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Homogeneous Lyapunov functionals and orbit-controlled convergence for coupled fixed point iterations

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Abstract: We investigate the convergence of coupled Picard iterations in complete metric spaces using positively homogeneous Lyapunov functionals. We introduce a unified parameterization of admissible homogeneous invariants, namely continuous positively homogeneous functionals equivalent to the maximum norm, through continuous profile functions on the unit simplex. This framework includes many standard choices, such as the sum, maximum, weighted norms, and ℓ_p norms. Building on this representation, we develop an abstract orbit-controlled convergence framework based on two complementary conditions: geometric decay of a homogeneous invariant and a projective ratio condition describing the asymptotic balance of the iteration. Under these assumptions, we prove convergence of the coupled Picard iteration to a coupled fixed point. For asymptotically linear systems, we identify a distinguished Lyapunov functional by incorporating the classical Perron–Frobenius optimization of induced absolute norms. The resulting Perron-weighted absolute ℓ_1 norm minimizes the induced contraction factor among absolute norms for positive linear operators, linking the homogeneous-functional framework with classical matrix analysis. A nonlinear example illustrates the applicability of the proposed approach.

Keywords: Coupled fixed point, homogeneous invariant, Perron–Frobenius theory, positive matrix, asymptotically linear iteration

MSC: 47H10, 47H09, 47A16, 15B48.

1. Introduction

Fixed point theory plays a central role in nonlinear analysis and has a long history originating from Banach's contraction principle [1]. Over the past decades, this framework has been significantly extended to ordered metric spaces, coupled systems, and mixed monotone operators, with important developments due to Bhaskar and Lakshmikantham [2], Guo and Lakshmikantham [3], and for more recent references and their applications to this topic, see for instance [4–8]. In particular, coupled fixed point iterations of the form

$$x_{n+1} = F(x_n, y_n), \quad y_{n+1} = F(y_n, x_n),$$

have become a standard tool for studying symmetric systems arising in differential equations, integral equations, and dynamical systems. Despite this extensive literature, most convergence analyses rely on scalar reductions such as the maximum norm, which often conceal the interaction between the two error components.

In this paper, we study the convergence of coupled Picard iterations through positively homogeneous Lyapunov functionals. Rather than analyzing the successive increments

$$a_n = d(x_n, x_{n+1}), \quad b_n = d(y_n, y_{n+1}),$$

independently, we decompose the dynamics into two complementary quantities. The first is a positively homogeneous functional $\Psi(a_n, b_n)$ measuring the overall magnitude of the error, while the second is the ratio

$$R_n = \frac{a_n}{b_n},$$

which describes the relative distribution of the two error components and may be viewed as a projective coordinate on the positive cone. This separation naturally distinguishes the decay of the error magnitude from its internal geometric evolution.

To support this approach, we first introduce a unified parameterization of admissible positively homogeneous functionals equivalent to the maximum norm on the positive cone. Although the normalization of homogeneous functions by restriction to a cross-section of the cone is classical, representing such functionals through a single profile function provides a convenient framework for comparing the Lyapunov quantities used throughout the paper. This representation also serves as the basis for the optimization problem considered later for asymptotically linear systems.

Building on this representation, we develop an abstract orbit-controlled convergence framework based on two complementary conditions. The first requires geometric decay of the homogeneous Lyapunov functional, while the second governs the evolution of the projective ratio. Under these assumptions, we establish convergence of the coupled Picard iteration to a coupled fixed point and prove an asymptotic balancing property for the successive error increments. This framework separates the choice of Lyapunov functional from the convergence analysis and provides a flexible setting applicable to both linear and nonlinear examples.

We also investigate the asymptotically linear case. By incorporating the classical Perron–Frobenius optimization of induced absolute norms for positive matrices, we identify a distinguished weighted absolute norm that minimizes the induced contraction factor among all absolute norms. While this optimization result is classical, its incorporation into the present homogeneous-functional framework provides a natural connection between matrix analysis and the convergence theory developed here.

It is useful to compare the present approach with existing work on multidimensional fixed points. Many contributions, beginning with Bhaskar and Lakshmikantham [2] and continuing with Karapinar and Van Luong [9], Rashid et al. [10], and Berzig and Samet [11], study higher-order fixed points in ordered metric spaces using mixed monotonicity and order-dependent contractive conditions. A key observation of Samet et al. [12] and Roldán et al. [13] is that many such results can be obtained by reducing the problem to classical fixed-point theory on suitable product spaces. The present framework is formulated differently. It does not rely on order structures or monotonicity assumptions, but instead imposes orbit-controlled conditions verified directly along the coupled Picard iteration through the successive increments. In addition to convergence, this approach yields information on the asymptotic balance of the two components of the iteration, while, in the asymptotically linear setting, it naturally identifies the Perron-optimal homogeneous norm.

