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Existence and Uniqueness of Solutions for Initial Value Problems of First-Order Differential Equations with Nonlinear Riemann–Liouville Variable-Order Fractional Derivative Terms on the Half-Line

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Abstract: In recent years, numerous studies have addressed the existence of solutions to initial value problems for first-order differential equations that involve nonlinear terms of variable-order fractional derivatives. However, owing to the distinctive characteristics of variable-order fractional derivatives, these existing results cannot be directly generalized to nonlinear cases that incorporate nonlinear terms of such derivatives. Building upon established conclusions concerning Riemann-Liouville variable-order fractional derivatives and employing standard analytical techniques, this paper investigates the existence and uniqueness of solutions to the initial value problem for first-order differential equations with nonlinear terms of Riemann-Liouville variable-order fractional derivatives on the half-line. Finally, two illustrative examples are provided to illustrate the applicability of the results developed in the previous sections.

Keywords: First-order differential equation, Riemann-Liouville variable-order fractional derivative, initial value problem, existence and uniqueness, half-line

MSC: 34A08, 34A12, 26A33.

1. Introduction

In this paper, we investigate the existence and uniqueness of solutions to the initial value problem for first-order differential equations that involve nonlinear terms of the Riemann-Liouville variable-order fractional derivative defined on the half-line

$$\begin{cases} x'(t) = f(t, x(t), D_{0+}^{\beta(t)} x(t)), \ \ 0 < t < +\infty, \\ x(0) = 0, \end{cases} \tag{1}$$

where $\beta(t)$ and $f(t, x, y)$ are both given real-valued functions, $0 < \beta(t) < 1$, $D_{0+}^{\beta(t)} x(t)$ denotes the Riemann-Liouville variable-order fractional derivative

$$D_{0+}^{\beta(t)} x(t) = \frac{\mathrm{d}}{\mathrm{d}t} I_{0+}^{1-\beta(t)} x(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{(t-s)^{-\beta(s)}}{\Gamma(1-\beta(s))} x(s) \mathrm{d}s, \tag{2}$$

and $I_{0+}^{1-\beta(t)} x(t)$ is the Riemann-Liouville fractional integral

$$I_{0+}^{1-\beta(t)} x(t) = \int_0^t \frac{(t-s)^{-\beta(s)}}{\Gamma(1-\beta(s))} x(s) \mathrm{d}s. \tag{3}$$

Several definitions of variable-order fractional differentiation must be distinguished when formulating this problem. Valério and Sá da Costa [1] compare variable-order derivatives and their numerical approximations, showing why the choice of definition matters. Sun et al. [2] review the mathematical

foundations, physical models, numerical methods, and applications of variable-order equations, providing a broader context for the operators considered here. Samko and Ross [3] develop integration and differentiation of variable fractional order and examine their fundamental properties. Samko [4] further studies variable-order integration and differentiation, contributing to the operator theory underlying this subject. The constant-order semigroup law should not be assumed to hold for arbitrary variable-order operators. In particular, the order in (2) and (3) depends on the integration variable s, which specifies the history-state-based convention used throughout the present analysis.

Variable-order fractional operators, whose orders depend on selected variables, extend the classical constant-order calculus. The monograph by Kilbas et al. [5] provides the constant-order theory of fractional differential equations against which this extension can be understood. Variable orders allow the memory kernel to change with the relevant variable and can describe processes with evolving hereditary behavior. Exact solutions are generally difficult to obtain, making numerical methods an important complementary approach.

Zhao et al. [6] develop second-order approximations of variable-order fractional derivatives and apply them to fractional evolution equations. Coronel-Escamilla et al. [7] construct numerical solutions for a variable-order reaction–diffusion model with a nonlocal, nonsingular kernel; their work illustrates a different operator convention from the singular kernel used here. Gómez-Aguilar [8] studies a nonlinear alcoholism model through variable-order fractional equations, including solution analysis and numerical calculations. Its relevance here is the treatment of nonlinear variable-order equations, rather than the specific application. Babaei et al. [9] use sixth-kind Chebyshev collocation to approximate nonlinear quadratic integro-differential equations of variable order. Chen et al. [10] develop schemes with high spatial accuracy for variable-order anomalous subdiffusion, illustrating the discretization challenges created by variable orders. Hassani, Tenreiro Machado, and Naraghirad [11] present a numerical technique for a nonlinear time-fractional Klein–Gordon equation with variable order. Hassani, Avazzadeh, and Tenreiro Machado [12] use transcendental Bernstein series to approximate a space–time fractional telegraph equation of variable order. These studies address distinct numerical problems and provide computational context rather than direct proofs of the half-line results established below.

Existence and uniqueness have also been investigated for several classes of variable-order equations. Razminia et al. [13] study solution existence for nonautonomous variable-order fractional differential equations. Zhang [14] establishes an existence result for variable-order equations subject to nonlinear boundary conditions. In a separate study, Zhang [15] investigates existence and uniqueness for variable-order fractional initial value problems. Zhang and Hu [16] consider a nonlinear fractional initial value problem containing a variable-order derivative and a parameter. Zhang and Su [17] establish unique solvability for an initial value problem involving a fractional derivative of variable order. Xu and He [18] obtain existence and uniqueness results for a variable-order fractional Cauchy problem. These contributions motivate the present investigation, while their conclusions must be interpreted with respect to their particular operators, hypotheses, and time intervals. Zhang and Wang [19] treat a boundary value problem involving a history-state-based Riemann–Liouville variable-order derivative, including derivative dependence in the nonlinear term on a finite interval. The present problem instead concerns nonlinear derivative dependence on the half-line.

In recent years, a number of research papers have emerged that focus on the existence of solutions to initial value problems for first-order differential equations involving a linear term of variable-order fractional derivatives with orders between 0 and 1.

Wang and Zheng [20] proved the well-posedness of an initial value problem for a first-order differential equation involving a linear term of a variable-order fractional derivative

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + k(t) D_{0+}^{1-\alpha(t)} u = f(t,u), \ \ 0 < t \leqslant T, \\ u(0) = u_0, \end{cases} \tag{4}$$

where $0 < \alpha(t) \leqslant 1$, $D_{0+}^{1-\alpha(t)}$ is defined by

$$D_{0+}^{1-\alpha(t)} u(t) = \left[\frac{1}{\Gamma(\alpha(t))} \frac{\mathrm{d}}{\mathrm{d}\xi} \int_0^{\xi} u(s)(\xi - s)^{\alpha(t)-1} \mathrm{d}s \right] \Big|_{\xi=t}.$$

Muhammad Rizki Fadillah and Kirane [21] investigated the necessary conditions for the finite-time blow-up of solutions to a system of first-order differential equations involving linear terms of variable-order fractional derivatives

$$\begin{cases} u'(t) + D_{0+}^{\alpha(t)}(u(t) - u_0) = |v(t)|^q, t > 0, q > 1, \\ v'(t) + D_{0+}^{\beta(t)}(v(t) - v_0) = |u(t)|^p, t > 0, p > 1, \\ u(0) = u_0 > 0, v(0) = v_0 > 0, \end{cases} \tag{5}$$

where α, β are $C^1(0, +\infty)$ functions such that $0 < \alpha_m \leqslant \alpha(t) \leqslant \alpha_M < 1$ and $0 < \beta_m \leqslant \beta(t) \leqslant \beta_M < 1$, the operators $D_{0+}^{\alpha(t)}$ and $D_{0+}^{\beta(t)}$ are the Riemann-Liouville variable-order fractional derivatives

$$D_{0+}^{\alpha(t)}u(t) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\Gamma(1-\alpha(t))}\int_0^t (t-s)^{-\alpha(t)}u(s)\mathrm{d}s,$$

$$D_{0+}^{\beta(t)}v(t) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\Gamma(1-\beta(t))}\int_0^t (t-s)^{-\beta(t)}v(s)\mathrm{d}s.$$

Guedim et al. [22] studied the following finite-delay initial value problem for a first-order differential equation involving a linear term of a variable-order fractional derivative

$$\begin{cases} D_{0+}^{\mu(t)}x(t) + lx'(t) = f(t, x_t), 0 \leqslant t \leqslant T, \\ x(t) = \phi(t), t \in [-a, 0], \end{cases} \tag{6}$$

where $\mu : [0, T] \to (0, 1)$, $D_{0+}^{\mu(t)}$ denotes the variable-order fractional derivative used in that study, $\phi : [-a, 0] \to R$ is a continuous function, with $\phi(0) = 0$, and $f : [0, T] \times C([-a, 0], R) \to R$ is a given function.

We notice that the variable-order fractional derivative terms contained in the problems displayed above enter linearly. Furthermore, owing to the distinctive characteristics of variable-order fractional derivatives, these existing results cannot be directly generalized to nonlinear cases that incorporate nonlinear terms of such derivatives. Therefore, inspired by the studies discussed above, in this paper we will study the existence and uniqueness of solutions to initial value problem (1), which contain nonlinear terms of the Riemann-Liouville variable-order fractional derivatives. The paper is organized as follows. In §2, some basic notations, propositions, and lemmas are presented. We also establish some results on the Riemann-Liouville variable-order fractional integral (3) and derivative (2), which will play important roles in obtaining the existence of solutions to (1). In §3, the existence and uniqueness of solutions to initial value problem (1) are proved, and two examples are given to substantiate the theoretical results.

2. Preliminaries

In this section, we introduce some basic notations, propositions, and lemmas that will be used throughout this paper.

Let $C^1[0, +\infty)$ be the space of all functions that are continuously differentiable on $[0, +\infty)$. For $0 < \eta_1 < \eta$, we define the space

$$E = \left\{ u \in C^1[0, +\infty) : \lim_{t\to+\infty} \frac{u(t)}{\mathrm{e}^{\eta t}} = 0, \lim_{t\to+\infty} \frac{u'(t)}{\mathrm{e}^{\eta_1 t}} = 0 \right\}, \tag{7}$$

endowed with the norm

$$\|u\| = \max\left\{ \sup_{t\geqslant 0} \frac{|u(t)|}{\mathrm{e}^{\eta t}}, \sup_{t\geqslant 0} \frac{|u'(t)|}{\mathrm{e}^{\eta_1 t}} \right\}, u \in E. \tag{8}$$

Kou et al. [23] use weighted function spaces to investigate fractional initial value problems on the half-axis. Adapting their compactness method, we can prove that $(E, \|\cdot\|)$ is a Banach space and that the following results hold.

