

## Article

# Constraint internalization through parametric reconstruction: A legality-aware framework for geometric optimization problems

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**Abstract:** Geometric optimization problems such as  $PA + k \cdot PB$  are commonly solved using reflection methods, projection techniques, or calculus-based optimization. In elementary treatments, these approaches often assume that signed algebraic expressions directly represent non-negative geometric distances across the entire real domain. When the variable extends beyond the traditional geometric boundary, this implicit assumption fails, creating what we describe as a *representation legality gap*. In this paper, we propose a parametric reconstruction framework that internalizes geometric admissibility constraints into the algebraic representation. Rather than partitioning the variable domain through piecewise case distinctions, we introduce an admissible auxiliary scaffold parameter  $L \geq x$  to decompose the objective function into a geometrically legal core path and an algebraically compensating residual. The auxiliary parameter  $L$  cancels identically in the reconstructed representation, ensuring global validity across  $\mathbb{R}$ . We demonstrate this framework on a classical Hu Bugui-type optimization problem, establish the global minimum via dynamic projection inequalities, prove the exact invariance of the minimizer, and extend the result to an isomorphic parametric family.

**Keywords:** geometric optimization, constraint internalization, parametric reconstruction, representation legality gap, Hu Bugui problem, algebraic invariance

MSC: 51M16, 51N20

## 1. Introduction

**G**eometric extremum problems play an important role in secondary and undergraduate mathematics education. Classical methods include reflection transformations, inequality-based reasoning, and calculus techniques [1]. A representative example is the so-called Hu Bugui model, which relates to classical weighted path optimization and optical trajectory problems [2].

Despite their pedagogical value, geometric interpretations often rely on implicit domain assumptions. In particular, expressions of the form  $C - x$  are treated as geometric segment lengths without explicit verification of non-negativity. When the dynamic variable  $x$  ranges over the unconstrained continuum  $\mathbb{R}$ , the quantity  $C - x$  can become negative, producing intermediate expressions that lack direct Euclidean interpretation.

In elementary treatments, this issue is often handled implicitly by restricting the admissible position of the dynamic point, or explicitly through piecewise case distinctions. We introduce the term *representation legality gap* to describe this structural inconsistency between an unconstrained algebraic variable and a non-negative geometric quantity.

This paper proposes an alternative approach: instead of enforcing feasibility externally via piecewise domain subdivisions, we reconstruct the representation by lifting the geometric constraint into an auxiliary scaffolding parameter. This construction ensures that every geometric component remains non-negative and interpretable, allowing global optimization to proceed uniformly over  $\mathbb{R}$ .

## 2. Related work

The reflection method originates from classical geometric optimization problems such as Heron's shortest path problem [3] and is closely related to Fermat's principle and Snell's law in optics [4].

In mathematics education, research on semiotic representations emphasizes cognitive transformations between algebraic, graphical, and geometric registers [5]. However, existing literature primarily focuses on translation between fixed representations rather than algebraic reconstruction to preserve geometric semantics.

In broader mathematical modeling, handling constraints typically involves slack variables, extended formulations, or penalty reformulations [6]. Unlike reflection methods that alter spatial geometric configurations, calculus-based methods that solve the optimization without preserving the underlying geometric representation, or piecewise analysis that fragments the variable domain, our framework maintains the unconstrained objective across  $\mathbb{R}$  by reconstructing its internal representation, embedding geometric feasibility directly into the algebraic decomposition.

## 3. Framework: Parametric reconstruction

**Definition 1** (Geometric Legality). Whenever an algebraic expression is interpreted as a Euclidean distance, its value must be non-negative and must correspond to an admissible geometric segment.

**Definition 2** (Legality Gap). A representation legality gap occurs when an algebraic objective is defined over a domain  $D \subseteq \mathbb{R}$ , but intermediate components fail to maintain geometric legality on subsets of  $D$ .

**Definition 3** (Constraint Internalization Principle). Constraint internalization is the process of absorbing geometric admissibility conditions into the construction of an auxiliary parametric representation, such that geometric operations act exclusively on legal geometric quantities without requiring piecewise domain restrictions.