The remainder of the paper is organized as follows. §2 introduces the notation and preliminary concepts used throughout the paper. §3 develops the parameterization of admissible homogeneous invariants. §4 studies optimal admissible norms for positive operators. §5 develops an abstract orbit-controlled convergence framework. §6 establishes the main convergence theorem. §7 presents a Banach-space application to asymptotically linear coupled systems. Finally, §8 provides illustrative linear and nonlinear examples demonstrating the applicability of the proposed framework.

2. Preliminaries

Let (X, d) be a complete metric space. For a mapping $F: X \times X \rightarrow X$, the coupled Picard iteration is

$$x_{n+1} = F(x_n, y_n), \quad y_{n+1} = F(y_n, x_n).$$

Define

$$a_n := d(x_n, x_{n+1}), \quad b_n := d(y_n, y_{n+1}).$$

A pair $(z, w) \in X \times X$ with $F(z, w) = z$ and $F(w, z) = w$ is a coupled fixed point.

Throughout the paper, whenever the ratio

$$R_n = \frac{a_n}{b_n},$$

is considered, we assume that

$$a_n > 0, \quad b_n > 0.$$

This assumption is needed only to ensure that the ratio variable is well defined. No claim is made that a vanishing increment necessarily implies that the corresponding coordinate has reached a fixed point, since this is false in general without additional assumptions on the mapping F .

3. Parameterization of homogeneous invariants

Instead of studying the two error sequences (a_n, b_n) separately, we seek a single scalar quantity that measures their joint size while preserving their relative behavior. Such quantities will be required to scale linearly and to remain equivalent to the maximum norm. We call them admissible homogeneous invariants.

Definition 1. A function $\Psi: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is an *admissible homogeneous invariant* if:

- (i) Ψ is continuous and positive on $\mathbb{R}_+^2 \setminus \{(0,0)\}$;
- (ii) $\Psi(\lambda s, \lambda t) = \lambda \Psi(s, t)$ for all $\lambda > 0$;
- (iii) there exist constants $0 < c \leq C$ such that for all $s, t \geq 0$,

$$c \max\{s, t\} \leq \Psi(s, t) \leq C \max\{s, t\};$$

- (iv) $\Psi(0, 0) = 0$.

Remark 1. The term *homogeneous invariant* is used in the sense of an invariantly defined scalar quantity associated with the pair (a_n, b_n) , rather than a conserved quantity. Along the iteration, $\Psi(a_n, b_n)$ is generally decreasing rather than constant.

Lemma 1 (Representation). Let $\Psi: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ be an admissible homogeneous invariant. Then there exists a unique continuous function

$$h: [0, 1] \longrightarrow (0, \infty),$$

such that

$$\Psi(s, t) = (s + t) h\left(\frac{s}{s + t}\right), \quad (s, t) \neq (0, 0).$$

Moreover, if

$$c_1 \max\{s, t\} \leq \Psi(s, t) \leq C_1 \max\{s, t\},$$

then

$$\frac{c_1}{2} \leq h(u) \leq C_1, \quad 0 \leq u \leq 1.$$

Conversely, if

$$m \leq h(u) \leq M, \quad 0 \leq u \leq 1,$$

then the above formula defines an admissible homogeneous invariant satisfying

$$m \max\{s, t\} \leq \Psi(s, t) \leq 2M \max\{s, t\}.$$

Proof. For every $(s, t) \neq (0, 0)$, define

$$u = \frac{s}{s + t} \in [0, 1].$$

Then

$$s = (s + t)u, \quad t = (s + t)(1 - u).$$

Using positive homogeneity,

$$\Psi(s, t) = (s + t) \Psi(u, 1 - u).$$

Define

$$h(u) = \Psi(u, 1 - u), \quad 0 \leq u \leq 1.$$

Since Ψ is continuous, h is continuous on $[0, 1]$, and the representation formula follows immediately. Next observe that

$$\frac{1}{2} \leq \max\{u, 1 - u\} \leq 1, \quad 0 \leq u \leq 1.$$

Therefore,

$$c_1 \max\{u, 1 - u\} \leq h(u) \leq C_1 \max\{u, 1 - u\},$$

which implies

$$\frac{c_1}{2} \leq h(u) \leq C_1.$$

Conversely, let h be continuous with

$$m \leq h(u) \leq M.$$

Define

$$\Psi(0, 0) = 0,$$

and for $(s, t) \neq (0, 0)$,

$$\Psi(s, t) = (s + t)h\left(\frac{s}{s + t}\right).$$

Since

$$\max\{s, t\} \leq s + t \leq 2 \max\{s, t\},$$

we obtain

$$m \max\{s, t\} \leq \Psi(s, t) \leq 2M \max\{s, t\}.$$

Finally, since h is bounded,

$$|\Psi(s, t)| \leq 2M \max\{s, t\} \longrightarrow 0 \quad \text{as } (s, t) \rightarrow (0, 0),$$

showing continuity at the origin. Uniqueness follows immediately from

$$h(u) = \Psi(u, 1 - u), \quad 0 \leq u \leq 1. \quad \square$$

Proposition 1 (Classification). *The correspondence*

$$\Psi \longmapsto h, \quad h(u) = \Psi(u, 1 - u),$$

is a bijection between admissible homogeneous invariants and continuous functions

$$h : [0, 1] \rightarrow (0, \infty).$$

Under this correspondence, the equivalence constants of Ψ and the bounds of h are related by the estimates established in Lemma 1. Concrete examples:

- $\Psi(s, t) = s + t$: $h(u) = 1$.
- $\Psi(s, t) = \max\{s, t\}$: $h(u) = \max\{u, 1 - u\}$.
- $\Psi(s, t) = \|(s, t)\|_p$: $h(u) = (u^p + (1 - u)^p)^{1/p}$.
- $\Psi(s, t) = \alpha s + \beta t$: $h(u) = \alpha u + \beta(1 - u)$.

