant E decides the second-order inequality locally.
(iv) For distinct rational roots, an exact reduction $\mathcal{S}(P) = R - \Lambda$ with $R \in \mathbb{Q}$ and Λ a linear form in $\psi(\tau_a)$, $\tau_a \in (0,1]$, showing that $\mathcal{S}(P)$ is always either rational or transcendental (Theorem 7); a summation-by-parts criterion for $\Lambda \neq 0$ (Theorem 8); and a complete decision when the roots occupy at most two classes modulo 1, in every degree (Corollary 3). In particular, rationality is governed by the class sums of the residues and *not* by congruence of the roots: $\sum_{n\ge 0} 1/\big((n+1)(n+2)(2n+1)(2n+5)\big) = 1/9$ although the roots fall into two classes (Example 5).

The remainder of this paper is organized as follows. §2 fixes notation. §3 proves the expansion and compares it with the literature, §4 gives the general recurrence, §5 presents the remainder bounds, and §6 discusses other centers. The cubic and quartic cases occupy §7–§9, and §10 contains reproducible numerical checks. §11–§12 relate the expansion to divided differences and prove a global lower bound. §13–§14 treat the arithmetic questions.

2. Notation and Preliminaries

Throughout, $\mathbb{N} = \{0, 1, 2, \dots\}$ and $-\mathbb{N} = \{0, -1, -2, \dots\}$. The letter N (not to be confused with $\mathbb{N}$) denotes a truncation order. The centered shifts are always written y_j, with $\eta = \max_j |y_j|$ and $\rho = \eta/\mu$; the letters r_j are reserved for roots of a polynomial P (§13–§14).

For $s > 1$ and $x \in \mathbb{C} \setminus (-\mathbb{N})$ the Hurwitz zeta function is

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}; \tag{5}$$

for $x > 0$ this is the usual definition, and for other $x \notin -\mathbb{N}$ the series still converges absolutely. Clearly

$$\zeta(s,x) = \zeta(s,x+1) + x^{-s}. \tag{6}$$

The digamma function $\psi = \Gamma'/\Gamma$ satisfies

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \tag{7}$$

$$\psi(z) = -\gamma + \sum_{n=0}^{\infty} \Big(\frac{1}{n+1} - \frac{1}{n+z} \Big), \qquad z \in \mathbb{C} \setminus (-\mathbb{N}), \tag{8}$$

where γ is Euler's constant, and ψ is strictly increasing on $(0,\infty)$ because $\psi'(x) = \zeta(2,x) > 0$.

For $s > 1$ and $x > 0$, comparison with an integral gives

$$\zeta(s,x) \le x^{-s} + \int_0^{\infty} (x+t)^{-s}\, \mathrm{d}t = x^{-s}\Big(1 + \frac{x}{s-1}\Big). \tag{9}$$

The complete homogeneous symmetric polynomials $h_k = h_k(y_1, \dots, y_d)$ are defined by

$$\sum_{k=0}^{\infty} h_k(y_1, \dots, y_d)\, t^k = \prod_{j=1}^{d} \frac{1}{1 - y_j t}, \qquad h_k = \sum_{\substack{m_1 + \cdots + m_d = k \\ m_j \geq 0}} y_1^{m_1} \cdots y_d^{m_d}, \tag{10}$$

and the elementary symmetric polynomials are denoted by $e_j = e_j(y_1, \dots, y_d)$, $e_0 = 1$. After centering, $e_1 = 0$.

Lemma 1. *For every $k \geq 0$,*

$$|h_k(y_1, \dots, y_d)| \leq \binom{k + d - 1}{d - 1} \eta^k.$$

If $y_1, \dots, y_d$ are pairwise distinct, then

$$h_k(y_1, \dots, y_d) = \sum_{j=1}^{d} \frac{y_j^{k+d-1}}{\prod_{i \neq j}(y_j - y_i)}, \tag{11}$$

and hence

$$|h_k| \leq K_y\, \eta^{k+d-1}, \qquad K_y = \sum_{j=1}^{d} \prod_{i \neq j} \frac{1}{|y_j - y_i|}.$$

Proof. The sum in (10) has $\binom{k+d-1}{d-1}$ monomials, each of modulus at most η^k. For distinct y_j, the partial-fraction decomposition $\prod_j (1 - y_j t)^{-1} = \sum_j A_j (1 - y_j t)^{-1}$ with $A_j = y_j^{d-1} / \prod_{i \neq j}(y_j - y_i)$ gives (11) (see also [1, Ch. I]). □

3. The Barycentric Hurwitz-Zeta Expansion

Theorem 1 (Barycentric expansion)**.** *Let $d \geq 2$, $\alpha_1, \dots, \alpha_d > 0$, and let μ, y_j, η be as in (2)–(4). If $\eta < \mu$, then*

$$\mathcal{S}_d(\alpha_1, \dots, \alpha_d) = \sum_{k=0}^{\infty} (-1)^k h_k(y_1, \dots, y_d)\, \zeta(d + k, \mu), \tag{12}$$

and the series on the right converges absolutely.

Proof. Put $u = n + \mu \geq \mu$. Since $|y_j / u| \leq \rho < 1$, the generating function (10) at $t = -1/u$ gives

$$\frac{1}{\prod_j (n + \alpha_j)} = \sum_{k=0}^{\infty} \frac{(-1)^k h_k(y_1, \dots, y_d)}{(n + \mu)^{d+k}}. \tag{13}$$

By Lemma 1 and (9),

$$\sum_{n \geq 0} \sum_{k \geq 0} \frac{|h_k|}{(n + \mu)^{d+k}} \leq \sum_{k \geq 0} \binom{k + d - 1}{d - 1} \eta^k \zeta(d + k, \mu) \leq \frac{1 + \mu}{\mu^d} \sum_{k \geq 0} \binom{k + d - 1}{d - 1} \rho^k < \infty.$$

Hence the double series converges absolutely, the summations may be interchanged, and (12) follows. □

The hypothesis $\eta < \mu$ is sufficient for the expansion; it is not needed for the convergence of $\mathcal{S}_d$ itself.

Proposition 1 (Uniqueness of the barycentric center)**.** *Let $c > 0$ and $\delta_j = \alpha_j - c$. If $\max_j |\delta_j| < c$, then*

$$\mathcal{S}_d(\alpha_1, \dots, \alpha_d) = \sum_{k=0}^{\infty} (-1)^k h_k(\delta_1, \dots, \delta_d)\, \zeta(d + k, c), \tag{14}$$

and $h_1(\delta_1, \dots, \delta_d) = \sum_j \alpha_j - dc$. Consequently, the first-order coefficient vanishes if and only if $c = \mu$.