Lemma 1. *Let Ω be a subset of E. Then Ω is precompact if and only if the following conditions are satisfied:*

(i) $\left\{ \frac{u(t)}{\mathrm{e}^{\eta t}} : u \in \Omega \right\}$ *and* $\left\{ \frac{u'(t)}{\mathrm{e}^{\eta_1 t}} : u \in \Omega \right\}$ *are uniformly bounded;*

(ii) $\left\{ \frac{u(t)}{\mathrm{e}^{\eta t}} : u \in \Omega \right\}$ *and* $\left\{ \frac{u'(t)}{\mathrm{e}^{\eta_1 t}} : u \in \Omega \right\}$ *are equicontinuous on* $[0, +\infty)$*;*

(iii) $\left\{\frac{u(t)}{e^{\eta t}} : u \in \Omega\right\}$ *and* $\left\{\frac{u'(t)}{e^{\eta_1 t}} : u \in \Omega\right\}$ *converge uniformly to zero as* $t \to +\infty$, *i.e. for any* $\varepsilon > 0$, *there exists* $T_0 > 0$, *such that for all* $u \in \Omega$ *and any* $t \geqslant T_0$, *we have* $\frac{|u(t)|}{e^{\eta t}} < \varepsilon$ *and* $\frac{|u'(t)|}{e^{\eta_1 t}} < \varepsilon$.

We assume that

(A_1) $\beta : [0, +\infty) \to [\beta_*, \beta^*]$ $(0 < \beta_* < \beta^* < 1)$ is a continuously differentiable function and $\sup\limits_{t \geqslant 0} \frac{|\beta'(t)|}{e^{\eta_2 t}} < +\infty$ $(\eta_2 = \eta - \eta_1)$.

Let $H_\lambda[0, L](0 < \lambda < 1, 0 < L < +\infty)$ be the space of functions $x(t)$ such that $x(t)$ is Hölder continuous with index λ on the interval $[0, L]$. Banaś and Nalepa [24] study spaces controlled by a modulus of continuity, providing functional-space context for the following Hölder norm

$$\|x\|_\lambda = |x(0)| + \sup_{\substack{t_1, t_2 \in [0,L] \\ t_1 \neq t_2}} \frac{|x(t_1) - x(t_2)|}{|t_1 - t_2|^\lambda}, \quad x \in H_\lambda[0, L].$$

Zheng and Wang [25] investigate well-posedness and smoothing for history-state-based variable-order diffusion equations. Their treatment of variable-order integrals supplies the differentiation identity recalled in the following lemma.

Lemma 2. *Assume that* $p \in C^1[0, L](0 < \min\limits_{0 \leqslant t \leqslant L} p(t) = p_* \leqslant p(t) < 1, 0 \leqslant t \leqslant L)$ *and* $x \in H_\lambda[0, L](p_* + \lambda > 1)$. *Then* $I_{0+}^{p(t)}(x(t) - x(0)) \in C^1[0, L]$ *and*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} I_{0+}^{p(t)}(x(t) - x(0)) =& \frac{x(t) - x(0)}{\Gamma(p(t))} \int_0^t (t - s)^{p(s)-1} \left[p'(s) \ln(t - s) + \frac{p(t) - p(s)}{t - s}\right] \mathrm{d}s \\ &+ \int_0^t \left[\frac{x(s) - x(0)}{\Gamma(p(s) - 1)} - \frac{x(t) - x(0)}{\Gamma(p(t) - 1)}\right] (t - s)^{p(s)-2} \mathrm{d}s + \frac{x(t) - x(0)}{\Gamma(p(t))} t^{p(0)-1}, \end{aligned} \tag{9}$$

where $I_{0+}^{p(t)}$ *is the Riemann-Liouville variable-order fractional integral defined in (3).*

Lemma 3. *Assume that* (A_1) *holds and* $x \in C^1[0, +\infty)$ *with* $x(0) = 0$. *Then* $I_{0+}^{1-\beta(t)} x(t) \in C^1[0, +\infty)$ *and*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} I_{0+}^{1-\beta(t)} x(t) =& \frac{x(t)}{\Gamma(1 - \beta(t))} \int_0^t (t - s)^{-\beta(s)} \left[-\beta'(s) \ln(t - s) + \frac{\beta(s) - \beta(t)}{t - s}\right] \mathrm{d}s \\ &+ \int_0^t \left[\frac{x(s)}{\Gamma(-\beta(s))} - \frac{x(t)}{\Gamma(-\beta(t))}\right] (t - s)^{-\beta(s)-1} \mathrm{d}s + \frac{x(t)}{\Gamma(1 - \beta(t))} t^{-\beta(0)}, \quad 0 \leqslant t < +\infty. \end{aligned} \tag{10}$$

where $I_{0+}^{1-\beta(t)}$ *is the Riemann-Liouville variable-order fractional integral defined in (3).*

Proof. It is sufficient to prove that $I_{0+}^{1-\beta(s)} x(s) \in C^1[0, t]$ for any fixed $0 < t < +\infty$. Note that $C^1[0, t] \subset H_\lambda[0, t]$ $(\beta^* < \lambda < 1)$. From the facts that $1 - \beta(t) \geqslant 1 - \beta^*, 0 \leqslant t < +\infty$ and $\lambda > \beta^*$ we immediately obtain that $\lambda + 1 - \beta^* > 1$. Then, according to Lemma 2 for $p(s) = 1 - \beta(s)$, we have that $I_{0+}^{1-\beta(s)} x(s) \in C^1[0, t]$ and (10) holds. □

3. Main Results

This section is devoted to dealing with the existence and uniqueness of solutions to the initial value problem (1).

We assume that

(A_2) Let $f : [0, +\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(t, x(t), D_{0+}^{\beta(t)} x(t))$ belongs to $C[0, +\infty)$ for any $x(t) \in E$.

Lemma 4. *Assume that* (A_1) *and* (A_2) *hold. Then* $x \in E$ *is a solution to the initial value problem (1) if and only if* $x(t)$ *is a solution of the integral equation*

$$x(t) = \int_0^t f\left(s, x(s), \sum_{i=1}^{3} \omega_i(x(s))\right) \mathrm{d}s, \ \ 0 \leqslant t < +\infty, \tag{11}$$

where

$$\omega_1(x(t)) = \frac{x(t)}{\Gamma(1-\beta(t))} \int_0^t (t-s)^{-\beta(s)} \left[-\beta'(s)\ln(t-s) + \frac{\beta(s)-\beta(t)}{t-s}\right] \mathrm{d}s, \tag{12}$$

$$\omega_2(x(t)) = \int_0^t \left[\frac{x(s)}{\Gamma(-\beta(s))} - \frac{x(t)}{\Gamma(-\beta(t))}\right] (t-s)^{-\beta(s)-1} \mathrm{d}s, \tag{13}$$

and

$$\omega_3(x(t)) = \frac{x(t)}{\Gamma(1-\beta(t))} t^{-\beta(0)}. \tag{14}$$

Proof. If $x \in E$ is a solution to the initial value problem (1), then $x(t)$ satisfies $x(0) = 0$ and

$$x'(t) = f(t, x, D_{0+}^{\beta(t)} x(t)), \ \ 0 < t < +\infty. \tag{15}$$

For $x \in C^1[0, +\infty)$, according to (2), (3) and Lemma 3, we have

$$D_{0+}^{\beta(t)} x(t) = \frac{\mathrm{d}}{\mathrm{d}t} I_{0+}^{1-\beta(t)} x(t) = \sum_{i=1}^{3} \omega_i(x(t)), \ \ 0 < t < +\infty, \tag{16}$$

where $\omega_i(x(t))$ $(i = 1, 2, 3)$ is given by (12)-(14). Based on assumption (A_2) and (16), integrating from 0 to t on both sides of the equation (15), we have

$$\begin{aligned} x(t) &= \int_0^t f\left(s, x(s), D_{0+}^{\beta(s)}(x(s))\right) \mathrm{d}s \\ &= \int_0^t f\left(s, x(s), \sum_{i=1}^{3} \omega_i(x(s))\right) \mathrm{d}s, \ \ 0 \leqslant t < +\infty. \end{aligned}$$

Therefore, $x(t)$ is a solution of the integral equation (11).
Conversely, if $x \in E$ is a solution of the integral equation (11), then

$$x(t) = \int_0^t f\left(s, x(s), \sum_{i=1}^{3} \omega_i(x(s))\right) \mathrm{d}s, \ \ 0 \leqslant t < +\infty.$$

Based on assumption (A_1) and Lemma 3, if $x \in C^1[0, +\infty)$, it holds that

$$\sum_{i=1}^{3} \omega_i(x(s)) = \frac{\mathrm{d}}{\mathrm{d}s} I_{0+}^{1-\beta(s)}(x(s)) = D_{0+}^{\beta(s)} x(s), \ \ 0 \leqslant s < +\infty.$$

As a result, we have

$$x(t) = \int_0^t f\left(s, x(s), D_{0+}^{\beta(s)} x(s)\right) \mathrm{d}s, \ \ 0 \leqslant t < +\infty. \tag{17}$$

Then we have

$$x'(t) = f(t, x(t), D_{0+}^{\beta(t)} x(t)), \ \ 0 < t < +\infty.$$

Evaluating the integral equation at $t = 0$ gives the required initial condition $x(0) = 0$. Therefore, we conclude that $x(t)$ is a solution to the initial value problem (1). □

For convenience, in the following analysis, we let

$$a_1 = \sup_{t\geq 0}\left|\frac{1}{\Gamma(1-\beta(t))}\right|,\ a_2 = \sup_{t\geq 0}\left|\frac{1}{\Gamma(-\beta(t))}\right|,\ a_3 = \max_{-\beta^*\leq\omega\leq-\beta_*}\left|\left(\frac{1}{\Gamma(\omega)}\right)'\right|, \tag{18}$$

$$M_{\beta'} = \sup_{t\geq 0}\frac{|\beta'(t)|}{e^{\eta_2 t}},\ B = \max\left\{\frac{M_1 T^{1+\beta^*-\beta_*-\mu}}{1-\mu}, \frac{M_2}{1+\beta^*-\beta_*}, \frac{T^{1-\beta_*}}{1-\beta^*}\right\}, \tag{19}$$

$$D = \max\{a_1, 2a_1BM_{\beta'}, a_2B, a_3BM_{\beta'}\}, \tag{20}$$

where $\beta^* < \mu < 1$ and $T > 1$, $M_1 > 0, M_2 > 0$ are constants satisfying the following inequalities

$$|t^{\mu-\beta^*}\ln t| < M_1,\ 0 < t \leqslant T;\quad |t^{-\beta^*}\ln t| < M_2,\ T \leqslant t < +\infty. \tag{21}$$

$$\sup_{t\geqslant 0}\frac{t^\tau}{e^{\omega t}} = \left(\frac{\tau}{\omega}\right)^\tau e^{-\tau} < \left(\frac{\tau}{\omega}\right)^\tau,\quad \tau > 0, \omega > 0. \tag{22}$$

Remark 1. Based on (A_1), it is easy to see that a_1, a_2, a_3 and $M_{\beta'}$ exist.