Remark 2. The representation established above is obtained by the classical normalization of positively homogeneous functions to the unit simplex. Its purpose here is not to introduce a new representation theorem, but rather to provide a convenient parameterization of all admissible homogeneous functionals relevant to the coupled fixed-point framework developed in the remainder of the paper.

The representation allows different Lyapunov functionals to be compared through their profile functions and naturally connects the abstract classification with the optimization problem studied in §4.

Remark 3. It is important to distinguish admissible homogeneous functionals from norms.

The assumptions of Definition 1 require only continuity, positivity, positive homogeneity and equivalence with the maximum norm. In particular, no convexity, monotonicity, symmetry or subadditivity is assumed.

Consequently, many admissible homogeneous functionals are not norms. §4 therefore restricts attention to the subclass of absolute (lattice) norms, since the Perron–Frobenius optimization theorem relies on standard properties of induced operator norms.

4. Perron-optimal absolute norms for positive operators

In this section we abuse notation slightly by writing Ψ for an absolute norm, since every absolute norm restricts to an admissible homogeneous invariant on the positive cone.

When the dynamics is asymptotically linear, different admissible invariants lead to different contraction factors. A natural question is therefore whether one invariant is optimal. This section answers this question by showing that the Perron-weighted ℓ_1 norm minimizes the induced contraction constant.

Definition 2. An absolute norm (or lattice norm) on $\mathbb{R}^2$ is a norm $\|\cdot\|$ satisfying

$$\|(x_1, x_2)\| = \|(|x_1|, |x_2|)\|,$$

for every $(x_1, x_2) \in \mathbb{R}^2$. Equivalently,

$$|x_i| \leq |y_i|, i = 1, 2, \implies \|x\| \leq \|y\|.$$

Let $\mathcal{N}$ denote the class of all absolute norms on $\mathbb{R}^2$. For $A \in \mathbb{R}^{2 \times 2}$ define

$$\gamma_\Psi(A) = \sup_{x \neq 0} \frac{\Psi(Ax)}{\Psi(x)},$$

where $\Psi \in \mathcal{N}$. The optimal contraction factor is

$$\gamma^*(A) = \inf_{\Psi \in \mathcal{N}} \gamma_\Psi(A).$$

Remark 4. The restriction to absolute norms is essential. For arbitrary norms, the value of the induced operator norm generally cannot be recovered from its restriction to the positive cone. If A is a nonnegative matrix and Ψ is an absolute norm, then

$$|Ax| \leq A|x|,$$

(componentwise), and monotonicity of Ψ yields

$$\Psi(Ax) \leq \Psi(A|x|).$$

Consequently,

$$\gamma_\Psi(A) = \sup_{x > 0} \frac{\Psi(Ax)}{\Psi(x)},$$

so the optimization may be carried out entirely on the positive cone. This reduction is a standard property of absolute norms in matrix analysis.

Remark 5. §3 considers the broader class of admissible homogeneous functionals, which need not satisfy convexity or the triangle inequality. §4 restricts attention to absolute norms because the Perron–Frobenius optimization theorem relies on standard properties of induced operator norms. Thus every norm considered in this section is an admissible homogeneous functional, but the converse is generally false.

Theorem 1 (Perron-optimal absolute norm). Let A be an irreducible nonnegative matrix satisfying

$$\rho(A) < 1.$$

Let $\mathcal{N}$ denote the class of absolute norms on $\mathbb{R}^2$. Then

$$\gamma^*(A) = \rho(A),$$

where

$$\gamma^*(A) = \inf_{\Psi \in \mathcal{N}} \gamma_\Psi(A).$$

Moreover, if $\mathbf{w}^T A = \rho(A) \mathbf{w}^T$, is a positive left Perron eigenvector, then the weighted absolute ℓ_1 norm

$$\|x\|_{\mathbf{w}} = w_1|x_1| + w_2|x_2|,$$

belongs to $\mathcal{N}$ and satisfies

$$\|A\|_{\mathbf{w}} = \rho(A).$$

Consequently, $\|\cdot\|_{\mathbf{w}}$ is an optimizer for the above minimization problem.