Proof. The proof of Theorem 1 applies verbatim with μ replaced by c; the last assertion is immediate. □

Remark 1 (Relation to prior work)**.** For a single shift, the analogous statement is the classical Taylor expansion of $\zeta(s,x)$ in its second variable, in which the coefficients are binomial and the values $\zeta(s+k)$ appear (see [2, §25.11]). Theorem 1 is the symmetric multi-shift analogue: the binomial coefficients are replaced by complete homogeneous symmetric polynomials of the centered shifts. Many expansions of reciprocal products and series of Hurwitz zeta values are known; see Coffey [3], Paris [4], Srivastava and Choi [5], and Wang [6], who expands $\prod_k (1-z/a_k)^{-1}$ when (a_k) is the infinite arithmetic sequence being summed. The same series (1) is, up to sign, the divided difference $\psi[\alpha_1,\dots,\alpha_d]$ (§11), whose monotonicity and convexity properties were studied by Guo and Qi [7], Yang and Tian [8], and Tian and Yang [9]. We do not claim novelty for (12) as an identity: it follows in a few lines from (10). We are not aware of the consequences (i)–(iv) listed in the Introduction in the works cited, but given the density of closely related constructions we make no stronger claim than this.

4. The General Centered Recurrence

Since $e_1 = 0$ after centering,

$$\prod_{j=1}^{d}(1-y_j t) = 1 + \sum_{j=2}^{d}(-1)^j e_j t^j. \tag{15}$$

Theorem 2 (Centered recurrence)**.** *With $h_0 = 1$ and $h_k = 0$ for $k < 0$, for every $k \geq 1$,*

$$h_k = \sum_{j=2}^{\min(d,k)} (-1)^{j+1} e_j h_{k-j}. \tag{16}$$

In particular, $h_1 = 0$, and the entire sequence (h_k) is determined by $e_2,\dots,e_d$.

Proof. Multiply (10) by (15) and compare coefficients of t^k; this is the classical identity $\sum_{j=0}^{k}(-1)^j e_j h_{k-j} = 0$ [1, Ch. I, (2.6′)] specialized to $e_1 = 0$. □

The cubic and quartic cases below are the specializations $d = 3,4$, with $e_2 = -V/2$, where $V = \sum_j y_j^2$ (because $0 = e_1^2 = V + 2e_2$).

5. Explicit Remainder Estimates

For $N \geq 0$ let

$$T_N = \sum_{k=0}^{N}(-1)^k h_k(y_1,\dots,y_d)\,\zeta(d+k,\mu), \qquad R_N = \mathcal{S}_d(\alpha_1,\dots,\alpha_d) - T_N. \tag{17}$$

Theorem 3 (Remainder bound)**.** *Under the hypotheses of Theorem 1, let $\rho = \eta/\mu$. If*

$$q_N = \rho\,\frac{N+d+1}{N+2} < 1, \tag{18}$$

then

$$|R_N| \leq \Big(1 + \frac{\mu}{N+2}\Big)\frac{1}{\mu^d}\binom{N+d}{d-1}\frac{\rho^{N+1}}{1-q_N}. \tag{19}$$

Proof. For $k \geq N+1$ we have $d+k-1 \geq N+2$, so (9) gives $\zeta(d+k,\mu) \leq \mu^{-d-k}\big(1+\mu/(N+2)\big)$. With Lemma 1,

$$|R_N| \leq \Big(1+\frac{\mu}{N+2}\Big)\mu^{-d}\sum_{k=N+1}^{\infty} A_k, \qquad A_k = \binom{k+d-1}{d-1}\rho^k.$$

For $k \geq N+1$, $A_{k+1}/A_k = \rho\,(k+d)/(k+1) \leq q_N$, so the tail is at most $A_{N+1}/(1-q_N)$. □

The binomial factor in (19) comes from counting monomials and can substantially overestimate the actual error. When the shifts are distinct, the second part of Lemma 1 removes it.

Proposition 2 (Separation-sensitive bound). *Under the hypotheses of Theorem 1, assume moreover that $\alpha_1, \dots, \alpha_d$ are pairwise distinct, and let K_y be as in (11). Then for every $N \geq 0$,*

$$|R_N| \leq K_y\, \eta^{d-1} \left(1 + \frac{\mu}{N+2}\right) \frac{1}{\mu^d} \frac{\rho^{N+1}}{1-\rho}. \tag{20}$$

Proof. As in the preceding proof, we use $|h_k| \leq K_y \eta^{k+d-1}$ and then sum the geometric series $\sum_{k \geq N+1} \rho^k$ to obtain the stated bound. □

Both bounds certify the *truncation* error only. In numerical applications, the values $\zeta(d+k, \mu)$ are computed numerically, and that evaluation error must be controlled separately (§10).

6. Choice of Center Beyond the Barycentric Convergence Region

It may happen that $\max_j |\alpha_j - \mu| \geq \mu$ although all α_j are positive; then (12) is not asserted. A center $c > 0$ is admissible whenever

$$\rho(c) := \frac{\max_j |\alpha_j - c|}{c} < 1, \tag{21}$$

and Proposition 1 then gives a convergent expansion. For instance, $c = \max_j \alpha_j$ is always admissible, since $0 \leq c - \alpha_j < c$. A numerically efficient center minimizes $\rho(c)$, and does not necessarily coincide with the arithmetic mean. There are thus two different criteria: (i) minimizing the geometric ratio $\rho(c)$; (ii) eliminating the linear symmetric term. Only the second singles out $c = \mu$.

7. The Cubic Case and Its Two Invariants

Write $\mathcal{S}_3 = \mathcal{S}_3(\alpha, \beta, \gamma)$, $\mu = (\alpha + \beta + \gamma)/3$ and $x = \alpha - \mu$, $y = \beta - \mu$, $z = \gamma - \mu$, so $x + y + z = 0$. Put

$$V = x^2 + y^2 + z^2, \qquad E = xyz. \tag{22}$$

Since $xy + yz + zx = -V/2$,

$$(1 - xt)(1 - yt)(1 - zt) = 1 - \frac{V}{2}t^2 - Et^3, \qquad \sum_{k=0}^{\infty} h_k(x, y, z)\, t^k = \frac{1}{1 - \frac{V}{2}t^2 - Et^3}, \tag{23}$$

and Theorem 2 with $d = 3$, $e_2 = -V/2$, $e_3 = E$ gives

$$h_k = \frac{V}{2} h_{k-2} + E h_{k-3}, \qquad h_0 = 1, \quad h_{-1} = h_{-2} = 0. \tag{24}$$

The first coefficients, computed from (24) and checked independently by expanding (23) and by direct symbolic expansion of (10) with $z = -x - y$, are

k	0	1	2	3	4
h_k	1	0	$\frac{V}{2}$	E	$\frac{V^2}{4}$

k	5	6	7	8	9
h_k	VE	$\frac{V^3}{8} + E^2$	$\frac{3V^2E}{4}$	$\frac{V^4}{16} + \frac{3VE^2}{2}$	$\frac{V^3E}{2} + E^3$

(25)

Theorem 4 (Cubic barycentric expansion). *If $\max\{|x|, |y|, |z|\} < \mu$, then*

$$\begin{aligned}\mathcal{S}_3(\alpha, \beta, \gamma) = {} & \zeta(3, \mu) + \frac{V}{2}\zeta(5, \mu) - E\zeta(6, \mu) + \frac{V^2}{4}\zeta(7, \mu) - VE\,\zeta(8, \mu) \\ & + \Big(\frac{V^3}{8} + E^2\Big)\zeta(9, \mu) - \frac{3V^2E}{4}\zeta(10, \mu) + \Big(\frac{V^4}{16} + \frac{3VE^2}{2}\Big)\zeta(11, \mu) - \Big(\frac{V^3E}{2} + E^3\Big)\zeta(12, \mu) + \cdots,\end{aligned} \tag{26}$$

and the truncation after the term $k = N$ satisfies (19) and (20) with $d = 3$.