Remark 2. It follows from $\lim_{t\to 0} t^{\mu-\beta^*}\ln t = 0 (\beta^* < \mu < 1)$ and $\lim_{t\to+\infty} t^{-\beta^*}\ln t = 0$ that there exist constants $T > 1, M_1 > 0, M_2 > 0$ such that (21) holds.

Theorem 1. *Let (A_1) and (A_2) hold. Suppose there exist positive constants L_1, L_2, L_3, α, γ with $0 < \alpha, \gamma < 1$ such that*

$$|f(t, e^{\eta t}x, e^{\eta t}y)| \leqslant L_1 + L_2|x|^\alpha + L_3|y|^\gamma,\ t \in [0, +\infty), x, y \in \mathbb{R}. \tag{23}$$

Then the initial value problem (1) has a solution $x \in E$.

Proof. From Lemma 4, it is sufficient to consider the existence of a fixed point of operator $F : E \to E$ defined by

$$Fx(t) = \int_0^t f\left(s, x(s), \sum_{i=1}^3 \omega_i(x(s))\right)\mathrm{d}s,\ 0 \leqslant t < +\infty.$$

We shall use the Schauder fixed point theorem to complete the proof and divide it into four steps. We examine the mapping, continuity, invariance, and compactness properties of this operator in turn.

Step 1: We first prove that the operator F maps E into itself.

It is sufficient to prove

$$Fx(t) = \int_0^t f\left(s, x(s), \sum_{i=1}^3 \omega_i(x(s))\right)\mathrm{d}s \in C^1[0, +\infty), \tag{24}$$

and

$$\lim_{t\to+\infty}\frac{Fx(t)}{e^{\eta t}} = 0,\ \lim_{t\to+\infty}\frac{(Fx(t))'}{e^{\eta_1 t}} = 0. \tag{25}$$

For every $x \in E$, we have $f\left(t, x(t), \sum_{i=1}^3 \omega_i(x(t))\right) \in C[0, +\infty)$; therefore, the derivative of $F(x(t))$ exists and

$$(Fx(t))' = f\left(t, x(t), \sum_{i=1}^3 \omega_i(x(t))\right) \in C[0, +\infty), \tag{26}$$

that is $F(x(t)) \in C^1[0, +\infty)$.

Firstly, for $0 \leqslant t \leqslant T$ (T is the constant in (21)), from (21), we have

$$\int_0^t (t-s)^{-\beta(s)}|\ln(t-s)|\mathrm{d}s = \int_0^t T^{-\beta(s)}\left(\frac{t-s}{T}\right)^{-\beta(s)}|\ln(t-s)|\mathrm{d}s$$

$$
\begin{aligned}
&\leqslant \int_0^t T^{-\beta(s)} \left(\frac{t-s}{T}\right)^{-\beta^*} |\ln(t-s)| \mathrm{d}s \\
&= \int_0^t T^{\beta^*-\beta(s)} (t-s)^{-\beta^*} |\ln(t-s)| \mathrm{d}s \\
&\leqslant T^{\beta^*-\beta_*} \int_0^t (t-s)^{-\beta^*} |\ln(t-s)| \mathrm{d}s \\
&= T^{\beta^*-\beta_*} \int_0^t r^{-\mu} |r^{\mu-\beta^*} \ln r| \mathrm{d}r \\
&\leqslant T^{\beta^*-\beta_*} M_1 \int_0^t r^{-\mu} \mathrm{d}r \\
&= \frac{T^{\beta^*-\beta_*} M_1 t^{1-\mu}}{1-\mu} \\
&\leqslant \frac{M_1 T^{1+\beta^*-\beta_*-\mu}}{1-\mu}.
\end{aligned} \tag{27}
$$

$$
\int_0^t (t-s)^{-\beta(s)} \mathrm{d}s \leqslant T^{\beta^*-\beta_*} \int_0^t (t-s)^{-\beta^*} \mathrm{d}s \leqslant \frac{T^{1-\beta_*}}{1-\beta^*}. \tag{28}
$$

Secondly, for $t > T$, we have

$$
\begin{aligned}
\int_0^t (t-s)^{-\beta(s)} |\ln(t-s)| \mathrm{d}s &= \int_0^t r^{-\beta(t-r)} |\ln r| \mathrm{d}r \\
&= \int_0^T r^{\beta^*-\beta(t-r)} r^{-\mu} r^{\mu-\beta^*} |\ln r| \mathrm{d}r + \int_T^t r^{\beta^*-\beta(t-r)} |r^{-\beta^*} \ln r| \mathrm{d}r \\
&\leqslant \int_0^T T^{\beta^*-\beta(t-r)} r^{-\mu} r^{\mu-\beta^*} |\ln r| \mathrm{d}r + \int_T^t r^{\beta^*-\beta_*} |r^{-\beta^*} \ln r| \mathrm{d}r \\
&\leqslant T^{\beta^*-\beta_*} M_1 \int_0^T r^{-\mu} \mathrm{d}r + M_2 \int_T^t r^{\beta^*-\beta_*} \mathrm{d}r \\
&= \frac{T^{1+\beta^*-\beta_*-\mu} M_1}{1-\mu} + \frac{M_2 (t^{1+\beta^*-\beta_*} - T^{1+\beta^*-\beta_*})}{1+\beta^*-\beta_*} \\
&\leqslant \frac{T^{1+\beta^*-\beta_*-\mu} M_1}{1-\mu} + \frac{M_2 t^{1+\beta^*-\beta_*}}{1+\beta^*-\beta_*}.
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
\int_0^t (t-s)^{-\beta(s)} \mathrm{d}s &= \int_0^t r^{-\beta(t-r)} \mathrm{d}r \\
&= \int_0^T r^{\beta^*-\beta(t-r)} r^{-\beta^*} \mathrm{d}r + \int_T^t r^{-\beta(t-r)} \mathrm{d}r \\
&\leqslant \int_0^T T^{\beta^*-\beta(t-r)} r^{-\beta^*} \mathrm{d}r + \int_T^t r^{-\beta_*} \mathrm{d}r \\
&\leqslant T^{\beta^*-\beta_*} \int_0^T r^{-\beta^*} \mathrm{d}r + \frac{t^{1-\beta_*} - T^{1-\beta_*}}{1-\beta_*} \\
&\leqslant \frac{T^{1-\beta_*}}{1-\beta^*} + \frac{t^{1-\beta_*}}{1-\beta_*}.
\end{aligned} \tag{30}
$$

Then from (27)-(30), for $0 \leq t < +\infty$, we have

$$
\int_0^t (t-s)^{-\beta(s)} |\ln(t-s)| \mathrm{d}s \leqslant \frac{T^{1+\beta^*-\beta_*-\mu} M_1}{1-\mu} + \frac{M_2 t^{1+\beta^*-\beta_*}}{1+\beta^*-\beta_*} \leqslant B + B t^{1+\beta^*-\beta_*}. \tag{31}
$$

$$
\int_0^t (t-s)^{-\beta(s)} \mathrm{d}s \leqslant \frac{T^{1-\beta_*}}{1-\beta^*} + \frac{t^{1-\beta_*}}{1-\beta_*}. \tag{32}
$$

Therefore, for all $x \in E$, by (12)-(14), (18)-(22) and (31)-(32), we have

$$\begin{aligned}
\frac{|\omega_1(x(t))|}{\mathrm{e}^{\eta t}} \leqslant & \frac{|x(t)|}{\mathrm{e}^{\eta t}\Gamma(1-\beta(t))}\int_0^t [(t-s)^{-\beta(s)}|\beta'(s)||\ln(t-s)| + (t-s)^{-\beta(s)}|\beta'(\xi)|]\mathrm{d}s \\
= & \frac{|x(t)-x(0)|}{\mathrm{e}^{\eta_1 t}\Gamma(1-\beta(t))}\int_0^t \left[(t-s)^{-\beta(s)}\frac{|\beta'(s)|}{\mathrm{e}^{\eta_2 t}}|\ln(t-s)| + (t-s)^{-\beta(s)}\frac{|\beta'(\xi)|}{\mathrm{e}^{\eta_2 t}}\right]\mathrm{d}s \\
\leqslant & \frac{|x'(\vartheta)t|}{\mathrm{e}^{\eta_1 \vartheta}\Gamma(1-\beta(t))}\int_0^t \left[(t-s)^{-\beta(s)}\frac{|\beta'(s)|}{\mathrm{e}^{\eta_2 s}}|\ln(t-s)| + (t-s)^{-\beta(s)}\frac{|\beta'(\xi)|}{\mathrm{e}^{\eta_2 \xi}}\right]\mathrm{d}s \\
\leqslant & a_1 M_{\beta'}\|x\| t\left(\int_0^t (t-s)^{-\beta(s)}|\ln(t-s)|\mathrm{d}s + \int_0^t (t-s)^{-\beta(s)}\mathrm{d}s\right) \\
\leqslant & a_1 M_{\beta'}\|x\| t(B + Bt^{1+\beta^*-\beta_*} + B + Bt^{1-\beta_*}) \\
\leqslant & D(t + t^{2-\beta_*} + t^{2+\beta^*-\beta_*})\|x\|,
\end{aligned} \tag{33}$$

where $0 < \vartheta < t < +\infty$, $0 \leqslant s < \xi < t < +\infty$.