Proof. For every induced operator norm,

$$\rho(A) \leq \|A\|,$$

hence

$$\gamma^*(A) \geq \rho(A).$$

Let $\mathbf{w}$ be the positive left Perron eigenvector,

$$\mathbf{w}^T A = \rho(A) \mathbf{w}^T.$$

Define the weighted absolute norm

$$\|x\|_{\mathbf{w}} = w_1|x_1| + w_2|x_2| = \mathbf{w}^T|x|.$$

Since A is nonnegative,

$$|Ax| \leq A|x|,$$

(componentwise). Therefore,

$$\begin{aligned} \|Ax\|_{\mathbf{w}} &= \mathbf{w}^T |Ax| \\ &\leq \mathbf{w}^T A|x| \\ &= \rho(A) \mathbf{w}^T |x| \\ &= \rho(A) \|x\|_{\mathbf{w}}. \end{aligned}$$

Hence

$$\|A\|_{\mathbf{w}} \leq \rho(A).$$

On the other hand, let $v > 0$ be a right Perron eigenvector,

$$Av = \rho(A)v.$$

Then

$$\|Av\|_{\mathbf{w}} = \rho(A) \|v\|_{\mathbf{w}},$$

so equality is attained. Therefore

$$\|A\|_{\mathbf{w}} = \rho(A).$$

Combining this with the lower bound yields

$$\gamma^*(A) = \rho(A),$$

and $\|\cdot\|_{\mathbf{w}}$ is an optimizer. $\square$

Remark 6. The theorem asserts only that the Perron-weighted absolute ℓ_1 norm is an optimizer for the induced operator norm. The paper does not address uniqueness of the optimizer, and different absolute norms may attain the same optimal value. For the symmetric case $A = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$, the left Perron eigenvector is $\mathbf{w} = (1, 1)^T$, so the optimal invariant is $\Psi(s, t) = s + t$.

Remark 7. Theorem 1 relies on classical Perron–Frobenius theory and standard facts about induced operator norms. Its role in the present paper is not to establish a new matrix-theoretic result, but to identify a distinguished optimizer within the family of admissible homogeneous functionals introduced in §3. This optimizer becomes the natural Lyapunov functional for the asymptotically linear coupled iterations studied later in the paper, thereby linking the abstract functional framework with classical matrix analysis.

5. An abstract orbit-controlled convergence framework

The convergence framework developed below is intentionally formulated at the level of the orbit generated by the coupled Picard iteration. This abstraction allows the same theory to be applied to different classes of operators once suitable estimates on the generated increments have been established. The subsequent sections illustrate how such orbit conditions arise in important situations.

Definition 3 (Orbit-Controlled Conditions). Let Ψ be an admissible homogeneous invariant. We say that the orbit generated from (x_0, y_0) satisfies the orbit-controlled conditions if:

(O1) There exist $\lambda \in (0, 1)$ and an index $N_1 \geq 0$ such that

$$\Psi(a_{n+1}, b_{n+1}) \leq \lambda \Psi(a_n, b_n), \quad \forall n \geq N_1.$$

(O2) There exists an index $N_2 \geq 0$ such that

$$a_n > 0 \quad \text{and} \quad b_n > 0 \quad \text{for every } n \geq N_2,$$

so that

$$R_n = \frac{a_n}{b_n},$$

is well defined for every $n \geq N_2$. Moreover, there exist $\rho \in (0, 1)$ and an index $N \geq N_2$ such that

$$|R_{n+1} - 1| \leq \rho |R_n - 1|, \quad \forall n \geq N.$$

The conditions introduced in this paper are formulated along the generated orbit rather than directly on the mapping. Their purpose is to separate the dynamical properties responsible for convergence from the structural assumptions imposed on the operator itself.

In concrete applications, these orbit conditions should be verified from properties of the underlying mapping. §7 illustrates this requirement for asymptotically linear systems, where condition (O1) follows naturally from matrix estimates.

Remark 8. Condition (O2) concerns only the relative evolution of the two increment sequences and therefore requires the ratio

$$R_n = \frac{a_n}{b_n},$$

to be well defined. Accordingly, throughout the paper, (O2) is understood to apply only along those indices for which $a_n, b_n > 0$. No assumptions are imposed on indices at which one of the increments vanishes.

Remark 9. Conditions (O1) and (O2) are orbit assumptions rather than operator assumptions. Their verification depends on the structure of the underlying mapping.

For example, asymptotically linear systems naturally satisfy (O1) under the hypotheses of §7. More generally, deriving (O2) from intrinsic assumptions on the mapping remains an interesting problem that depends on the specific analytical setting under consideration.

Remark 10. The conditions (O1) and (O2) are checked along the specific Picard orbit. (O1) is required only eventually; the finite initial segment can be discarded. This makes the framework applicable to asymptotically linear dynamics. (O2) is assumed for all n to simplify the exposition; the same argument works if it holds eventually.

Condition (O1) controls the decay of the overall error, whereas (O2) governs the relative evolution of the two coordinates. The two conditions are logically independent: (O1) alone implies convergence of the error size, while (O2) yields the additional asymptotic balancing $a_n/b_n \rightarrow 1$.