### 3.1. Concrete formulation and parametric scaffolding

To illustrate this mechanism, consider the classical Hu Bugui-type optimization problem over the unconstrained real domain  $x \in \mathbb{R}$ :

$$f(x) = \sqrt{x^2 + 9} + \frac{3\sqrt{3} - x}{2}. \quad (1)$$

In the Cartesian coordinate plane with origin  $O(0,0)$ , let  $A(0,3)$  be a fixed point on the  $y$ -axis, and let  $P(x,0)$  be a dynamic point sliding along the  $x$ -axis. The term  $\sqrt{x^2 + 9}$  corresponds to the Euclidean distance:

$$PA = \sqrt{(x-0)^2 + (0-3)^2} = \sqrt{x^2 + 9}.$$

For the second term  $\frac{3\sqrt{3}-x}{2}$ , a direct interpretation as  $\frac{1}{2}PB$  with fixed  $B(3\sqrt{3},0)$  fails for  $x > 3\sqrt{3}$ . To maintain geometric legality for all  $x \in \mathbb{R}$ , we define the *admissible scaffold set* for any  $x \in \mathbb{R}$ :

$$\mathcal{L}(x) = \{L \in \mathbb{R} : L \geq x\}.$$

For every  $x \in \mathbb{R}$ ,  $\mathcal{L}(x)$  is non-empty (for instance, choosing  $L = x + \epsilon$  with  $\epsilon \geq 0$ ).

For any chosen  $L \in \mathcal{L}(x)$ , we place an auxiliary anchor point  $B(L,0)$  on the  $x$ -axis. Because  $L \geq x$ , the segment length  $PB = L - x$  is guaranteed to be non-negative ( $PB \geq 0$ ). We construct an auxiliary reference line  $l_L$  passing through  $B(L,0)$ , such that the ray from  $B$  toward  $x < L$  makes an angle of  $\alpha = 30^\circ$  with the negative  $x$ -axis:

$$l_L : x - \sqrt{3}y - L = 0.$$

Dropping an orthogonal perpendicular  $PH \perp l_L$  at  $H$  creates a right triangle  $\triangle PBH$  where  $\sin 30^\circ = \frac{PH}{PB} = \frac{1}{2}$ , establishing the exact geometric scaling:

$$PH = \frac{1}{2}PB = \frac{L - x}{2}.$$

We now partition the original objective function into an active geometric core  $g(x, L)$  and an algebraic compensating residual:

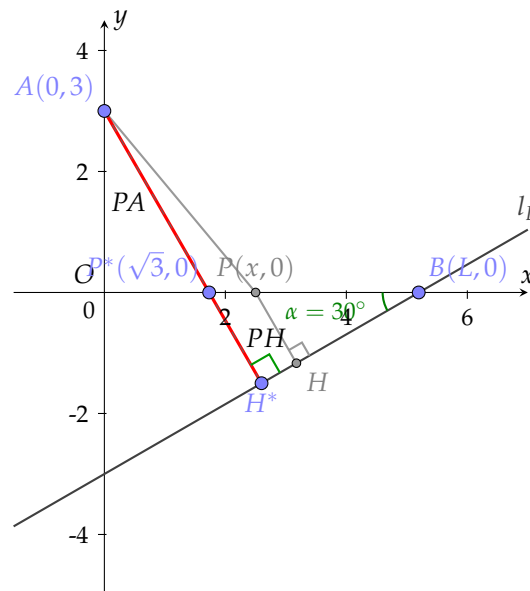
$$f(x) = \underbrace{\left( \sqrt{x^2 + 9} + \frac{L - x}{2} \right)}_{g(x, L)} + \frac{3\sqrt{3} - L}{2}. \quad (2)$$

In this formulation, *scaffold validity* is explicitly verified by the admissibility condition  $L \in \mathcal{L}(x)$  (which guarantees  $PB \geq 0$ ), while *reconstruction equivalence* is guaranteed unconditionally by the algebraic identity in Eq. (2).

Under this reconstruction, the components of  $g(x, L)$  map faithfully to physical Euclidean segments:

$$g(x, L) \equiv PA + PH, \quad \text{where } PA = \sqrt{x^2 + 9} \text{ and } PH = \frac{L - x}{2} \geq 0.$$

The residual  $\frac{3\sqrt{3} - L}{2}$  serves as an algebraic compensation and is not required to carry geometric segment semantics. Thus, the reconstructed representation preserves exact algebraic invariance with respect to  $L$ . The geometric path minimization on the reference line  $l_L$ , illustrated for  $L = 3\sqrt{3}$ , is shown in Figure 1.