$$\begin{aligned}
\frac{|\omega_2(x(t))|}{\mathrm{e}^{\eta t}} \leqslant & \frac{1}{\mathrm{e}^{\eta t}}\int_0^t \left|\frac{x(s)}{\Gamma(-\beta(s))} - \frac{x(t)}{\Gamma(-\beta(t))}\right|(t-s)^{-\beta(s)-1}\mathrm{d}s \\
\leqslant & \frac{1}{\mathrm{e}^{\eta t}}\int_0^t \left[\left|\frac{x(s)}{\Gamma(-\beta(s))} - \frac{x(t)}{\Gamma(-\beta(s))}\right| + \left|\frac{x(t)}{\Gamma(-\beta(s))} - \frac{x(t)}{\Gamma(-\beta(t))}\right|\right](t-s)^{-\beta(s)-1}\mathrm{d}s \\
\leqslant & \frac{a_2}{\mathrm{e}^{(\eta-\eta_1)t}}\int_0^t \frac{|x'(\xi)|}{\mathrm{e}^{\eta_1 t}}(t-s)^{-\beta(s)}\mathrm{d}s + \frac{|x(t)|}{\mathrm{e}^{\eta_1 t}}\int_0^t (t-s)^{-\beta(s)}\left|\left(\frac{1}{\Gamma(\omega)}\right)'\right|\frac{|\beta'(\theta)|}{\mathrm{e}^{\eta_2 t}}\mathrm{d}s \\
\leqslant & \frac{a_2}{\mathrm{e}^{(\eta-\eta_1)t}}\int_0^t \frac{|x'(\xi)|}{\mathrm{e}^{\eta_1 \xi}}(t-s)^{-\beta(s)}\mathrm{d}s + \frac{a_3 t|x'(\zeta)|}{\mathrm{e}^{\eta_1 \zeta}}\int_0^t (t-s)^{-\beta(s)}\frac{|\beta'(\theta)|}{\mathrm{e}^{\eta_2 \theta}}\mathrm{d}s \\
\leqslant & a_2\|x\|\int_0^t (t-s)^{-\beta(s)}\mathrm{d}s + a_3 M_{\beta'}\|x\| t\int_0^t (t-s)^{-\beta(s)}\mathrm{d}s \\
\leqslant & (a_2\|x\| + a_3 M_{\beta'}\|x\| t)(B + Bt^{1-\beta_*}) \\
= & (a_2 B + a_2 Bt^{1-\beta_*} + a_3 M_{\beta'} Bt + a_3 M_{\beta'} Bt^{2-\beta_*})\|x\| \\
\leqslant & D(1 + t^{1-\beta_*} + t + t^{2-\beta_*})\|x\|,
\end{aligned} \tag{34}$$

where ω lies between $-\beta(s)$ and $-\beta(t)$, $0 \leqslant s < \xi < t$, $0 < \zeta < t$, $0 \leqslant s < \theta < t$.

$$\begin{aligned}
\frac{|\omega_3(x(t))|}{\mathrm{e}^{\eta t}} \leqslant & \frac{|x(t)|}{\mathrm{e}^{\eta t}\Gamma(1-\beta(t))}t^{-\beta(0)} \\
\leqslant & a_1\frac{|x(t)-x(0)|}{\mathrm{e}^{\eta t}}t^{-\beta(0)} \\
= & a_1\frac{|x'(\xi)|}{\mathrm{e}^{(\eta-\eta_1)t}\mathrm{e}^{\eta_1 t}}t^{1-\beta(0)} \\
\leqslant & a_1\left(\frac{|x'(\xi)|}{\mathrm{e}^{(\eta-\eta_1)t}\mathrm{e}^{\eta_1 \xi}}t^{1-\beta(0)}\right. \\
\leqslant & a_1\|x\|\frac{t^{1-\beta(0)}}{\mathrm{e}^{(\eta-\eta_1)t}} \\
\leqslant & D\|x\|t^{1-\beta(0)},
\end{aligned} \tag{35}$$

where $0 < \xi < t$. As a result, for $x \in E$, from (33)-(35) and the inequality $(a+b)^\delta \leqslant a^\delta + b^\delta$ ($a, b \geqslant 0, 0 < \delta \leqslant 1$), we have

$$\begin{aligned}
\left(\frac{\left|\sum_{i=1}^3 \omega_i(x(s))\right|}{\mathrm{e}^{\eta s}}\right)^\gamma \leqslant & \left(\frac{|\omega_1(x(s))|}{\mathrm{e}^{\eta s}} + \frac{|\omega_2(x(s))|}{\mathrm{e}^{\eta s}} + \frac{|\omega_3(x(s))|}{\mathrm{e}^{\eta s}}\right)^\gamma \\
\leqslant & \left(\frac{|\omega_1(x(s))|}{\mathrm{e}^{\eta s}}\right)^\gamma + \left(\frac{|\omega_2(x(s))|}{\mathrm{e}^{\eta s}}\right)^\gamma + \left(\frac{|\omega_3(x(s))|}{\mathrm{e}^{\eta s}}\right)^\gamma
\end{aligned}$$

$$\leqslant D^{\gamma}\|x\|^{\gamma}\left[((s+s^{2-\beta_*}+s^{2+\beta^*-\beta_*})^{\gamma}+(1+s^{1-\beta_*}+s+s^{2-\beta_*})^{\gamma}+s^{(1-\beta(0))\gamma}\right]$$
$$\leqslant D^{\gamma}(1+s^{\gamma(1-\beta_*)}+2s^{\gamma}+2s^{\gamma(2-\beta_*)}+s^{\gamma(1-\beta(0))}+s^{\gamma(2+\beta^*-\beta_*)})||x||^{\gamma}. \tag{36}$$

From (23) and (36), we obtain

$$\begin{aligned}
&\int_0^t \left| f\left(s, x(s), \sum_{i=1}^{3}\omega_i(x(s))\right)\right| \mathrm{d}s \\
&\leqslant \int_0^t \left[L_1 + L_2\left(\frac{|x(s)|}{\mathrm{e}^{\eta s}}\right)^{\alpha} + L_3\left(\frac{\left|\sum_{i=1}^{3}\omega_i(x(s))\right|}{\mathrm{e}^{\eta s}}\right)^{\gamma}\right] \mathrm{d}s \\
&\leqslant \int_0^t \left[L_1 + L_2\|x\|^{\alpha} + L_3 D^{\gamma}\left(1+s^{\gamma(1-\beta_*)}+2s^{\gamma}+2s^{\gamma(2-\beta_*)}+s^{\gamma(1-\beta(0))}+s^{\gamma(2+\beta^*-\beta_*)}\right)\|x\|^{\gamma}\right] \mathrm{d}s \\
&= (L_1 + L_2\|x\|^{\alpha} + L_3 D^{\gamma}\|x\|^{\gamma})t + L_3 D^{\gamma}\|x\|^{\gamma}\left(\frac{t^{1+\gamma(1-\beta_*)}}{1+\gamma(1-\beta_*)} + \frac{2t^{1+\gamma}}{1+\gamma} + \frac{t^{1+\gamma(1-\beta(0))}}{1+\gamma(1-\beta(0))}\right. \\
&\quad \left. + \frac{2t^{1+\gamma(2-\beta_*)}}{1+\gamma(2-\beta_*)} + \frac{t^{1+\gamma(2+\beta^*-\beta_*)}}{1+\gamma(2+\beta^*-\beta_*)}\right) \\
&\leqslant (L_1 + L_2\|x\|^{\alpha} + L_3 D^{\gamma}\|x\|^{\gamma})t + L_3 D^{\gamma}\|x\|^{\gamma}(t^{1+\gamma(1-\beta_*)} + 2t^{1+\gamma} + t^{1+\gamma(1-\beta(0))} + 2t^{1+\gamma(2-\beta_*)} + t^{1+\gamma(2+\beta^*-\beta_*)}).
\end{aligned} \tag{37}$$

Hence from (23) and (37), we have

$$\begin{aligned}
\frac{|Fx(t)|}{\mathrm{e}^{\eta t}} &\leqslant \frac{1}{\mathrm{e}^{\eta t}}\int_0^t \left| f\left(s, x(s), \sum_{i=1}^{3}\omega_i(x(s))\right)\right| \mathrm{d}s \\
&\leqslant \frac{(L_1 + L_2\|x\|^{\alpha})t}{\mathrm{e}^{\eta t}} + L_3 D^{\gamma}\|x\|^{\gamma}\left[\frac{t + t^{1+\gamma(1-\beta_*)} + 2t^{1+\gamma}}{\mathrm{e}^{\eta t}} + \frac{t^{1+\gamma(1-\beta(0))} + 2t^{1+\gamma(2-\beta_*)} + t^{1+\gamma(2+\beta^*-\beta_*)}}{\mathrm{e}^{\eta t}}\right].
\end{aligned} \tag{38}$$

From $\lim\limits_{t\to+\infty}\dfrac{t^{\sigma}}{\mathrm{e}^{\eta t}} = 0 (0 \leqslant \sigma < +\infty)$, we have

$$\lim_{t\to+\infty}\frac{Fx(t)}{\mathrm{e}^{\eta t}} = 0. \tag{39}$$

From (26) and (37), we have

$$\begin{aligned}
\frac{|(Fx(t))'|}{\mathrm{e}^{\eta_1 t}} &= \frac{\left| f\left(t, x(t), \sum_{i=1}^{3}\omega_i(x(t))\right)\right|}{\mathrm{e}^{\eta_1 t}} \\
&\leqslant \frac{L_1 + L_2\left(\frac{|x(t)|}{\mathrm{e}^{\eta t}}\right)^{\alpha} + L_3\left(\frac{\left|\sum_{i=1}^{3}\omega_i(x(t))\right|}{\mathrm{e}^{\eta t}}\right)^{\gamma}}{\mathrm{e}^{\eta_1 t}} \\
&\leqslant \frac{L_1 + L_2\|x\|^{\alpha} + L_3 D^{\gamma}(1+t^{\gamma(1-\beta_*)}+2t^{\gamma}+2t^{\gamma(2-\beta_*)}+t^{\gamma(1-\beta(0))}+t^{\gamma(2+\beta^*-\beta_*)})\|x\|^{\gamma}}{\mathrm{e}^{\eta_1 t}}.
\end{aligned} \tag{40}$$

From $\lim\limits_{t\to+\infty}\dfrac{t^{\sigma}}{\mathrm{e}^{\eta t}} = 0 (0 \leqslant \sigma < +\infty)$, we have

$$\lim_{t\to+\infty}\frac{(Fx(t))'}{\mathrm{e}^{\eta_1 t}} = 0. \tag{41}$$

Therefore, the required conclusion is true.