6. Main convergence theorem

Theorem 2. Let (X, d) be a complete metric space and let

$$F : X \times X \longrightarrow X,$$

be continuous. Suppose that the orbit generated from (x_0, y_0) satisfies (O1) with respect to an admissible homogeneous invariant Ψ . Then the coupled Picard iteration converges to a coupled fixed point (z, w) . Moreover,

$$a_n \longrightarrow 0, \quad b_n \longrightarrow 0.$$

If, in addition, (O2) holds, then

$$\frac{a_n}{b_n} \longrightarrow 1.$$

Proof. Set

$$Q_n = \Psi(a_n, b_n).$$

By (O1),

$$Q_{n+1} \leq \lambda Q_n, \quad n \geq N.$$

Hence

$$Q_n \leq \lambda^{n-N} Q_N,$$

so that

$$Q_n \longrightarrow 0.$$

Since Ψ is equivalent to the maximum norm, there exists a constant $c > 0$ such that

$$\max\{a_n, b_n\} \leq \frac{1}{c} Q_n.$$

Therefore,

$$a_n, b_n \leq \frac{Q_N}{c} \lambda^{n-N},$$

for every $n \geq N$. Consequently,

$$\sum_{n=N}^{\infty} a_n < \infty, \quad \sum_{n=N}^{\infty} b_n < \infty.$$

Hence both (x_n) and (y_n) are Cauchy sequences. Since (X, d) is complete, there exist $z, w \in X$ such that

$$x_n \rightarrow z, \quad y_n \rightarrow w.$$

Using the continuity of F , we obtain

$$F(z, w) = z, \quad F(w, z) = w,$$

so (z, w) is a coupled fixed point. Finally, if (O2) holds, then

$$|R_n - 1| \leq \rho^n |R_0 - 1|,$$

which immediately implies

$$R_n = \frac{a_n}{b_n} \longrightarrow 1. \quad \square$$

7. A Banach-space application: asymptotically linear coupled systems

In this section we illustrate the abstract orbit framework developed in the preceding sections in the setting of Banach spaces. Unlike the previous results, which are formulated in complete metric spaces, the present section assumes a normed linear structure and considers coupled iterations whose error vectors satisfy an asymptotic linear recurrence.

The purpose of this section is to show that such systems naturally satisfy condition (O1) of the abstract convergence theory. No claim is made that this recurrence follows automatically from Fréchet differentiability. Instead, it is taken as an explicit hypothesis.

Definition 4 (Asymptotically Linear Iteration). Let E be a Banach space. A coupled iteration is said to be *asymptotically linear* if its error vectors

$$\mathbf{e}_n = (a_n, b_n)^T,$$

satisfy

$$\mathbf{e}_{n+1} = A\mathbf{e}_n + \mathbf{r}_n,$$

where

$$A \in \mathbb{R}^{2 \times 2}, \quad \rho(A) < 1,$$

and

$$\|\mathbf{r}_n\| = o(\|\mathbf{e}_n\|).$$

The above relation is regarded as an explicit asymptotic hypothesis. Here $\|\cdot\|$ denotes any norm on $\mathbb{R}^2$ (e.g., the Euclidean norm). We assume throughout this section that the error vectors $\mathbf{e}_n$ remain in the positive cone $\mathbb{R}_+^2$; this is natural in many applications and ensures that the weighted functional $\Psi_{\mathbf{w}}(\mathbf{e}_n)$ is a norm on the relevant cone.

Remark 11. Such asymptotic recurrences arise in many Banach-space problems after an appropriate local linearization. The next proposition requires only the recurrence itself and does not depend on the particular construction by which it is obtained.

Proposition 2 (Eventual Contraction for Asymptotically Linear Iterations). *Let the error vector of a coupled iteration be asymptotically linear with strictly positive matrix A , and assume $\mathbf{e}_n \in \mathbb{R}_+^2$ for all n . Then, for the optimal invariant $\Psi_{\mathbf{w}}(s, t) = w_1 s + w_2 t$ where $\mathbf{w}$ is the left Perron eigenvector of A , the following holds: for every $\varepsilon > 0$, there exists N such that for all $n \geq N$,*

$$\Psi_{\mathbf{w}}(\mathbf{e}_{n+1}) \leq (\rho(A) + \varepsilon) \Psi_{\mathbf{w}}(\mathbf{e}_n).$$

Proof. Since $w_i > 0$, the functional $\Psi_{\mathbf{w}}$ is equivalent to the ℓ_1 -norm on the positive cone:

$$\min(w_1, w_2) \|(x, y)\|_1 \leq \Psi_{\mathbf{w}}(x, y) \leq \max(w_1, w_2) \|(x, y)\|_1.$$

Thus $\Psi_{\mathbf{w}}(\mathbf{e}_n) \asymp \|\mathbf{e}_n\|$ for $\mathbf{e}_n \in \mathbb{R}_+^2$, so $\mathbf{r}_n = o(\|\mathbf{e}_n\|) = o(\Psi_{\mathbf{w}}(\mathbf{e}_n))$.