Proof. The expansion follows from Theorem 1 together with the coefficients in (25). The remainder statements follow from Theorem 3 and Proposition 2, subject to their respective hypotheses. □

The invariants have a direct meaning:

$$V = \frac{(\alpha-\beta)^2 + (\beta-\gamma)^2 + (\gamma-\alpha)^2}{3}, \tag{27}$$

measures dispersion, and E measures signed asymmetry. The centered cubic $t^3 - \frac{V}{2}t - E$ with roots x, y, z has discriminant $\text{Disc} = \frac{V^3}{2} - 27E^2 \geq 0$, since its roots are real; hence

$$|E| \leq \frac{V^{3/2}}{3\sqrt{6}}, \tag{28}$$

so V and E are constrained rather than independent parameters.

A Symmetric Cubic Family

If one centered shift vanishes, say $(x, y, z) = (-\delta, 0, \delta)$ with $0 < |\delta| < \mu$, then $V = 2\delta^2$, $E = 0$, and (24) gives $h_{2m} = \delta^{2m}$, $h_{2m+1} = 0$.

Theorem 5. *For* $0 < |\delta| < \mu$,

$$\sum_{n=0}^{\infty} \frac{1}{(n+\mu-\delta)(n+\mu)(n+\mu+\delta)} = \sum_{m=0}^{\infty} \delta^{2m}\zeta(2m+3, \mu). \tag{29}$$

Proof. This is the case $E = 0$ of Theorem 4. Directly, $\frac{1}{(u-\delta)u(u+\delta)} = \frac{1}{u}\sum_{m\geq 0} \delta^{2m}u^{-2m-2}$ for $u = n + \mu > |\delta|$. □

Example 1. For $(\alpha, \beta, \gamma) = (1, 2, 3)$ we have $\mu = 2$, $\delta = 1$, and the left-hand side of (29) equals $1/4$. Since $\zeta(s, 2) = \zeta(s) - 1$, this gives $\sum_{m\geq 0}\big(\zeta(2m+3) - 1\big) = \frac{1}{4}$.

8. A Sign-Controlled Double Expansion and One-Sided Bounds

As formal power series,

$$\frac{1}{1 - \frac{V}{2}t^2 - Et^3} = \sum_{j\geq 0}\Big(\frac{V}{2}t^2 + Et^3\Big)^j = \sum_{p,q\geq 0}\binom{p+q}{p}\Big(\frac{V}{2}\Big)^p E^q\, t^{2p+3q}, \tag{30}$$

and comparison with (23) therefore gives

$$h_k(x, y, z) = \sum_{2p+3q=k}\binom{p+q}{p}\Big(\frac{V}{2}\Big)^p E^q. \tag{31}$$

In the expansion of $\mathcal{S}_3$ the variable is $t = -1/(n+\mu)$, and $(-1)^{2p+3q} = (-1)^q$. This sign is responsible for the correct statement below.

Proposition 3. *If* $\max\{|x|, |y|, |z|\} < \mu$, *then*

$$\mathcal{S}_3(\alpha, \beta, \gamma) = \sum_{p,q\geq 0}\binom{p+q}{p}\Big(\frac{V}{2}\Big)^p (-E)^q\, \zeta(3 + 2p + 3q, \mu), \tag{32}$$

where the double series converges absolutely, so it may be summed in any order. If $E \leq 0$, *every term on the right is nonnegative.*

Proof. Let $\sigma = 1$ if $E \geq 0$ and $\sigma = -1$ if $E < 0$, and put $(\tilde{x}, \tilde{y}, \tilde{z}) = \sigma(x, y, z)$. Then $\tilde{x} + \tilde{y} + \tilde{z} = 0$, the invariants of the new triple are V and $\sigma^3 E = |E|$, and $\max\{|\tilde{x}|, |\tilde{y}|, |\tilde{z}|\} = \eta$. By (31) applied to this triple,

$$\sum_{2p+3q=k} \binom{p+q}{p} \left(\frac{V}{2}\right)^p |E|^q = h_k(\tilde{x}, \tilde{y}, \tilde{z}) \leq \binom{k+2}{2} \eta^k,$$

by Lemma 1. Therefore the triple series

$$\sum_{n\geq 0} \sum_{p,q\geq 0} \binom{p+q}{p} \left(\frac{V}{2}\right)^p |E|^q (n+\mu)^{-3-2p-3q} = \sum_{k\geq 0} h_k(\tilde{x}, \tilde{y}, \tilde{z})\, \zeta(3+k, \mu),$$

(a rearrangement of nonnegative terms) is finite, by the estimate in the proof of Theorem 1. Consequently, the triple series with the signed terms $\binom{p+q}{p}(V/2)^p(-E)^q(n+\mu)^{-3-2p-3q}$ converges absolutely and may be summed in any order. Summing first over p, q with $2p + 3q = k$ and using (31) together with $(-E)^q = (-1)^k E^q$ for $2p + 3q = k$, then over k, gives $\sum_k (-1)^k h_k\, (n+\mu)^{-3-k}$, which is (13); summing over n yields $\mathcal{S}_3$. Summing first over n yields the right-hand side of (32). The last assertion is clear since $V \geq 0$. □

Corollary 1 (Certified lower bounds for $E \leq 0$)**.** *Assume* $\max\{|x|, |y|, |z|\} < \mu$ *and* $E \leq 0$. *Then* $(-1)^k h_k(x, y, z) \geq 0$ *for every* k, *the partial sums* T_N *of* (26) *are nondecreasing in* N, *and*

$$T_N \leq \mathcal{S}_3(\alpha, \beta, \gamma) \leq T_N + B_N,$$

for every N, *where* B_N *is either of the right-hand sides of* (19) (*when* $q_N < 1$) *or* (20) *with* $d = 3$. *In particular,*

$$\mathcal{S}_3(\alpha, \beta, \gamma) \geq \zeta(3, \mu) + \frac{V}{2}\zeta(5, \mu) + |E|\, \zeta(6, \mu) + \frac{V^2}{4}\zeta(7, \mu),$$

with strict inequality unless $\alpha = \beta = \gamma$.

Proof. By (31), $(-1)^k h_k = \sum_{2p+3q=k} \binom{p+q}{p}(V/2)^p(-E)^q \geq 0$. Hence all terms of (26) are nonnegative, which gives monotonicity and $T_N \leq \mathcal{S}_3$; the upper bound is $R_N \leq |R_N|$. If the shifts are not all equal, then $V > 0$, and the displayed bound is T_4; the omitted term $k = 6$ is at least $\frac{V^3}{8}\zeta(9, \mu) > 0$, so the inequality is strict. □

The variance-corrected bound of Corollary 1 does not extend to $E > 0$.