Step 2: We show that the operator F is continuous.

Let (x_n) be a sequence in E converging to $x \in E$, so that $\lim_{n\to\infty} \| x_n - x \| = 0$. Then, in view of (23) and (38), we have

$$\begin{aligned}\frac{|Fx_n(t) - Fx(t)|}{e^{\eta t}} \leqslant & \frac{1}{e^{\eta t}} \int_0^t \left| f\left(s, x_n(s), \sum_{i=1}^{3} \omega_i(x_n(s))\right) - f\left(s, x(s), \sum_{i=1}^{3} \omega_i(x(s))\right)\right| ds \\ \leqslant & \frac{1}{e^{\eta t}} \int_0^t \left| f\left(s, x_n(s), \sum_{i=1}^{3} \omega_i(x_n(s))\right)\right| + \left| f\left(s, x(s), \sum_{i=1}^{3} \omega_i(x(s))\right)\right| ds \\ \leqslant & \frac{2(L_1 + L_2\|x\|^{\alpha})t}{e^{\eta t}} + 2L_3 D^{\gamma}\|x\|^{\gamma} \left[\frac{t + t^{1+\gamma(1-\beta_*)} + 2t^{1+\gamma}}{e^{\eta t}}\right. \\ & \left. + \frac{t^{1+\gamma(1-\beta(0))} + 2t^{1+\gamma(2-\beta_*)} + t^{1+\gamma(2+\beta^*-\beta_*)}}{e^{\eta t}}\right]. \end{aligned} \tag{42}$$

From (42) and $\lim_{t\to+\infty} \frac{t^{\sigma}}{e^{\eta t}} = 0 (\sigma > 0)$, for every $\varepsilon > 0$, there exists a constant $T > 0$ such that for any $t > T$, we have

$$\frac{|Fx_n(t) - Fx(t)|}{e^{\eta t}} < \varepsilon. \tag{43}$$

On the other hand, from $\lim_{n\to\infty} \| x_n - x \| = 0$ and the uniform continuity of f on every compact subset of its domain, we have

$$\sup_{0\leqslant t\leqslant T} \left| f\left(s, x_n(s), \sum_{i=1}^{3} \omega_i(x_n(s))\right) - f\left(s, x(s), \sum_{i=1}^{3} \omega_i(x(s))\right)\right| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{44}$$

Thus, there exists an $N > 0$ such that, for any $n > N$,

$$\begin{aligned}\frac{|Fx_n(t) - Fx(t)|}{e^{\eta t}} \leqslant & \frac{1}{e^{\eta t}} \int_0^t \left| f\left(s, x_n(s), \sum_{i=1}^{3} \omega_i(x_n(s))\right) - f\left(s, x(s), \sum_{i=1}^{3} \omega_i(x(s))\right)\right| ds \\ = & \frac{1}{e^{\eta t}} \int_0^T \left| f\left(s, x_n(s), \sum_{i=1}^{3} \omega_i(x_n(s))\right) - f\left(s, x(s), \sum_{i=1}^{3} \omega_i(x(s))\right)\right| ds \\ & + \frac{1}{e^{\eta t}} \int_T^t \left| f\left(s, x_n(s), \sum_{i=1}^{3} \omega_i(x_n(s))\right) - f\left(s, x(s), \sum_{i=1}^{3} \omega_i(x(s))\right)\right| ds \\ < & T \sup_{0\leqslant t\leqslant T} \left| f\left(s, x_n(s), \sum_{i=1}^{3} \omega_i(x_n(s))\right) - f\left(s, x(s), \sum_{i=1}^{3} \omega_i(x(s))\right)\right| + \varepsilon \\ < & (T+1)\varepsilon, \end{aligned} \tag{45}$$

$$\begin{aligned}\frac{|(Fx_n)'(t) - (Fx)'(t)|}{e^{\eta_1 t}} \leqslant & \left| f\left(s, x_n(s), \sum_{i=1}^{3} \omega_i(x_n(s))\right) - f\left(s, x(s), \sum_{i=1}^{3} \omega_i(x(s))\right)\right| \\ \leqslant & \sup_{0\leqslant t\leqslant T} \left| f\left(s, x_n(s), \sum_{i=1}^{3} \omega_i(x_n(s))\right) - f\left(s, x(s), \sum_{i=1}^{3} \omega_i(x(s))\right)\right| \\ & + \sup_{T<t<+\infty} \left| f\left(s, x_n(s), \sum_{i=1}^{3} \omega_i(x_n(s))\right) - f\left(s, x(s), \sum_{i=1}^{3} \omega_i(x(s))\right)\right| \\ < & 2\varepsilon. \end{aligned} \tag{46}$$

From (45) and (46), we have

$$\| Fx_n - Fx \| = \max \left\{ \sup_{0\leqslant t<+\infty} \frac{|Fx_n(t) - Fx(t)|}{e^{\eta t}}, \sup_{0\leqslant t<+\infty} \frac{|(Fx_n)'(t) - (Fx)'(t)|}{e^{\eta_1 t}} \right\} \longrightarrow 0, \ \ n \longrightarrow \infty. \tag{47}$$

Step 3: We prove that the operator F maps $\Omega = \{x \in E : \|x\| \leqslant R\}$ into itself, where

$$R > \left\{ 3L_1K, (3L_2K)^{\frac{1}{1-\alpha}}, (18L_3KD^{\gamma})^{\frac{1}{1-\gamma}} \right\}; \tag{48}$$

$$\begin{aligned} K = \max \Bigg\{ & 1, \frac{1}{\eta}, \left(\frac{1+\gamma(1-\beta_*)}{\eta}\right)^{\gamma(1-\beta_*)}, 2\left(\frac{1+\gamma}{\eta}\right)^{1+\gamma}, \left(\frac{1+\gamma(1-\beta(0))}{\eta}\right)^{1+\gamma(1-\beta(0))}, \\ & 2\left(\frac{1+\gamma(2-\beta_*)}{\eta}\right)^{1+\gamma(2-\beta_*)}, \left(\frac{1+\gamma(2+\beta^*-\beta_*)}{\eta}\right)^{1+\gamma(2+\beta^*-\beta_*)}, \left(\frac{\gamma(1-\beta_*)}{\eta_1}\right)^{\gamma(1-\beta_*)}, \\ & 2\left(\frac{\gamma}{\eta_1}\right)^{\gamma}, 2\left(\frac{\gamma(2-\beta_*)}{\eta_1}\right)^{\gamma(2-\beta_*)}, \left(\frac{\gamma(1-\beta(0))}{\eta_1}\right)^{\gamma(1-\beta(0))}, \left(\frac{\gamma(2+\beta^*-\beta_*)}{\eta_1}\right)^{\gamma(2+\beta^*-\beta_*)} \Bigg\}. \end{aligned} \tag{49}$$

For any $x \in \Omega$, by (22), (38), (40), (48) and (49), we have

$$\begin{aligned} \frac{|Fx(t)|}{e^{\eta t}} \leqslant & (L_1 + L_2\|x\|^{\alpha})\frac{1}{\eta} + L_3D^{\gamma}\|x\|^{\gamma}\left[\frac{1}{\eta} + \left(\frac{1+\gamma(1-\beta_*)}{\eta}\right)^{1+\gamma(1-\beta_*)}\right. \\ & + 2\left(\frac{1+\gamma}{\eta}\right)^{1+\gamma} + \left(\frac{1+\gamma(1-\beta(0))}{\eta}\right)^{1+\gamma(1-\beta(0))} + 2\left(\frac{1+\gamma(2-\beta_*)}{\eta}\right)^{1+\gamma(2-\beta_*)} \\ & \left. + \left(\frac{1+\gamma(2+\beta^*-\beta_*)}{\eta}\right)^{1+\gamma(2+\beta^*-\beta_*)}\right] \\ \leqslant & K(L_1 + L_2\|x\|^{\alpha}) + 6L_3KD^{\gamma}\|x\|^{\gamma} \\ \leqslant & L_1K + L_2KR^{\alpha} + 6L_3KD^{\gamma}R^{\gamma} \\ = & L_1K + L_2KR^{\alpha-1}R + 6L_3KD^{\gamma}R^{\gamma-1}R \\ \leqslant & \frac{R}{3} + \frac{R}{3} + \frac{R}{3} = R, \end{aligned} \tag{50}$$

$$\begin{aligned} \frac{|(Fx)'(t)|}{e^{\eta_1 t}} \leqslant & L_1 + L_2\|x\|^{\alpha} + L_3D^{\gamma}\|x\|^{\gamma}\left[1 + \left(\frac{\gamma(1-\beta_*)}{\eta_1}\right)^{\gamma(1-\beta_*)} + 2\left(\frac{\gamma}{\eta_1}\right)^{\gamma}\right. \\ & \left. + 2\left(\frac{\gamma(2-\beta_*)}{\eta_1}\right)^{\gamma(2-\beta_*)} + \left(\frac{\gamma(1-\beta(0))}{\eta_1}\right)^{\gamma(1-\beta(0))} + \left(\frac{\gamma(2+\beta^*-\beta_*)}{\eta_1}\right)^{\gamma(2+\beta^*-\beta_*)}\right] \\ \leqslant & L_1 + L_2\|x\|^{\alpha} + 6L_3KD^{\gamma}\|x\|^{\gamma} \\ \leqslant & L_1 + L_2R^{\alpha} + 6L_3KD^{\gamma}R^{\gamma} \\ \leqslant & L_1K + L_2KR^{\alpha} + 6L_3KD^{\gamma}R^{\gamma} \\ = & L_1K + L_2KR^{\alpha-1}R + 6L_3KD^{\gamma}R^{\gamma-1}R \\ \leqslant & \frac{R}{3} + \frac{R}{3} + \frac{R}{3} = R, \end{aligned} \tag{51}$$

so (50) and (51) show that $F : \Omega \to \Omega$.

Step 4: We prove that $F : \Omega \to \Omega$ is a completely continuous operator.

It is sufficient to prove that $F(\Omega)$ satisfies the conditions in Lemma 1.

By virtue of (50),(51), (39) and (41), we can easily obtain that (i) and (iii) in Lemma 1 are satisfied.