Then:

$$\Psi_{\mathbf{w}}(\mathbf{e}_{n+1}) = \mathbf{w}^T A \mathbf{e}_n + \mathbf{w}^T \mathbf{r}_n = \rho(A) \Psi_{\mathbf{w}}(\mathbf{e}_n) + o(\Psi_{\mathbf{w}}(\mathbf{e}_n)).$$

Hence, for any $\varepsilon > 0$, there exists N such that for all $n \geq N$,

$$\Psi_{\mathbf{w}}(\mathbf{e}_{n+1}) \leq (\rho(A) + \varepsilon) \Psi_{\mathbf{w}}(\mathbf{e}_n).$$

Hence condition (O1) holds with contraction factor

$$\lambda = \rho(A) + \varepsilon,$$

where

$$0 < \varepsilon < 1 - \rho(A). \quad \square$$

The results of this section show that condition (O1) is not merely an abstract hypothesis but arises naturally for asymptotically linear coupled systems through standard matrix estimates.

The corresponding derivation of the projective condition (O2) generally requires additional information on the evolution of the relative error and is therefore left outside the scope of the present abstract framework.

Remark 12. Let

$$0 < \varepsilon < 1 - \rho(A),$$

and define

$$\lambda = \rho(A) + \varepsilon.$$

Then

$$0 < \lambda < 1.$$

Proposition 2 therefore establishes condition (O1) of the abstract framework. It does not establish the ratio condition (O2), which requires separate information on the evolution of the relative error.

The following examples verify the orbit-controlled conditions (O1)–(O2) and then apply the orbit-controlled nonlinear Perron–Frobenius framework; the spectral-radius and Perron-eigenvector facts used in the linear case come from classical Perron–Frobenius theory [14,15].

8. Examples

8.1. A linear example

We illustrate the abstract orbit-controlled framework by a simple linear coupled system.

Example 1. Let

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}, \quad 0 < \beta < \alpha, \quad \alpha + \beta < 1,$$

and define

$$F(x, y) = \alpha x + \beta y,$$

on the complete metric space $(\mathbb{R}, |\cdot|)$. The coupled Picard iteration is

$$x_{n+1} = F(x_n, y_n), \quad y_{n+1} = F(y_n, x_n).$$

Define the successive increments

$$a_n = |x_{n+1} - x_n|, \quad b_n = |y_{n+1} - y_n|.$$

Since

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} \leq A \begin{pmatrix} a_n \\ b_n \end{pmatrix},$$

the Perron–Frobenius theorem implies

$$\rho(A) = \alpha + \beta < 1.$$

Since A is symmetric, its positive left Perron eigenvector is

$$\mathbf{w} = (1, 1)^T,$$

and the associated weighted absolute norm is

$$\Psi(s, t) = s + t, \quad s, t \geq 0.$$

Therefore,

$$\Psi(a_{n+1}, b_{n+1}) \leq (\alpha + \beta) \Psi(a_n, b_n),$$

and condition (O1) holds with

$$\lambda = \alpha + \beta < 1.$$

Since condition (O2) only concerns indices for which the ratio is defined, assume that $b_n > 0$ for all sufficiently large n , and define

$$R_n = \frac{a_n}{b_n}.$$

A direct computation gives

$$R_{n+1} = g(R_n), \quad g(r) = \frac{\alpha r + \beta}{\beta r + \alpha}, \quad r > 0.$$

The unique fixed point of g is $r = 1$, and

$$g'(r) = \frac{\alpha^2 - \beta^2}{(\beta r + \alpha)^2}.$$

Since $0 < \beta < \alpha$, this derivative is strictly positive on $(0, \infty)$, so g is strictly increasing, and

$$g : (0, \infty) \longrightarrow \left(\frac{\beta}{\alpha}, \frac{\alpha}{\beta} \right).$$

Hence every positive initial ratio enters the invariant interval

$$I = \left[\frac{\beta}{\alpha}, \frac{\alpha}{\beta} \right],$$

after one iteration.

Since the denominator $(\beta r + \alpha)^2$ is strictly increasing in $r > 0$ and the numerator $\alpha^2 - \beta^2 > 0$ is constant, g' is positive and strictly decreasing on $(0, \infty)$; consequently

$$\sup_{r \in I} g'(r) = g'\left(\frac{\beta}{\alpha}\right) = \frac{\alpha^2(\alpha^2 - \beta^2)}{(\alpha^2 + \beta^2)^2} =: \rho.$$

Since $0 < \beta < \alpha$,

$$\rho = \frac{\alpha^2(\alpha^2 - \beta^2)}{(\alpha^2 + \beta^2)^2} < \frac{\alpha^2}{\alpha^2 + \beta^2} < 1.$$

Therefore, by the Mean Value Theorem,

$$|g(r) - g(s)| \leq \rho |r - s|, \quad r, s \in I,$$

and since $g(1) = 1$ and every positive ratio belongs to I after one iteration,

$$|R_{n+1} - 1| \leq \rho |R_n - 1|, \quad n \geq 1.$$

Thus condition (O2) holds (after discarding the first iterate).