**Figure 1.** Geometric path minimization onto reference line  $l_L$  (illustrated for  $L = 3\sqrt{3}$ )

### 3.2. Theoretical results and proofs

**Theorem 1** (Dynamic Geometric Projection Bound). *For any  $x \in \mathbb{R}$  and any scaffold parameter  $L \in \mathcal{L}(x)$ , the geometric core  $g(x, L) = PA + PH$  satisfies the dynamic projection lower bound:*

$$g(x, L) \geq \frac{|L + 3\sqrt{3}|}{2}. \quad (3)$$

**Proof.** In Euclidean plane geometry, for the fixed point  $A(0, 3)$  and any point  $H$  on the target reference line  $l_L : x - \sqrt{3}y - L = 0$ , the path length satisfies:

$$PA + PH \geq AH \geq d(A, l_L),$$

where the first inequality follows from the Euclidean triangle inequality on  $\triangle PAH$  (with equality holding if and only if  $P$  lies on the line segment  $AH$ ), and the second inequality represents the fundamental definition of the orthogonal point-to-line distance (with equality holding if and only if  $AH \perp l_L$ ).

Using the point-to-line orthogonal distance formula, the distance from  $A(0, 3)$  to  $l_L$  is given by:

$$d(A, l_L) = \frac{|0 - \sqrt{3}(3) - L|}{\sqrt{1^2 + (-\sqrt{3})^2}} = \frac{|-3\sqrt{3} - L|}{2} = \frac{|L + 3\sqrt{3}|}{2}.$$

Because the parametric construction ensures  $g(x, L) \equiv PA + PH$  for any  $L \geq x$ , the inequality  $g(x, L) \geq \frac{|L + 3\sqrt{3}|}{2}$  holds unconditionally for all  $x \in \mathbb{R}$  and  $L \in \mathcal{L}(x)$ .  $\square$

**Theorem 2** (Global Minimum via Scaffold Invariance). *The objective function  $f(x)$  attains a unique global minimum on  $\mathbb{R}$ :*

$$f_{\min} = 3\sqrt{3},$$

achieved uniquely at  $x^* = \sqrt{3}$ .

**Proof.** By Theorem 1, for any  $x \in \mathbb{R}$  and any choice of  $L \in \mathcal{L}(x)$ , we have  $g(x, L) \geq \frac{|L + 3\sqrt{3}|}{2}$ . Using the elementary real inequality  $|u| \geq u$ , we obtain:

$$\frac{|L + 3\sqrt{3}|}{2} \geq \frac{L + 3\sqrt{3}}{2}.$$

Substituting this into the reconstructed objective (2) yields:

$$\begin{aligned} f(x) &= g(x, L) + \frac{3\sqrt{3} - L}{2} \\ &\geq \frac{|L + 3\sqrt{3}|}{2} + \frac{3\sqrt{3} - L}{2} \\ &\geq \frac{L + 3\sqrt{3}}{2} + \frac{3\sqrt{3} - L}{2} \\ &= \frac{L + 3\sqrt{3} + 3\sqrt{3} - L}{2} \\ &= \frac{6\sqrt{3}}{2} = 3\sqrt{3}. \end{aligned}$$

The auxiliary parameter  $L$  cancels identically from the inequality, establishing  $f(x) \geq 3\sqrt{3}$  globally for all  $x \in \mathbb{R}$ .

To establish equality and uniqueness, all intermediate inequalities must be tight:

- Equality in  $PA + PH \geq d(A, l_L)$  requires points  $A(0, 3)$ ,  $P(x, 0)$ , and  $H$  to be collinear with  $P$  lying on the line segment  $AH$ , together with  $AH \perp l_L$ . Since the line  $l_L$  has slope  $\frac{1}{\sqrt{3}}$ , the vector  $\vec{AP} = (x, -3)$  must be orthogonal to the direction vector of  $l_L$ , which is  $(\sqrt{3}, 1)$ :

$$(x, -3) \cdot (\sqrt{3}, 1) = 0 \implies \sqrt{3}x - 3 = 0 \implies x^* = \sqrt{3}.$$

- Equality in  $|L + 3\sqrt{3}| \geq L + 3\sqrt{3}$  requires  $L + 3\sqrt{3} \geq 0$ , which is satisfied for any choice  $L \in \mathcal{L}(x^*)$  since  $L \geq \sqrt{3} > -3\sqrt{3}$ .