Moreover, for any $x \in \Omega$ and $t_1, t_2 \in [0, +\infty)$, $t_1 < t_2$, by virtue of (37), (50) and (51), we have

$$\begin{aligned}\left|\frac{Fx(t_2)}{e^{\eta t_2}} - \frac{Fx(t_1)}{e^{\eta t_1}}\right| \leqslant & \frac{e^{\eta t_2} - e^{\eta t_1}}{e^{\eta t_2}} \frac{1}{e^{\eta t_1}} \int_0^{t_1} \left| f\left(s, x(s), \sum_{i=1}^{3} \omega_i(x(s))\right)\right| ds + \frac{1}{e^{\eta t_2}} \int_{t_1}^{t_2} \left| f\left(s, x(s), \sum_{i=1}^{3} \omega_i(x(s))\right)\right| ds \\ \leqslant & R\left(e^{\eta t_2} - e^{\eta t_1}\right) + (L_1 + L_2 R^{\alpha} + L_3 D^{\gamma} R^{\gamma})(t_2 - t_1) \\ & + L_3 D^{\gamma} R^{\gamma} \left[(t_2 - t_1)^{1+\gamma(1-\beta_*)} + 2(t_2 - t_1)^{1+\gamma} + (t_2 - t_1)^{1+\gamma(1-\beta(0))}\right. \\ & \left. + 2(t_2 - t_1)^{1+\gamma(2-\beta_*)} + (t_2 - t_1)^{1+\gamma(2+\beta^* - \beta_*)}\right] \longrightarrow 0, \quad t_1 \longrightarrow t_2, \end{aligned} \tag{52}$$

$$\begin{aligned}\left|\frac{(Fx)'(t_2)}{e^{\eta_1 t_2}} - \frac{(Fx)'(t_1)}{e^{\eta_1 t_1}}\right| = & \left| \frac{f\left(t_2, x(t_2), \sum\limits_{i=1}^{3} \omega_i(x(t_2))\right)}{e^{\eta_1 t_2}} - \frac{f\left(t_1, x(t_1), \sum\limits_{i=1}^{3} \omega_i(x(t_1))\right)}{e^{\eta_1 t_1}} \right| \\ \leqslant & \left| \frac{f\left(t_2, x(t_2), \sum\limits_{i=1}^{3} \omega_i(x(t_2))\right) - f\left(t_1, x(t_1), \sum\limits_{i=1}^{3} \omega_i(x(t_1))\right)}{e^{\eta_1 t_2}} \right| \\ & + \frac{e^{\eta_1 t_2} - e^{\eta_1 t_1}}{e^{\eta_1 t_2}} \frac{1}{e^{\eta_1 t_1}} \left| f\left(t_1, x(t_1), \sum_{i=1}^{3} \omega_i(x(t_1))\right)\right| \\ \leqslant & \left| f\left(t_2, x(t_2), \sum_{i=1}^{3} \omega_i(x(t_2))\right) - f\left(t_1, x(t_1), \sum_{i=1}^{3} \omega_i(x(t_1))\right)\right| + R\left(e^{\eta_1 t_2} - e^{\eta_1 t_1}\right). \end{aligned} \tag{53}$$

According to the condition (A_2), we obtain that

$$\left|\frac{(Fx)'(t_2)}{e^{\eta_1 t_2}} - \frac{(Fx)'(t_1)}{e^{\eta_1 t_1}}\right| \longrightarrow 0, \quad t_1 \longrightarrow t_2. \tag{54}$$

Therefore, $F(\Omega)$ is equicontinuous. Thus $F(\Omega)$ is precompact according to Lemma 1.

As a result, F is a completely continuous operator. By the Schauder fixed point theorem, we deduce that the operator F possesses a fixed point $x \in E$, which implies that the initial value problem (1) has a solution $x \in E$. □

Remark 3. In Theorem 1, if $\alpha = 1$ and $\gamma = 1$ in (23), we can establish the existence of solutions to the initial value problem (1) in the same way, provided that we impose restrictions on the parameters L_1, L_2, L_3, D, and K.

In the following analysis, let

$$\begin{aligned} G = \max \Bigg\{ & 1, \frac{\left(\frac{2}{\eta}\right)^2}{e^2}, \frac{\left(\frac{2-\beta_*}{\eta}\right)^{2-\beta_*}}{(2-\beta_*)e^{2-\beta_*}}, \frac{2\left(\frac{3-\beta_*}{\eta}\right)^{3-\beta_*}}{(3-\beta_*)e^{3-\beta_*}}, \frac{\left(\frac{2-\beta(0)}{\eta}\right)^{2-\beta(0)}}{(2-\beta(0))e^{2-\beta(0)}}, \\ & \frac{\left(\frac{3+\beta^*-\beta_*}{\eta}\right)^{3+\beta^*-\beta_*}}{(3+\beta^*-\beta_*)e^{3+\beta^*-\beta_*}}, \frac{2}{\eta_1 e}, \frac{\left(\frac{1-\beta_*}{\eta_1}\right)^{1-\beta_*}}{e^{1-\beta_*}}, \frac{2\left(\frac{2-\beta_*}{\eta_1}\right)^{2-\beta_*}}{e^{2-\beta_*}}, \frac{\left(\frac{1-\beta(0)}{\eta_1}\right)^{1-\beta(0)}}{e^{1-\beta(0)}}, \frac{\left(\frac{2+\beta^*-\beta_*}{\eta_1}\right)^{2+\beta^*-\beta_*}}{e^{2+\beta^*-\beta_*}} \Bigg\}. \end{aligned} \tag{55}$$

Theorem 2. *Let (A_1) and (A_2) hold. Suppose there exist positive constants C_1, C_2, and $\eta_3 (\eta_3 \leqslant \eta_1)$, such that for all $t \in [0, +\infty)$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$, the following conditions hold:*

$$|f(t, e^{\eta t} x_1, e^{\eta t} y_1) - f(t, e^{\eta t} x_2, e^{\eta t} y_2)| \leqslant C_1 |x_1 - x_2| + C_2 |y_1 - y_2|, \tag{56}$$

and

$$\sup_{t\geqslant 0}\frac{|f(t,0,0)|}{\mathrm{e}^{\eta_3 t}}<+\infty;\quad (C_1+6C_2D)\,G<1, \tag{57}$$

are satisfied, where D and G are defined in (20) and (55), respectively. Then the initial value problem (1) has a unique solution $x\in E$.

Proof. From Lemma 4, it is sufficient to show that the operator $F:E\to E$, defined by

$$Fx(t)=\int_0^t f\left(s,x(s),\sum_{i=1}^{3}\omega_i(x(s))\right)\mathrm{d}s,\ \ 0\leqslant t<+\infty,$$

has a unique fixed point.

Step 1: We show that $F:E\to E$ is well defined.

For any $x\in E$, by similar arguments to those of Theorem 1, we can obtain

$$F(x(t))\in C^1[0,+\infty),\quad (Fx)'(t)\in C[0,+\infty).$$

$$\frac{|\omega_1(x(t))|}{\mathrm{e}^{\eta t}}\leqslant D(t+t^{2-\beta_*}+t^{2+\beta^*-\beta_*})\|x\|,$$

$$\frac{|\omega_2(x(t))|}{\mathrm{e}^{\eta t}}\leqslant D(1+t^{1-\beta_*}+t+t^{2-\beta_*})\|x\|,$$

$$\frac{|\omega_3(x(t))|}{\mathrm{e}^{\eta t}}\leqslant Dt^{1-\beta(0)}\|x\|.$$

Therefore, we have

$$\frac{\sum\limits_{i=1}^{3}|\omega_i(x(t))|}{\mathrm{e}^{\eta t}}\leqslant D(1+t^{1-\beta(0)}+t^{1-\beta_*}+2t+2t^{2-\beta_*}+t^{2+\beta^*-\beta_*})\|x\|. \tag{58}$$

Let $M_f=\sup\limits_{t\geqslant 0}\frac{|f(t,0,0)|}{\mathrm{e}^{\eta_3 t}}$, by virtue of (22), (55)-(58), we obtain

$$\begin{aligned}
\frac{|Fx(t)|}{\mathrm{e}^{\eta t}}\leqslant&\frac{1}{\mathrm{e}^{\eta t}}\int_0^t\left|f\left(s,x(s),\sum_{i=1}^{3}\omega_i(x(s))\right)\right|\mathrm{d}s\\
=&\frac{1}{\mathrm{e}^{\eta t}}\int_0^t\left|f\left(s,x(s),\sum_{i=1}^{3}\omega_i(x(s))\right)-f(s,0,0)+f(s,0,0)\right|ds\\
\leqslant&\frac{1}{\mathrm{e}^{\eta t}}\int_0^t\left(C_1\frac{|x(s)|}{\mathrm{e}^{\eta s}}+C_2\left|\sum_{i=1}^{3}\frac{\omega_i(x(s))}{\mathrm{e}^{\eta s}}\right|+|f(s,0,0)|\right)\mathrm{d}s\\
\leqslant&\frac{1}{\mathrm{e}^{\eta t}}\int_0^t\left(C_1\frac{|x(s)|}{\mathrm{e}^{\eta s}}+C_2\sum_{i=1}^{3}\frac{|\omega_i(x(s))|}{\mathrm{e}^{\eta s}}\right)\mathrm{d}s+\frac{1}{\mathrm{e}^{(\eta-\eta_3)t}}\int_0^t\frac{|f(s,0,0)|}{\mathrm{e}^{\eta_3 t}}\mathrm{d}s\\
\leqslant&\int_0^t\frac{(C_1+C_2D(1+s^{1-\beta_*}+s^{1-\beta(0)}+s+s^{2-\beta_*}+s^{2+\beta^*-\beta_*}))\|x\|}{e^{\eta t}}\mathrm{d}s+\frac{1}{\mathrm{e}^{(\eta-\eta_3)t}}\int_0^t\frac{|f(s,0,0)|}{\mathrm{e}^{\eta_3 s}}\mathrm{d}s\\
\leqslant&\frac{C_1\|x\|t}{\mathrm{e}^{\eta t}}+C_2D\|x\|\left(\frac{t}{\mathrm{e}^{\eta t}}+\frac{t^{2-\beta_*}}{\mathrm{e}^{\eta t}(2-\beta_*)}+\frac{t^2}{\mathrm{e}^{\eta t}}+\frac{2t^{3-\beta_*}}{\mathrm{e}^{\eta t}(3-\beta_*)}+\frac{t^{2-\beta(0)}}{\mathrm{e}^{\eta t}(2-\beta(0))}\right.\\
&\left.+\frac{t^{3+\beta^*-\beta_*}}{\mathrm{e}^{\eta t}(3+\beta^*-\beta_*)}\right)+\frac{M_f t}{\mathrm{e}^{(\eta-\eta_3)t}}\longrightarrow 0,\ \ t\longrightarrow+\infty.
\end{aligned}\tag{59}$$

Step 2: We show that $F:E\longrightarrow E$ is a contraction operator.