Remark 13. The hypothesis $\beta < \alpha$ is not merely a simplification. The function $|g'(r)| = |\alpha^2 - \beta^2|/(\beta r + \alpha)^2$ is strictly decreasing in r regardless of the sign of $\alpha^2 - \beta^2$, so its supremum on I is always attained at the endpoint of I closer to 0. When $\beta < \alpha$ this endpoint is β/α , giving the bound above, which is always below 1. When $\beta > \alpha$, the relevant endpoint is instead α/β , and the corresponding value is $(\beta^2 - \alpha^2)/(4\alpha^2)$, which need not be less than 1: for instance $\alpha = 0.1$, $\beta = 0.3$ (so that $\alpha + \beta = 0.4 < 1$) gives the value 2. The elementary Mean-Value-Theorem argument used here therefore does not extend to $\beta > \alpha$, and we do not claim that it does.

Consequently, both orbit-controlled conditions (O1) and (O2) hold. Applying Theorem 2, the coupled Picard iteration converges to the unique coupled fixed point. Moreover,

$$\frac{a_n}{b_n} \longrightarrow 1,$$

showing that the two successive error sequences become asymptotically balanced.

8.2. A nonlinear example

The previous example shows that linear systems satisfy (O1) and (O2) naturally. The following example illustrates that both orbit conditions can also be verified directly from estimates on a genuinely nonlinear mapping, without invoking the asymptotically linear framework of §7. Unlike the previous linear example, the increments a_n and b_n are not identically equal, and the ratio condition is obtained from the different asymptotic decay rates of the symmetric and antisymmetric components of the iteration.

Example 2. Let $X = \mathbb{R}$ with the Euclidean metric. Fix

$$\beta \in \left(0, \frac{1}{2}\right), \quad R > 0,$$

and choose $\eta > 0$ small enough that

$$\eta R^2 < \frac{1 - 2\beta}{3}.$$

This single condition implies, since $\beta < \frac{1}{2}$, both

$$\eta R^2 < \frac{1}{3} \quad \text{and} \quad 2\beta + \eta R^2 < 1,$$

as well as the strengthened bound

$$2\beta + 3\eta R^2 < 1,$$

that will be needed for condition (O1) below.

Define

$$F(x, y) = \beta(x + y) + \eta x^3, \quad x, y \in [-R, R].$$

The coupled iteration is

$$x_{n+1} = F(x_n, y_n), \quad y_{n+1} = F(y_n, x_n).$$

Observe that

$$F(x, y) + F(y, x) = 2\beta(x + y) + \eta(x^3 + y^3),$$

so the sum is no longer preserved, and consequently the increments a_n and b_n need not coincide.

Step 1. Invariance of the region. For every $x, y \in [-R, R]$,

$$|F(x, y)| \leq 2\beta R + \eta R^3 = R(2\beta + \eta R^2) \leq R.$$

Hence every orbit starting in $[-R, R]^2$ remains in $[-R, R]^2$.

Step 2. Uniqueness of the coupled fixed point. Suppose $F(z, w) = z$, $F(w, z) = w$. Subtracting, $\eta(z^3 - w^3) = z - w$. If $z \neq w$, then $\eta(z^2 + zw + w^2) = 1$, which contradicts $\eta R^2 < \frac{1}{3}$ since $z^2 + zw + w^2 \leq 3R^2$. Hence $z = w$. The equation $F(z, z) = z$ reduces to $z[(2\beta - 1) + \eta z^2] = 0$; since $2\beta + \eta R^2 < 1$, the only solution in $[-R, R]$ is $z = 0$. Thus $(0, 0)$ is the unique coupled fixed point in $[-R, R]^2$.

Step 3. Verification of (O1). Define $a_n = |x_{n+1} - x_n|$, $b_n = |y_{n+1} - y_n|$. By the Mean Value Theorem, $|x_n^3 - x_{n-1}^3| \leq 3R^2 a_{n-1}$, so

$$a_n \leq (\beta + 3\eta R^2)a_{n-1} + \beta b_{n-1}, \quad b_n \leq \beta a_{n-1} + (\beta + 3\eta R^2)b_{n-1}.$$

Adding,

$$a_n + b_n \leq \lambda(a_{n-1} + b_{n-1}), \quad \lambda = 2\beta + 3\eta R^2 < 1,$$

by the choice of η above. Taking $\Psi(s, t) = s + t$ gives $\Psi(a_n, b_n) \leq \lambda \Psi(a_{n-1}, b_{n-1})$, which proves (O1).