Proposition 4. *Let* $\mu = 1$ *and* $(\alpha, \beta, \gamma) = (1 - \varepsilon, 1 - \varepsilon, 1 + 2\varepsilon)$, *so that* $V = 6\varepsilon^2$ *and* $E = 2\varepsilon^3 > 0$. *For* $0 < \varepsilon < \zeta(6)/480$,

$$\mathcal{S}_3(\alpha, \beta, \gamma) < \zeta(3) + \frac{V}{2}\zeta(5). \tag{33}$$

Proof. Here $\eta = 2\varepsilon$ and $\rho = 2\varepsilon$. For $\varepsilon \leq \frac{1}{4}$ we have $\rho \leq \frac{1}{2}$ and $q_3 = \frac{7}{5}\rho \leq \frac{7}{10}$, so Theorem 3 with $d = 3$, $N = 3$ gives $|R_3| \leq \frac{6}{5}\binom{6}{2}\frac{(2\varepsilon)^4}{3/10} = 960\,\varepsilon^4$. Since $T_3 = \zeta(3) + 3\varepsilon^2\zeta(5) - 2\varepsilon^3\zeta(6)$,

$$\mathcal{S}_3 - \big(\zeta(3) + 3\varepsilon^2\zeta(5)\big) \leq -2\varepsilon^3\zeta(6) + 960\,\varepsilon^4 < 0,$$

for $0 < \varepsilon < \zeta(6)/480$. □

Remark 2 (Consistency of the two statements)**.** The family of Proposition 4 has $E > 0$, so it lies outside the scope of Corollary 1; the mirror family $(1 + \varepsilon, 1 + \varepsilon, 1 - 2\varepsilon)$ has the same V and $E = -2\varepsilon^3 < 0$, and satisfies the bound. Numerically, for $\varepsilon = 0.1$ the difference $\mathcal{S}_3 - \zeta(3) - \frac{V}{2}\zeta(5)$ equals -1.22×10^{-3} for the first family and $+3.10 \times 10^{-3}$ for the second. Thus, locally near the diagonal, the sign of E decides whether the second-order variance bound holds. The failure for $E > 0$ is genuinely local: for $\varepsilon = 0.4$ the difference is $+6.26 \times 10^{-2}$ even though $E > 0$. (With the sign of E in (32) reversed, one would wrongly obtain a lower bound for $E \geq 0$, contradicting Proposition 4.)

9. The Quartic Case

Let $d = 4$, $y_1 + \cdots + y_4 = 0$, and put $V = \sum_j y_j^2$, $E_3 = e_3(y_1, \ldots, y_4)$, $E_4 = e_4(y_1, \ldots, y_4)$. Then

$$\prod_{j=1}^{4}(1 - y_j t) = 1 - \frac{V}{2}t^2 - E_3 t^3 + E_4 t^4, \tag{34}$$

$$h_k = \frac{V}{2}h_{k-2} + E_3 h_{k-3} - E_4 h_{k-4}, \tag{35}$$

and

$$h_0 = 1,\ \ h_1 = 0,\ \ h_2 = \frac{V}{2},\ \ h_3 = E_3,\ \ h_4 = \frac{V^2}{4} - E_4,\ \ h_5 = VE_3,\ \ h_6 = \frac{V^3}{8} + E_3^2 - VE_4. \tag{36}$$

These coefficients were also checked independently by direct symbolic expansion. When $y_4 = 0$, so that $E_4 = 0$ and $E_3 = y_1 y_2 y_3$ with $y_1 + y_2 + y_3 = 0$, they reduce to the cubic coefficients (25); for example $h_5 = VE_3$ matches $h_5 = VE$, which provides an independent check of the cubic coefficient array in (25). Thus, if $\max_j |y_j| < \mu$,

$$\begin{aligned}\mathcal{S}_4(\alpha_1, \ldots, \alpha_4) = \zeta(4, \mu) + \frac{V}{2}\zeta(6, \mu) - E_3\zeta(7, \mu) + \Big(\frac{V^2}{4} - E_4\Big)\zeta(8, \mu) \\ - VE_3\,\zeta(9, \mu) + \Big(\frac{V^3}{8} + E_3^2 - VE_4\Big)\zeta(10, \mu) + \cdots .\end{aligned} \tag{37}$$

10. Numerical Illustration

The following checks are reproducible with any multiprecision library; we used `mpmath` with 50 significant digits. The coefficients h_k were generated in exact rational arithmetic from (24), so the only numerical error in T_N comes from the values $\zeta(3 + k, \mu)$. The reference value of $\mathcal{S}_3$ was computed independently in two ways: from the divided-difference formula (39) below, i.e. from digamma values only, and by direct summation with convergence acceleration (`nsum`); the two agree to within 10^{-49}, and recomputation at 70 digits changes none of the digits displayed. Hence the numerical evaluation error is below 10^{-45} in every entry, far smaller than the truncation errors shown, and the column "error" is the truncation error R_N itself.

Example 2 ($E > 0$). $(\alpha, \beta, \gamma) = (1, \frac{3}{2}, \frac{7}{2})$: $\mu = 2$, $(x, y, z) = (-1, -\frac{1}{2}, \frac{3}{2})$, $V = \frac{7}{2}$, $E = \frac{3}{4}$, $\rho = \frac{3}{4}$, $K_y = 2$. The reference value is $\mathcal{S}_3 = 0.27763\,11777\,70754\,17159\,90952\ldots$. The truncations, signed errors, and two remainder bounds are compared in Table 1.

Table 1. Truncation Errors and Remainder Bounds for $E > 0$

N	T_N	error R_N	bound (19)	bound (20)
10	0.27889 63705 51807	-1.27×10^{-3}	3.84	1.11×10^{-1}
20	0.27770 75365 62341	-7.64×10^{-5}	4.51×10^{-1}	5.84×10^{-3}
40	0.27763 14202 06203	-2.42×10^{-7}	4.16×10^{-3}	1.78×10^{-5}

Example 3 ($E < 0$, close to the convergence boundary). $(\alpha, \beta, \gamma) = (\frac{1}{20}, \frac{3}{2}, \frac{29}{20})$: $\mu = 1$, $(x, y, z) = (-\frac{19}{20}, \frac{1}{2}, \frac{9}{20})$, $V = \frac{271}{200}$, $E = -\frac{171}{800}$, $\rho = \frac{19}{20}$, $K_y = \frac{200}{7}$. Here (18) requires $N \geq 37$. The reference value is $\mathcal{S}_3 = 9.43248\,74211\,29457\,29701\,66336\ldots$. The corresponding values near the convergence boundary are presented in Table 2.

Table 2. Truncation Errors and Remainder Bounds for $E < 0$ Near the Convergence Boundary

N	T_N	error R_N	bound (19)	bound (20)
100	9.38247 64483 18778	5.00×10^{-2}	9.51×10^{2}	2.93
200	9.43219 13297 03594	2.96×10^{-4}	1.69×10^{1}	1.73×10^{-2}
400	9.43248 74107 50663	1.04×10^{-8}	2.09×10^{-3}	6.03×10^{-7}

Since $E < 0$, Corollary 1 applies: the errors are positive, T_N increases, and for instance $9.43248\,7410 \le \mathcal{S}_3 \le 9.43248\,8014$ is a certified enclosure obtained from $N = 400$ and (20).