For any $x, y \in E$, from the derivations of (33)-(35) in Theorem 1, we can obtain the following estimates

$$\sum_{i=1}^{3} \frac{|\omega_i(x(t)) - \omega_i(y(t))|}{e^{\eta t}} \leqslant D(1 + t^{1-\beta_*} + t^{1-\beta(0)} + 2t + 2t^{2-\beta_*} + t^{2+\beta^*-\beta_*})\|x - y\|. \tag{60}$$

By virtue of (22),(55), (56) and (60), we have

$$\begin{aligned}
\frac{|Fx(t) - Fy(t)|}{e^{\eta t}} \leqslant & \frac{1}{e^{\eta t}} \int_0^t \left| f\left(s, x(s), \sum_{i=1}^{3} \omega_i(x(s))\right) - f\left(s, y(s), \sum_{i=1}^{3} \omega_i(y(s))\right) \right| ds \\
\leqslant & \frac{1}{e^{\eta t}} \int_0^t \left(C_1 \frac{|x(s) - y(s)|}{e^{\eta s}} + C_2 \sum_{i=1}^{3} \frac{|\omega_i(x(s)) - \omega_i(y(s))|}{e^{\eta s}} \right) ds \\
\leqslant & \|x - y\| \int_0^t \frac{C_1 + C_2 D(1 + s^{1-\beta_*} + s^{1-\beta(0)} + 2s + 2s^{2-\beta_*} + s^{2+\beta^*-\beta_*})}{e^{\eta t}} ds \\
= & \frac{C_1 \|x - y\| t}{e^{\eta t}} + C_2 D\|x - y\| \left(\frac{t}{e^{\eta t}} + \frac{t^{2-\beta_*}}{e^{\eta t}(2 - \beta_*)} + \frac{t^2}{e^{\eta t}} + \frac{2t^{3-\beta_*}}{e^{\eta t}(3 - \beta_*)} \right. \\
& \left. + \frac{t^{2-\beta(0)}}{e^{\eta t}(2 - \beta(0))} + \frac{t^{3+\beta^*-\beta_*}}{e^{\eta t}(3 + \beta^* - \beta_*)} \right) \\
\leqslant & \frac{C_1 \|x - y\|}{\eta e} + C_2 D\|x - y\| \left[\frac{1}{\eta e} + \frac{(\frac{2-\beta_*}{\eta})^{2-\beta_*}}{(2 - \beta_*)e^{2-\beta_*}} + \frac{\left(\frac{2}{\eta}\right)^2}{e^2} + \frac{2(\frac{3-\beta_*}{\eta})^{3-\beta_*}}{(3 - \beta_*)e^{3-\beta_*}} \right. \\
& \left. + \frac{(\frac{2-\beta(0)}{\eta})^{2-\beta(0)}}{(2 - \beta(0))e^{2-\beta(0)}} + \frac{(\frac{3+\beta^*-\beta_*}{\eta})^{3+\beta^*-\beta_*}}{(3 + \beta^* - \beta_*)e^{3+\beta^*-\beta_*}} \right] \\
\leqslant & (C_1 + 6C_2 D)G\|x - y\|
\end{aligned} \tag{61}$$

$$\begin{aligned}
\frac{|(Fx)'(t) - (Fy)'(t)|}{e^{\eta_1 t}} \leqslant & \frac{\left| f(t, x(t), \sum_{i=1}^{3} \omega_i(x(t))) - f(t, y(t), \sum_{i=1}^{3} \omega_i(y(t))) \right|}{e^{\eta_1 t}} \\
\leqslant & \frac{1}{e^{\eta_1 t}} \left[C_1 \frac{|x(t) - y(t)|}{e^{\eta t}} + C_2 \frac{\left| \sum_{i=1}^{3} \omega_i(x(t)) - \sum_{i=1}^{3} \omega_i(y(t)) \right|}{e^{\eta t}} \right] \\
\leqslant & \frac{C_1 \|x - y\| + C_2 D\|x - y\|(1 + t^{1-\beta_*} + t^{1-\beta(0)} + 2t + 2t^{2-\beta_*} + t^{2+\beta^*-\beta_*})}{e^{\eta_1 t}} \\
\leqslant & C_1 \|x - y\| + C_2 D\|x - y\| \left[1 + \frac{\left(\frac{1-\beta_*}{\eta_1}\right)^{1-\beta_*}}{e^{1-\beta_*}} + \frac{\left(\frac{1-\beta(0)}{\eta_1}\right)^{1-\beta(0)}}{e^{1-\beta(0)}} \right. \\
& \left. + \frac{2}{\eta_1 e} + 2\frac{\left(\frac{2-\beta_*}{\eta_1}\right)^{2-\beta_*}}{e^{2-\beta_*}} + \frac{\left(\frac{2+\beta^*-\beta_*}{\eta_1}\right)^{2+\beta^*-\beta_*}}{e^{2+\beta^*-\beta_*}} \right] \\
\leqslant & (C_1 + 6C_2 D)G\|x - y\|,
\end{aligned} \tag{62}$$

$(C_1 + 6C_2 D)G < 1$ ensures that $F : E \to E$ is a contraction operator. By the Banach fixed point theorem, we deduce that the operator F possesses a unique fixed point $x \in E$ which implies that the initial value problem (1) has a unique solution $x \in E$. □

Example 1. We consider the following initial value problem

$$\begin{cases} x'(t) = \frac{|x|^{\frac{1}{3}}}{e^{\frac{2t}{3}}(1+x^2)} + \frac{x^2}{e^{\frac{14t}{3}}(1+x^2)} \left| D_{0+}^{\frac{3}{10} - \frac{t}{10(1+t^2)}} x(t) \right|^{\frac{1}{3}}, \ 0 < t < +\infty, \\ x(0) = 0. \end{cases} \tag{63}$$

Now we claim that the initial value problem (63) has a solution $x \in E$.

Let $\beta(t) = \frac{3}{10} - \frac{t}{10(1+t^2)}$, $f(t,x,y) = \frac{|x|^{\frac{1}{3}}}{e^{\frac{2t}{3}}(1+x^2)} + \frac{x^2|y|^{\frac{1}{3}}}{e^{\frac{14t}{3}}(1+x^2)}$. Obviously, $\beta(t) \in C^1[0,+\infty)$, $\beta_* = \frac{1}{4}$, $\beta^* = \frac{3}{10}$, $\beta(0) = 0.3$, $\sup\limits_{0\leqslant t<+\infty} \frac{|\beta'(t)|}{e^{\eta_2 t}} = \sup\limits_{0\leqslant t<+\infty} \frac{|1-t^2|}{10e^t(1+t^2)^2} = \frac{1}{10} < +\infty$ ($\eta_2 = 1$), thus $\beta(t)$ satisfies assumption (A_1). Since

$$\begin{aligned} \left| f(t, e^{2t}x, e^{2t}y) \right| &= \left| \frac{|e^{2t}x|^{\frac{1}{3}}}{e^{\frac{2t}{3}}(1+(e^{2t}x)^2)} + \frac{(e^{2t}x)^2|e^{2t}y|^{\frac{1}{3}}}{e^{\frac{14t}{3}}(1+(e^{2t}x)^2)} \right| \\ &= \frac{|x|^{\frac{1}{3}}}{(1+e^{4t}x^2)} + \frac{x^2|y|^{\frac{1}{3}}}{(1+e^{4t}x^2)} \\ &\leqslant 1 + |x|^{\frac{1}{3}} + |y|^{\frac{1}{3}}, \end{aligned}$$

then f satisfies the assumptions (A_2) and (23) with $\alpha = \gamma = \frac{1}{3}$, $L_1 = L_2 = L_3 = 1$, $\eta = 2$ and $\eta_1 = 1$. Let $\mu = \frac{8}{15}$. Direct calculation gives

$$\overline{a_1} = 0.820 > a_1 = \sup_{0\leqslant t<+\infty} \left| \frac{1}{\Gamma(1-\beta(t))} \right| = \sup_{0\leqslant t<+\infty} \left| \frac{1}{\Gamma(\frac{7}{10} + \frac{t}{10(1+t^2)})} \right| \approx 0.816,$$

$$\overline{a_2} = 0.240 > a_2 = \sup_{0\leqslant t<+\infty} \left| \frac{1}{\Gamma(-\beta(t))} \right| = \sup_{0\leqslant t<+\infty} \left| \frac{1}{\Gamma(\frac{t}{10(1+t^2)} - \frac{3}{10})} \right| \approx 0.231,$$

$$\overline{a_3} = 0.600 > a_3 = \sup_{-\beta^*\leqslant\omega\leqslant-\beta_*} \left| \left(\frac{1}{\Gamma(\omega)}\right)' \right| = \sup_{-\frac{3}{10}\leqslant\omega\leqslant-\frac{1}{4}} \left| \left(\frac{1}{\Gamma(\omega)}\right)' \right| \approx 0.595,$$

$$M_{\beta'} = \sup_{0\leqslant t<+\infty} \frac{|\beta'(t)|}{e^{\eta_2 t}} = \sup_{0\leqslant t<+\infty} \frac{|1-t^2|}{10e^t(1+t^2)^2} = \frac{1}{10} = 0.100,$$

moreover, we have $\frac{3}{10} < \mu = \frac{8}{15} < 1$, $T = 1.300 \times 10^{15}$, $M_1 = 1.190 \times 10^5$, $M_2 = 1.018 \times 10^{-3}$, such that

$$\left| t^{\mu-\beta^*} \ln t \right| = \left| t^{\frac{8}{15}-\frac{3}{10}} \ln t \right| = \left| t^{\frac{7}{30}} \ln t \right| \leqslant 1.190 \times 10^5 \ \ 0 < t \leqslant 1.300 \times 10^{15},$$