Evaluating at  $x^* = \sqrt{3}$  gives  $f(\sqrt{3}) = \sqrt{3 + 9} + \frac{3\sqrt{3} - \sqrt{3}}{2} = 2\sqrt{3} + \sqrt{3} = 3\sqrt{3}$ , confirming the unique global minimum.  $\square$

#### 4. Extension to isomorphic parametric family

**Theorem 3** (Parametric Invariant Family). *For any constant  $C \in \mathbb{R}$ , the global minimum of the parametric family  $f_C(x) = \sqrt{x^2 + 9} + \frac{C - x}{2}$  shifts linearly with  $C$ , while the global minimizer remains invariant:*

$$f_{\min} = \frac{C + 3\sqrt{3}}{2},$$

attained uniquely at  $x^* = \sqrt{3}$  for all  $C \in \mathbb{R}$ .

**Proof.** For any  $x \in \mathbb{R}$  and any scaffold parameter  $L \in \mathcal{L}(x)$ , we decompose  $f_C(x)$  into:

$$f_C(x) = \underbrace{\left( \sqrt{x^2 + 9} + \frac{L - x}{2} \right)}_{g(x, L)} + \frac{C - L}{2}.$$

Here, geometric legality is required only for the active core  $g(x, L)$ ; the constant residual  $\frac{C-L}{2}$  is purely algebraic and does not need to correspond to a physical geometric distance, thereby accommodating arbitrary  $C \in \mathbb{R}$ .

Applying Theorem 1 and the elementary inequality  $|u| \geq u$ , we obtain:

$$f_C(x) \geq \frac{|L + 3\sqrt{3}|}{2} + \frac{C - L}{2} \geq \frac{L + 3\sqrt{3}}{2} + \frac{C - L}{2} = \frac{C + 3\sqrt{3}}{2}.$$

The scaffold parameter  $L$  cancels identically for any constant  $C \in \mathbb{R}$ . The equality conditions are independent of  $C$ , since  $C$  appears only in the algebraic residual. Equality requires  $AP \perp l_L$ , whose direction is independent of  $L$ . Thus

$$(x, -3) \cdot (\sqrt{3}, 1) = 0,$$

uniquely gives  $x^* = \sqrt{3}$ . Since equality is attainable for every  $C \in \mathbb{R}$  (e.g., by taking any  $L \geq \sqrt{3}$ ), the unique minimizer remains  $x^* = \sqrt{3}$  for all  $C \in \mathbb{R}$ .  $\square$

## 5. Discussion

The framework presented here formalizes geometric optimization as a process of representation reconstruction.

Rather than attempting to force a global Euclidean distance interpretation directly onto a signed algebraic term, the parametric reconstruction decomposes the objective:

$$F(x) = \underbrace{G(x, L)}_{\text{geometrically legal}} + \underbrace{R(x, L)}_{\text{algebraic residual}}, \quad L \in \mathcal{L}(x).$$

This mechanism offers an illuminating methodological parallel:

- *Catalytic Representation Analogy.* In chemical kinetics, a catalyst participates in an intermediate transition state to enable a favorable pathway, but is fully recovered without appearing in the net stoichiometric reaction. Analogously, the auxiliary parameter  $L$  enables an intermediate state of geometric legality ( $PB \geq 0, PH \geq 0$ ), permitting Euclidean distance theorems to be deployed, before canceling out identically in the global lower bound.
- *Layer of Indirection.* In software architecture, David Wheeler's classic aphorism notes that "*all problems in computer science can be solved by another level of indirection.*" The scaffold parameter  $L$  acts precisely as an indirection layer, reconciling the unconstrained algebraic domain  $\mathbb{R}$  with the non-negative metric requirements of Euclidean geometry.

By lifting constraints into the representation structure itself, the optimization problem is solved uniformly on  $\mathbb{R}$  without piecewise branch analysis.

## 6. Conclusion

We investigated the representation legality gap in Hu Bugui-type optimization problems and proposed a parametric reconstruction framework based on constraint internalization. By introducing an auxiliary scaffold parameter, non-negative geometric admissibility is preserved during intermediate construction, while exact algebraic cancellation ensures an invariant global lower bound across  $\mathbb{R}$ . This perspective clarifies the boundary between algebraic expressions and geometric interpretations, offering a systematic template for geometric optimization.

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