The numerical results in Tables 1 and 2 show that both bounds hold in these examples and that (20) is sharper than (19) by a factor between about 30 and 3500 here, while still overestimating the true error, and that close to the boundary $\rho \to 1$ the expansion converges slowly, as the factor ρ^{N+1} predicts; there a different center (§6) or the closed form of §11 is preferable. The numerical checks support, but do not replace, the proofs.

11. Divided Differences and the Hermite–Genocchi Representation

Suppose first that $\alpha_1, \dots, \alpha_d > 0$ are distinct. Partial fractions give

$$\frac{1}{\prod_{j=1}^{d}(X+\alpha_j)} = \sum_{j=1}^{d} \frac{1}{(X+\alpha_j)\prod_{k\neq j}(\alpha_k - \alpha_j)}. \tag{38}$$

Using $\sum_{n\ge 0}\left(\frac{1}{n+x} - \frac{1}{n+y}\right) = \psi(y) - \psi(x)$, the divergent harmonic parts cancel and

$$\mathcal{S}_d(\alpha_1, \dots, \alpha_d) = (-1)^d \psi[\alpha_1, \dots, \alpha_d], \tag{39}$$

where $\psi[\alpha_1, \dots, \alpha_d]$ is the divided difference of order $d-1$.

Let $\Delta_{d-1} = \{(\lambda_1, \dots, \lambda_{d-1}) : \lambda_j \ge 0,\ \sum_{j<d} \lambda_j \le 1\}$, put $\lambda_d = 1 - \sum_{j<d} \lambda_j$, and let $\mathrm{d}\lambda$ be Lebesgue measure on $\Delta_{d-1} \subset \mathbb{R}^{d-1}$; its total mass is $1/(d-1)!$. The Hermite–Genocchi formula states that, for f of class C^{d-1},

$$f[\alpha_1, \dots, \alpha_d] = \int_{\Delta_{d-1}} f^{(d-1)}\Big(\sum_{j=1}^{d} \lambda_j \alpha_j\Big)\, \mathrm{d}\lambda. \tag{40}$$

It is convenient to use the probability measure

$$\mathrm{d}\nu = (d-1)!\, \mathrm{d}\lambda \quad \text{on } \Delta_{d-1}. \tag{41}$$

Since $\psi^{(d-1)}(x) = (-1)^d (d-1)!\, \zeta(d, x)$, we obtain

$$\mathcal{S}_d(\alpha_1, \dots, \alpha_d) = \int_{\Delta_{d-1}} \zeta\Big(d, \sum_{j=1}^{d} \lambda_j \alpha_j\Big)\, \mathrm{d}\nu(\lambda). \tag{42}$$

Both sides of (42) are continuous in $(\alpha_1, \dots, \alpha_d) \in (0, \infty)^d$, so (42) extends to repeated shifts.

The same series thus has three complementary descriptions: partial fractions, divided differences, and the barycentric expansion. The first two are global; the third is local around the barycenter.

12. Convexity and a Universal Lower Bound

For $d \ge 2$ and $x > 0$, $\frac{\partial^2}{\partial x^2}\zeta(d, x) = d(d+1)\zeta(d+2, x) > 0$, so $x \mapsto \zeta(d, x)$ is strictly convex.

Theorem 6. *For all $\alpha_1, \dots, \alpha_d > 0$,*

$$\mathcal{S}_d(\alpha_1, \dots, \alpha_d) \ge \zeta(d, \mu), \tag{43}$$

with equality if and only if $\alpha_1 = \cdots = \alpha_d$.

Proof. By symmetry of ν under permutations of $(\lambda_1, \dots, \lambda_d)$, $\int \lambda_j \, \mathrm{d}\nu = 1/d$ for each j, hence

$$\int_{\Delta_{d-1}} \sum_{j=1}^{d} \lambda_j \alpha_j \, \mathrm{d}\nu(\lambda) = \frac{1}{d} \sum_{j=1}^{d} \alpha_j = \mu.$$

Jensen's inequality for the probability measure ν and the convex function $\zeta(d,\cdot)$ applied to (42) gives (43). Equality forces $\sum_j \lambda_j \alpha_j$ to be ν-a.e. constant, i.e. $\alpha_1 = \cdots = \alpha_d$. □

Theorem 6 does not require $\eta < \mu$. For a polynomial $P(x) = a_d \prod_j (x+\alpha_j)$ with $a_d > 0$ and $\alpha_j > 0$ it reads $a_d \sum_{n\geq 0} 1/P(n) \geq \zeta(d,\mu)$. The barycentric expansion refines it locally: the first correction is quadratic because $h_1 = 0$, and in the cubic case the next one involves E, which explains §8.

13. Arithmetic Consequences for Rational Simple Roots

In this section and the next, the hypotheses are arithmetic rather than analytic: the roots of P need not be negative, and the condition $\eta < \mu$ plays no role. Let

$$P(x) = a_d \prod_{j=1}^{d} (x - r_j) \in \mathbb{Q}[x], \qquad d \geq 2, \quad r_j \in \mathbb{Q} \text{ pairwise distinct}, \quad r_j \notin \mathbb{N}, \tag{44}$$

so that $P(n) \neq 0$ for every $n \in \mathbb{N}$, and put

$$\mathcal{S}(P) = \sum_{n=0}^{\infty} \frac{1}{P(n)}. \tag{45}$$

Then

$$\frac{1}{P(x)} = \sum_{j=1}^{d} \frac{c_j}{x - r_j}, \qquad c_j = \frac{1}{P'(r_j)} \in \mathbb{Q}^{\times}, \qquad \sum_{j=1}^{d} c_j = 0, \tag{46}$$

where the last relation follows from $1/P(x) = O(x^{-2})$ as $x \to \infty$.

Lemma 2. *Under* (44),

$$\mathcal{S}(P) = -\sum_{j=1}^{d} c_j \, \psi(-r_j) = \frac{1}{P(0)} - \sum_{j=1}^{d} c_j \, \psi(1 - r_j). \tag{47}$$

Proof. Since $-r_j \notin -\mathbb{N}$, (8) applies at $z = -r_j$. Using $\sum_j c_j = 0$,

$$\frac{1}{P(n)} = \sum_{j=1}^{d} c_j \Big(\frac{1}{n - r_j} - \frac{1}{n+1} \Big),$$

and summing over $n \geq 0$ gives $\mathcal{S}(P) = -\sum_j c_j(\psi(-r_j) + \gamma) = -\sum_j c_j \psi(-r_j)$. By (7), $\psi(-r_j) = \psi(1 - r_j) + 1/r_j$ (note $r_j \neq 0$), and $-\sum_j c_j/r_j = \sum_j c_j/(0 - r_j) = 1/P(0)$. □

For instance, for $P(x) = (x+1)(x+2)$ we have $c_1 = 1$ at $r_1 = -1$ and $c_2 = -1$ at $r_2 = -2$, and (47) gives $\mathcal{S}(P) = \psi(2) - \psi(1) = 1$, as it should.