$$\left| t^{-\beta^*} \ln t \right| = \left| t^{-\frac{3}{10}} \ln t \right| < 1.018 \times 10^{-3}, \ \ t > 1.300 \times 10^{15},$$

$$\overline{B} = 1.650 \times 10^{13} > B = \max\left\{ \frac{M_1 T^{1+\beta^*-\beta_*-\mu}}{1-\mu}, \frac{M_2}{1+\beta^*-\beta_*}, \frac{T^{1-\beta_*}}{1-\beta^*} \right\} \approx 1.642 \times 10^{13},$$

$$\overline{D} = \max\{\overline{a_1}, 2\overline{a_1}\overline{B}M_{\beta'}, \overline{a_2}\overline{B}, \overline{a_3}\overline{B}M_{\beta'}\} = 3.96 \times 10^{12},$$

$$\overline{D} = 3.96 \times 10^{12} > D = \max\{a_1, 2a_1BM_{\beta'}, a_2B, a_3BM_{\beta'}\} \approx 3.793 \times 10^{12},$$

$$\overline{K} = 1.500 > K \approx 1.460,$$

$$\max\left\{ 3L_1K, (3L_2K)^{\frac{1}{1-\alpha}}, (18L_3KD^{\gamma})^{\frac{1}{1-\gamma}} \right\} \approx 2.624 \times 10^8.$$

$$\max\left\{ 3L_1\overline{K}, (3L_2\overline{K})^{\frac{1}{1-\alpha}}, (18L_3\overline{K}\,\overline{D}^{\gamma})^{\frac{1}{1-\gamma}} \right\} \approx 2.792 \times 10^8 > 2.624 \times 10^8$$

Then, according to Theorem 1, we obtain that the initial value problem (63) has a solution $x \in E$ with $\|x\| \leqslant R$, where

$$\begin{aligned} R = 3 \times 10^8 &> 2.792 \times 10^8 \approx \max\left\{3L_1\overline{K}, (3L_2\overline{K})^{\frac{1}{1-\alpha}}, (18L_3\overline{K}\,\overline{D}^{\gamma})^{\frac{1}{1-\gamma}}\right\} \\ &> 2.624 \times 10^8 \approx \max\left\{3L_1K, (3L_2K)^{\frac{1}{1-\alpha}}, (18L_3KD^{\gamma})^{\frac{1}{1-\gamma}}\right\}. \end{aligned}$$

Example 2. We consider the following initial value problem:

$$\begin{cases} x'(t) = t^2 + \frac{0.033}{e^{4t}}\ln(1+x^2) + \frac{0.0065}{e^{4t}}\ln(1+(D_{0+}^{\frac{2}{15}+\frac{2t}{15(1+t^2)}}x(t))^2), \ \ 0 < t < +\infty, \\ x(0) = 0. \end{cases} \tag{64}$$

Now we claim that the initial value problem (64) has a unique solution $x \in E$.

Let $\beta(t) = \frac{2}{15} + \frac{2t}{15(1+t^2)}$, $f(t,x,y) = t^2 + \frac{0.033}{e^{4t}}\ln(1+x^2) + \frac{0.0065}{e^{4t}}\ln(1+y^2)$, $\eta = 2$, $\eta_1 = 1$, $\eta_2 = 1$, $\eta_3 = 0.5$. Obviously, $\beta(t) \in C^1[0,+\infty)$, $\beta_* = \frac{2}{15}$, $\beta^* = \frac{1}{5}$, $\sup\limits_{0\leqslant t<+\infty}\frac{|\beta'(t)|}{e^{\eta_2 t}} = \sup\limits_{0\leqslant t<+\infty}\left|\frac{2(1-t^2)}{15e^t(1+t^2)^2}\right| = \frac{2}{15}$, thus $\beta(t)$ satisfies assumption (A_1). It follows from

$$f_x(t,x,y) = \frac{0.066x}{e^{4t}(1+x^2)}, \quad f_y(t,x,y) = \frac{0.013y}{e^{4t}(1+y^2)}$$

that

$$\begin{aligned} \left|f(t,e^{2t}x_1,e^{2t}y_1) - f(t,e^{2t}x_2,e^{2t}y_2)\right| &= \left|e^{2t}f_x(t,e^{2t}\xi_1,e^{2t}y_1)(x_1-x_2) + e^{2t}f_y(t,e^{2t}x_2,e^{2t}\xi_2)(y_1-y_2)\right| \\ &\leqslant \left|\frac{0.066\xi_1}{1+e^{4t}\xi_1^2}\right||x_1-x_2| + \left|\frac{0.013\xi_2}{1+e^{4t}\xi_2^2}\right||y_1-y_2| \\ &\leqslant 0.066|x_1-x_2| + 0.013|y_1-y_2|, \end{aligned}$$

where ξ_1 lies between x_1 and x_2, ξ_2 lies between y_1 and y_2. Then f satisfies the assumptions (A_2) and (56) with $C_1 = 0.066$, $C_2 = 0.013$. Let $\mu = \frac{2}{5}$. Direct calculation gives

$$M_f = \sup_{t\geqslant 0}\frac{|f(t,0,0)|}{e^{\eta_3 t}} = \sup_{t\geqslant 0}\frac{t^2}{e^{0.5t}} = \frac{16}{e^2} < +\infty,$$

$$\overline{a}_1 = 0.920 > a_1 = \sup_{0\leqslant t<+\infty}\left|\frac{1}{\Gamma(1-\beta(t))}\right| = \sup_{0\leqslant t<+\infty}\left|\frac{1}{\Gamma(\frac{13}{15}-\frac{2t}{15(1+t^2)})}\right| \approx 0.912,$$

$$\overline{a}_2 = 0.173 > a_2 = \sup_{0\leqslant t<+\infty}\left|\frac{1}{\Gamma(-\beta(t))}\right| = \sup_{0\leqslant t<+\infty}\left|\frac{1}{\Gamma(-\frac{2}{15}-\frac{2t}{15(1+t^2)})}\right| \approx 0.172,$$

$$\overline{a}_3 = 0.813 > a_3 = \sup_{-\beta^*\leqslant\omega\leqslant-\beta_*}\left|\left(\frac{1}{\Gamma(\omega)}\right)'\right| = \sup_{-\frac{1}{5}\leqslant\omega\leqslant-\frac{2}{15}}\left|\left(\frac{1}{\Gamma(\omega)}\right)'\right| \approx 0.812$$

$$M_{\beta'} = \sup_{0\leqslant t<+\infty}\frac{|\beta'(t)|}{e^{\eta_2 t}} = \frac{2}{15},$$

moreover, we have $\frac{1}{5} < \mu = \frac{2}{5} < 1$, $T = 3.700$, $M_1 = 1.840$, $M_2 = 1.840$, such that

$$\left|t^{\mu-\beta^*}\ln t\right| = \left|t^{\frac{2}{5}-\frac{1}{5}}\ln t\right| = \left|t^{\frac{1}{5}}\ln t\right| \leqslant 1.840 \ \ 0 < t \leqslant 3.700,$$

$$\left|t^{-\beta^*}\ln t\right| = \left|t^{-\frac{1}{5}}\ln t\right| < 1.840, \ \ t > 3.700,$$

$$\overline{B} = 7.400 > 7.336 \approx B = \max\left\{\frac{M_1 T^{1+\beta^*-\beta_*-\mu}}{1-\mu}, \frac{M_2}{1+\beta^*-\beta_*}, \frac{T^{1-\beta_*}}{1-\beta^*}\right\}.$$

$$\overline{D} = \max\left\{\overline{a}_1, 2\overline{a}_1\overline{B}M_{\beta'}, \overline{a}_2\overline{B}, \overline{a}_3\overline{B}M_{\beta'}\right\} \approx 1.816,$$

$$\overline{D} \approx 1.816 > 1.784 \approx D \approx \max\{a_1, 2a_1BM_{\beta'}, a_2B, a_3BM_{\beta'}\},$$

$$G = \max\left\{1, \frac{\left(\frac{2}{\eta}\right)^2}{\mathrm{e}^2}, \frac{\left(\frac{2-\beta_*}{\eta}\right)^{2-\beta_*}}{(2-\beta_*)\mathrm{e}^{2-\beta_*}}, \frac{2\left(\frac{3-\beta_*}{\eta}\right)^{3-\beta_*}}{(3-\beta_*)\mathrm{e}^{3-\beta_*}}, \frac{\left(\frac{2-\beta(0)}{\eta}\right)^{2-\beta(0)}}{(2-\beta(0))\mathrm{e}^{2-\beta(0)}}, \right.$$
$$\left. \frac{\left(\frac{3+\beta^*-\beta_*}{\eta}\right)^{3+\beta^*-\beta_*}}{(3+\beta^*-\beta_*)\mathrm{e}^{3+\beta^*-\beta_*}}, \frac{2}{\eta_1\mathrm{e}}, \frac{\left(\frac{1-\beta_*}{\eta_1}\right)^{1-\beta_*}}{\mathrm{e}^{1-\beta_*}}, \frac{2\left(\frac{2-\beta_*}{\eta_1}\right)^{2-\beta_*}}{\mathrm{e}^{2-\beta_*}}, \frac{\left(\frac{1-\beta(0)}{\eta_1}\right)^{1-\beta(0)}}{\mathrm{e}^{1-\beta(0)}}, \frac{\left(\frac{2+\beta^*-\beta_*}{\eta_1}\right)^{2+\beta^*-\beta_*}}{\mathrm{e}^{2+\beta^*-\beta_*}}\right\} = 1.$$

Since $(C_1 + 6C_2D)G \approx 0.205$, $(C_1 + 6C_2\overline{D})G \approx 0.208$, we obtain

$$(C_1 + 6C_2D)G < (C_1 + 6C_2\overline{D})G < 1,$$

Therefore, according to Theorem 2, we obtain that the initial value problem (64) has a unique solution $x \in E$.

4. Conclusion

This paper is devoted to investigating the existence of solutions to a first-order initial value problem involving nonlinear terms of the Riemann-Liouville variable-order fractional derivative defined on the half-line. Here, Theorem 1 establishes existence, whereas Theorem 2 establishes both existence and uniqueness. Together, these results provide a method for establishing the existence and uniqueness of solutions to such problems involving nonlinear terms of the Riemann-Liouville variable-order fractional derivative on the half-line.

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