Canonical Reduction

Put $s_j = -r_j \in \mathbb{Q} \setminus (-\mathbb{N})$ and group the indices j according to the class of s_j modulo 1; let $\mathcal{C}_1, \ldots, \mathcal{C}_m$ be the classes and $\tau_a \in (0,1]$ the unique representative of $\mathcal{C}_a$ in $(0,1]$. For $j \in \mathcal{C}_a$ write $s_j = \tau_a + k_j$ with $k_j \in \mathbb{Z}$; if $\tau_a = 1$ then $s_j \geq 1$, so $k_j \geq 0$. Iterating (7),

$$\psi(\tau + k) = \psi(\tau) + \varrho_k(\tau), \qquad \varrho_k(\tau) = \begin{cases} \displaystyle\sum_{i=0}^{k-1} \frac{1}{\tau + i}, & k \geq 0, \\ \displaystyle -\sum_{i=1}^{|k|} \frac{1}{\tau - i}, & k < 0, \end{cases} \tag{48}$$

where all denominators are nonzero (a vanishing $\tau - i$ would require $\tau = 1$ and $k < 0$). Hence $\varrho_{k_j}(\tau_a) \in \mathbb{Q}$ and (47) becomes

$$\left.\begin{aligned} \mathcal{S}(P) = R - \Lambda, \qquad \Lambda = \sum_{a=1}^{m} C_a\, \psi(\tau_a), \qquad C_a = \sum_{j \in \mathcal{C}_a} c_j, \\ R = -\sum_{a=1}^{m} \sum_{j \in \mathcal{C}_a} c_j\, \varrho_{k_j}(\tau_a) \in \mathbb{Q}, \end{aligned}\right\} \tag{49}$$

with

$$\sum_{a=1}^{m} C_a = 0. \tag{50}$$

The arguments $\tau_a \in (0,1]$ are pairwise distinct, and all rational corrections have been moved into R.

Example 4. For $P(x) = (x+1)(2x+1)$ we have $r = -1, -\frac{1}{2}$, $c = -1, 1$, $s = 1, \frac{1}{2}$; thus $m = 2$, $\tau = 1, \frac{1}{2}$, $R = 0$ and $\Lambda = -\psi(1) + \psi(\frac{1}{2}) = -2\log 2$, so $\mathcal{S}(P) = 2\log 2$. Had we used the representatives $1 - r_j$, the corresponding form $\psi(\frac{3}{2}) - \psi(2) = 1 - 2\log 2$ would contain a rational part; the canonical choice $\tau_a \in (0,1]$ avoids this.

The Rational/Transcendental Alternative

By Gauss's digamma theorem [10, §5.4(iii)], for integers $0 < p < q$,

$$\psi\Big(\frac{p}{q}\Big) + \gamma = -\log(2q) - \frac{\pi}{2}\cot\frac{\pi p}{q} + 2 \sum_{1 \le n \le (q-1)/2} \cos\frac{2\pi n p}{q}\, \log\sin\frac{\pi n}{q}, \tag{51}$$

and $\psi(1) + \gamma = 0$. Since $\pi = -i\log(-1)$ and the numbers $\cot(\pi p/q)$, $\cos(2\pi np/q)$, $\sin(\pi n/q)$ are algebraic, each $\psi(\tau) + \gamma$ with $\tau \in \mathbb{Q} \cap (0,1]$ is a linear form in logarithms of nonzero algebraic numbers with algebraic coefficients and without constant term. By (50),

$$\Lambda = \sum_{a=1}^{m} C_a\big(\psi(\tau_a) + \gamma\big), \tag{52}$$

so Λ is itself such a linear form; Euler's constant has disappeared, and no rational constant is hidden in Λ.

We use Baker's theorem [11, Theorem 2.1]: if $\beta_1, \dots, \beta_n$ are algebraic and $\alpha_1, \dots, \alpha_n$ are nonzero algebraic numbers, then $\beta_1 \log\alpha_1 + \cdots + \beta_n \log\alpha_n$ is either 0 or transcendental. (Baker's Theorem 2.1 asserts that 1 and $\mathbb{Q}$-linearly independent logarithms of algebraic numbers are linearly independent over $\overline{\mathbb{Q}}$; writing the given logarithms in terms of a $\mathbb{Q}$-basis of their $\mathbb{Q}$-span yields the stated form.)

Theorem 7 (Rational or transcendental)**.** *Under* (44)*, with R and Λ as in* (49)*:*

(a) *if $\Lambda = 0$, then $\mathcal{S}(P) = R \in \mathbb{Q}$;*
(b) *if $\Lambda \neq 0$, then $\mathcal{S}(P)$ is transcendental.*

In particular, $\mathcal{S}(P)$ is never an irrational algebraic number, and $\mathcal{S}(P) \in \mathbb{Q}$ if and only if $\Lambda = 0$.

Proof. (a) is (49). For (b), Λ is a nonzero linear form as above, hence transcendental by Baker's theorem, and so is $\mathcal{S}(P) = R - \Lambda$ because $R \in \mathbb{Q}$. □

The hypothesis in (b) concerns the canonical form (49), which contains no rational part. For non-canonical representatives (for example $1 - r_j$) the corresponding digamma combination may contain a rational part, and its non-vanishing would not suffice; see Example 4. What remains is to decide when $\Lambda = 0$.

Corollary 2 (Vanishing class sums)**.** *If $C_a = 0$ for every a, then $\mathcal{S}(P) \in \mathbb{Q}$. In particular, this holds when all roots are congruent modulo 1 ($m = 1$), since then $C_1 = \sum_j c_j = 0$.*

Theorem 8 (Summation-by-parts criterion)**.** *Order the classes so that* $\tau_1 < \tau_2 < \cdots < \tau_m$*, and put* $D_a = C_1 + \cdots + C_a$ $(1 \le a \le m)$*, so that* $D_m = 0$*. Then*

$$\Lambda = \sum_{a=1}^{m-1} D_a\big(\psi(\tau_a) - \psi(\tau_{a+1})\big). \tag{53}$$

Consequently, if $D_1, \dots, D_{m-1}$ *are all* ≥ 0*, or all* ≤ 0*, and not all zero, then* $\Lambda \neq 0$ *and* $\mathcal{S}(P)$ *is transcendental.*

Proof. Summation by parts with $C_a = D_a - D_{a-1}$, $D_0 = 0$, $D_m = 0$ gives (53). Since ψ is strictly increasing on $(0,\infty)$ and $0 < \tau_a < \tau_{a+1} \le 1$, every bracket in (53) is negative; under the sign hypothesis Λ is therefore nonzero, and Theorem 7 applies. □

The strict monotonicity of ψ is used here only on $(0,1]$, where it is valid; this is why the canonical arguments τ_a are needed.

Corollary 3 (At most two classes)**.** *Suppose the roots occupy at most two classes modulo* 1*. If* $m = 1$*, or* $m = 2$ *and* $C_1 = 0$*, then* $\mathcal{S}(P) \in \mathbb{Q}$*; if* $m = 2$ *and* $C_1 \neq 0$*, then* $\mathcal{S}(P)$ *is transcendental.*

Proof. For $m = 2$, (53) reads $\Lambda = C_1(\psi(\tau_1) - \psi(\tau_2))$ with $\psi(\tau_1) \neq \psi(\tau_2)$. □

Example 5 (Rational value with two classes)**.** The hypothesis "all roots congruent modulo 1" in Corollary 2 is sufficient but not necessary for rationality when $d \ge 4$. Let $P(x) = (x+1)(x+2)(2x+1)(2x+5)$, with roots $-1, -2, -\frac{1}{2}, -\frac{5}{2}$. The roots are symmetric about $-\frac{3}{2}$; for a polynomial of even degree with this symmetry $P(-3-x) = P(x)$, hence $P'(-3-x) = -P'(x)$, so c_j changes sign under the reflection. The reflection preserves each class ($-1 \leftrightarrow -2$ and $-\frac{1}{2} \leftrightarrow -\frac{5}{2}$), so $C_1 = C_2 = 0$ ($c = -\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, -\frac{1}{6}$), and Corollary 2 gives $\mathcal{S}(P) \in \mathbb{Q}$. Evaluating (48) and (49), with numerical summation providing an independent check, one obtains

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)(2n+1)(2n+5)} = \frac{1}{9}.$$

Example 6 (Three classes)**.** (a) $P(x) = (3x+1)(3x+2)(2x+3)$: the classes are $\tau = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ with (D_1, D_2) of the same sign (both positive after normalizing P to be monic), so $\mathcal{S}(P)$ is transcendental by Theorem 8.

(b) $P(x) = (x+1)(2x+1)(3x+1)$: here $\tau = \frac{1}{3}, \frac{1}{2}, 1$, and (D_1, D_2) have opposite signs, so Theorem 8 does not apply. A direct evaluation with (51) gives

$$\mathcal{S}(P) = \frac{\sqrt{3}}{4}\,\pi + \frac{9}{4}\log 3 - 4\log 2.$$

Since $\log(-1), \log 2, \log 3$ are $\mathbb{Q}$-linearly independent, Baker's theorem shows that this number is nonzero, hence transcendental.

Problem 1. For $m \ge 3$ classes, can $\Lambda = 0$ occur with $(C_1, \dots, C_m) \neq 0$?

The question is not empty, because the numbers $\psi(\tau) + \gamma$ satisfy nontrivial $\mathbb{Q}$-linear relations coming from the multiplication formula; for instance, by (51),

$$\big(\psi(\tfrac{1}{4}) + \gamma\big) + \big(\psi(\tfrac{3}{4}) + \gamma\big) = 3\big(\psi(\tfrac{1}{2}) + \gamma\big) = -6\log 2,$$

so the vector $(C_{1/4}, C_{1/2}, C_{3/4}, C_1) = (1, -3, 1, 1)$, which satisfies (50), gives $\Lambda = 0$. Hence non-vanishing of the class sums does not by itself imply $\Lambda \neq 0$; the question is whether such a relation can be realized by the residues $c_j = 1/P'(r_j)$ of a polynomial. An exact search over all polynomials whose roots $r_j = -s_j$ satisfy $s_j = \tau + k$ with $\tau \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$, using four roots with $0 \le k \le 24$ or five roots with $0 \le k \le 8$, found no such realization. We leave Problem 1 open, and we do not claim a rational/transcendental *decision procedure* in general; Theorem 7 shows that the dichotomy itself always holds.

For related transcendence results concerning individual digamma values at rational arguments, see Murty and Saradha [12] and Murty [13].

14. Quadratic Polynomials

Let $P(x) = ax^2 + bx + c \in \mathbb{Z}[x]$, $a \neq 0$, with $P(n) \neq 0$ for all $n \in \mathbb{N}$, and discriminant $D = b^2 - 4ac$.

Distinct Rational Roots

Suppose $D = m^2$ with $m > 0$ an integer. The roots are

$$r_{1,2} = \frac{-b \mp m}{2a}, \qquad r_2 - r_1 = \frac{m}{a}. \tag{54}$$

Theorem 9 (Quadratic rational-root dichotomy)**.** *Under these assumptions,*

$$\mathcal{S}(P) = \frac{1}{m}\big(\psi(-r_1) - \psi(-r_2)\big), \tag{55}$$

and

$$\mathcal{S}(P) \in \mathbb{Q} \iff r_2 - r_1 \in \mathbb{Z} \iff a \mid m. \tag{56}$$

If $a \nmid m$, then $\mathcal{S}(P)$ is transcendental.

Proof. Here $c_1 = 1/P'(r_1) = 1/\big(a(r_1 - r_2)\big) = -1/m$ and $c_2 = 1/m$, so (55) is (47). If $r_2 - r_1 \in \mathbb{Z}$, there is one class and $\mathcal{S}(P) \in \mathbb{Q}$ by Corollary 2. Otherwise there are two classes with $C_1 = c_1 \neq 0$, and Corollary 3 gives transcendence. The last equivalence follows from (54). □

Remark 3. In terms of the shifted arguments $1 - r_j$, (47) gives $\mathcal{S}(P) = \frac{1}{P(0)} + \frac{1}{m}\big(\psi(1 - r_1) - \psi(1 - r_2)\big)$, where $\frac{1}{P(0)} = \frac{1}{m}\big(\frac{1}{r_1} - \frac{1}{r_2}\big) \in \mathbb{Q}$; for $P(x) = (x+1)(x+2)$ the term $1/P(0) = \frac{1}{2}$ is exactly the difference between $\mathcal{S}(P) = 1$ and $\psi(3) - \psi(2) = \frac{1}{2}$. Since adding a rational number does not change whether a number is rational or transcendental, the classification is insensitive to this term. The transcendence statement also follows from Murty's result [13, Corollary 1.1] on differences $\psi(x) - \psi(y)$ of digamma values at rational arguments with $x - y \notin \mathbb{Z}$; the proof given here is self-contained apart from Baker's theorem.

The Double Root Case

Suppose $D = 0$, so $P(x) = a(x - r)^2$ with $r = -b/(2a) \notin \mathbb{N}$. Then

$$\mathcal{S}(P) = \frac{1}{a}\sum_{n=0}^{\infty} \frac{1}{(n-r)^2} = \frac{1}{a}\,\zeta(2, -r), \tag{57}$$

with $\zeta(2, \cdot)$ as in (5).

Proposition 5 (Double root, $a \mid b$)**.** *If $D = 0$ and $a \mid b$, then $\mathcal{S}(P) = \lambda\pi^2 + q$ with $\lambda \in \mathbb{Q}^\times$ and $q \in \mathbb{Q}$; in particular $\mathcal{S}(P)$ is transcendental.*

Proof. Write $b = aj$ with $j \in \mathbb{Z}$, so $-r = j/2$.

$j = 2t$ even. Then $-r = t$, and $r \notin \mathbb{N}$ forces $t \geq 1$. Hence $\zeta(2, t) = \sum_{i \geq t} i^{-2} = \zeta(2) - \sum_{i=1}^{t-1} i^{-2}$ and $\mathcal{S}(P) = \frac{\pi^2}{6a} - \frac{1}{a}\sum_{i=1}^{t-1} i^{-2}$.

$j = 2t + 1$ odd, $t \geq 0$. Then $\zeta(2, t + \frac{1}{2}) = 4\sum_{n \geq 0} (2n + 2t + 1)^{-2}$, and since $\sum_{i \text{ odd}} i^{-2} = \pi^2/8$,

$$\mathcal{S}(P) = \frac{\pi^2}{2a} - \frac{4}{a}\sum_{\substack{1 \leq i \leq 2t-1 \\ i \text{ odd}}} \frac{1}{i^2}.$$

$j = 2t+1$ *odd*, $t \le -1$. Put $s = -t-1 \ge 0$, so $r = s + \frac{1}{2}$. As n runs over $\mathbb{N}$, $2n-2s-1$ runs over $-(2s+1), \dots, -1$ and then over $1, 3, 5, \dots$, so

$$\mathcal{S}(P) = \frac{\pi^2}{2a} + \frac{4}{a} \sum_{\substack{1 \le i \le 2s+1 \\ i \text{ odd}}} \frac{1}{i^2}.$$

In every case $\lambda \in \{\frac{1}{6a}, \frac{1}{2a}\}$ and $q \in \mathbb{Q}$; since π^2 is transcendental, so is $\lambda\pi^2 + q$. □

When $a \nmid b$, write $-r = b/(2a)$ in lowest terms. Its denominator q satisfies $q \ge 3$: indeed $q \in \{1, 2\}$ means $2 \cdot \frac{b}{2a} = \frac{b}{a} \in \mathbb{Z}$, i.e. $a \mid b$. Write

$$-r = k + \frac{p}{q}, \qquad k \in \mathbb{Z}, \quad 0 < p < q, \quad \gcd(p, q) = 1.$$

By (6), the integer shift contributes only a finite rational correction:

$$\begin{aligned} \zeta\Big(2, \frac{p}{q} + k\Big) &= \zeta\Big(2, \frac{p}{q}\Big) - \sum_{i=0}^{k-1} \Big(\frac{p}{q} + i\Big)^{-2} \quad (k \ge 0), \\ \zeta\Big(2, \frac{p}{q} - k\Big) &= \zeta\Big(2, \frac{p}{q}\Big) + \sum_{i=1}^{k} \Big(\frac{p}{q} - i\Big)^{-2} \quad (k \ge 1). \end{aligned} \tag{58}$$

For instance $\zeta(2, \frac{5}{4}) = \zeta(2, \frac{1}{4}) - 16$. For the reduced argument p/q, orthogonality of Dirichlet characters modulo q [14, Ch. 6] gives

$$\zeta\Big(2, \frac{p}{q}\Big) = q^2 \sum_{\substack{n \ge 1 \\ n \equiv p \pmod q}} \frac{1}{n^2} = \frac{q^2}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi}(p)\, L(2, \chi), \tag{59}$$

where $\gcd(p, q) = 1$ is essential. The even characters contribute algebraic multiples of π^2, while for $q \ge 3$ the group of characters modulo q contains odd characters (because $-1 \not\equiv 1 \pmod q$), whose values $L(2, \chi)$ have no known closed form. Their total contribution is $\frac{1}{2}\big(\zeta(2, \frac{p}{q}) - \zeta(2, 1 - \frac{p}{q})\big) \ne 0$, since $p/q \ne \frac{1}{2}$ and $\zeta(2, \cdot)$ is strictly decreasing. This is the parity obstruction responsible for the open status of Catalan's constant $G = L(2, \chi_{-4})$ (cf. [15]).

Example 7. For $P(n) = (4n+1)^2$ we have $a = 16$, $b = 8$, $16 \nmid 8$, $r = -\frac{1}{4}$, and $-r = \frac{1}{4}$ is already reduced ($k = 0$, $q = 4$). Hence

$$\mathcal{S}(P) = \frac{1}{16}\,\zeta\Big(2, \frac{1}{4}\Big) = \frac{\pi^2}{16} + \frac{G}{2},$$

using $\zeta(2, \frac{1}{4}) = \psi'(\frac{1}{4}) = \pi^2 + 8G$. Thus $\mathcal{S}(P) \in \mathbb{Q}$ if and only if $G + \pi^2/8 \in \mathbb{Q}$, a question with the same open status as the irrationality of G. This example is consistent with, but is not a proof of, the general discussion above.

Complex Conjugate Roots

Suppose $D < 0$ and, for simplicity, $a > 0$. Put

$$r_{1,2} = u \pm iv, \qquad u = -\frac{b}{2a}, \qquad v = \frac{\sqrt{-D}}{2a} > 0.$$

Partial fractions give $\frac{1}{P(n)} = \frac{1}{2iav}\big(\frac{1}{n-u-iv} - \frac{1}{n-u+iv}\big)$, and $\sum_{n \ge 0}\big(\frac{1}{n+z_1} - \frac{1}{n+z_2}\big) = \psi(z_2) - \psi(z_1)$ gives

$$\mathcal{S}(P) = \frac{1}{av}\,\operatorname{Im}\psi(-u + iv), \tag{60}$$

the digamma function being evaluated at a non-real point. If moreover $u < 0$, so that $x = -u > 0$, the integral representation of ψ yields

$$\mathcal{S}(P) = \frac{1}{av}\int_0^\infty \frac{e^{-xt}\sin(vt)}{1-e^{-t}}\,\mathrm{d}t = \frac{1}{2av}\int_0^\infty e^{-(x-\frac{1}{2})t}\frac{\sin(vt)}{\sinh(t/2)}\,\mathrm{d}t. \tag{61}$$

(Both formulas were checked numerically, e.g. for $P(x) = x^2 + x + 1$.) Here $u \pm iv$ is a pair of non-real algebraic numbers, Gauss's formula does not apply, and we do not assert a rationality/transcendence classification.

15. Conclusion

For positive shifts $\alpha_1, \ldots, \alpha_d$ with $\max_j |\alpha_j - \mu| < \mu$, the barycentric expansion (12) expresses $\mathcal{S}_d$ through $\zeta(d+k, \mu)$ with coefficients determined by $e_2, \ldots, e_d$ of the centered shifts, and Theorem 3 and Proposition 2 certify its truncations. In the cubic case, the sign-controlled expansion (32) shows that for $E \leq 0$ every truncation is a lower bound, while for $E > 0$ the variance-corrected bound fails near the diagonal. Without any condition on η, Jensen's inequality gives the global bound $\mathcal{S}_d \geq \zeta(d, \mu)$, that is, $a_d\mathcal{S}(P) \geq \zeta(d, \mu)$ for $P(x) = a_d\prod_j(x + \alpha_j)$ with $a_d > 0$.

For polynomials with distinct rational roots outside $\mathbb{N}$, $\mathcal{S}(P)$ is always either rational or transcendental, and it is rational exactly when the canonical form Λ vanishes. This is decided completely when the roots lie in at most two classes modulo 1 (in particular for all quadratic polynomials with distinct rational roots), and by a sign criterion in many other cases; rationality is governed by the class sums of residues rather than by congruence of roots. Whether Λ can vanish with nonzero class sums (Problem 1), and the arithmetic of the double-root case with $a \nmid b$ and of the case $D < 0$, remain open. Natural further directions are sharper coefficient-sensitive remainder estimates and optimal inequalities in the cubic (V, E) region.

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