Step 4. Verification of (O2). Introduce $s_n = x_n + y_n$, $e_n = x_n - y_n$. A direct computation yields

$$s_{n+1} = s_n \left[2\beta + \frac{\eta}{4}(s_n^2 + 3e_n^2) \right], \quad e_{n+1} = \frac{\eta}{4}e_n(3s_n^2 + e_n^2).$$

Therefore

$$|s_{n+1}| \leq \tau |s_n|, \quad \tau = 2\beta + \eta R^2 < 1, \quad |e_{n+1}| \leq \nu |e_n|, \quad \nu = \eta R^2 < 1.$$

Assume $s_0 \neq 0$. Since $2\beta + \frac{\eta}{4}(s_n^2 + 3e_n^2) > 0$, the recurrence for s_n shows $s_{n+1} = 0 \iff s_n = 0$; hence $s_n \neq 0$ for every n , and

$$\theta_n := \frac{|e_n|}{|s_n|},$$

is well defined. Then

$$\theta_{n+1} = \mu_n \theta_n, \quad \mu_n = \frac{\frac{\eta}{4}(3s_n^2 + e_n^2)}{2\beta + \frac{\eta}{4}(s_n^2 + 3e_n^2)}.$$

Since $s_n, e_n \rightarrow 0$, we have $\mu_n \rightarrow 0$. Consequently, there exists, for every $\rho_0 \in (0, 1)$, an index N_{ρ_0} such that

$$\mu_n \leq \rho_0 \quad \text{for every } n \geq N_{\rho_0}.$$

Furthermore,

$$(x_{n+1} - x_n) - (y_{n+1} - y_n) = -e_n \left[1 - \frac{\eta}{4}(3s_n^2 + e_n^2) \right],$$

and since $\eta R^2 < \frac{1}{3}$, the bracket lies in $(\frac{2}{3}, 1]$ for every n .

We now show that $x_{n+1} - x_n$ and $y_{n+1} - y_n$ eventually share a sign. Indeed,

$$x_{n+1} - x_n = s_n \left(\beta - \frac{1}{2} \right) - \frac{e_n}{2} + \eta x_n^3, \quad y_{n+1} - y_n = s_n \left(\beta - \frac{1}{2} \right) + \frac{e_n}{2} + \eta y_n^3.$$

Since $\theta_n \rightarrow 0$, $e_n = o(s_n)$, and $x_n^3, y_n^3 = O(s_n^3) = o(s_n)$; hence both increments equal $-(\frac{1}{2} - \beta)s_n + o(s_n)$ and share the sign of $-s_n$ for n sufficiently large. Consequently, for such n ,

$$|a_n - b_n| = ||x_{n+1} - x_n| - |y_{n+1} - y_n|| = |(x_{n+1} - x_n) - (y_{n+1} - y_n)| \geq \frac{2}{3}|e_n|,$$

and together with the trivial bound $|a_n - b_n| \leq |e_n|$,

$$\frac{2}{3}|e_n| \leq |a_n - b_n| \leq |e_n|.$$

Also $b_n/|s_n| \rightarrow \frac{1}{2} - \beta > 0$, so there exist $c_0, C_0 > 0$ and N_0 such that

$$c_0|s_n| \leq b_n \leq C_0|s_n|, \quad n \geq N_0.$$

• *Comparison estimates.* For $n \geq N_0$,

$$\frac{2}{3}|e_n| \leq |a_n - b_n| \leq |e_n|, \quad c_0|s_n| \leq b_n \leq C_0|s_n|,$$

with c_0, C_0 fixed throughout. Dividing gives

$$\underbrace{\frac{2}{3C_0}}_{=: \kappa_1} \theta_n \leq |R_n - 1| \leq \underbrace{\frac{1}{c_0}}_{=: \kappa_2} \theta_n, \quad n \geq N_0.$$

For $n \geq N_{\rho_0}$ we have $\theta_{n+1} \leq \rho_0 \theta_n$, so

$$|R_{n+1} - 1| \leq \kappa_2 \theta_{n+1} \leq \kappa_2 \rho_0 \theta_n \leq \frac{\kappa_2}{\kappa_1} \rho_0 |R_n - 1|.$$

Choosing $\rho_0 < \kappa_1/\kappa_2$ and setting $\rho := \frac{\kappa_2}{\kappa_1}\rho_0 \in (0, 1)$ gives

$$|R_{n+1} - 1| \leq \rho |R_n - 1|, \quad n \geq N_{\rho_0},$$

which proves (O2).

Consequently, both orbit-controlled conditions (O1) and (O2) hold. Applying Theorem 2, the coupled iteration converges to the unique coupled fixed point $(0, 0)$ in $[-R, R]^2$, and

$$a_n \rightarrow 0, \quad b_n \rightarrow 0, \quad \frac{a_n}{b_n} \rightarrow 1.$$

9. Concluding remarks

This paper develops an abstract orbit-controlled framework for the convergence of coupled Picard iterations based on positively homogeneous Lyapunov functionals. The approach separates the decay of the error magnitude from the evolution of its projective geometry, yielding convergence to a coupled fixed point together with an asymptotic balancing property for the successive error increments.

The framework is supported by two complementary ingredients. First, admissible homogeneous functionals are conveniently parameterized through profile functions on the simplex, providing a unified language for constructing and comparing Lyapunov functionals. Second, in the asymptotically linear setting, the framework naturally incorporates the classical Perron–Frobenius optimization of induced absolute norms, thereby connecting the abstract convergence theory with positive matrix analysis.

Several directions remain open. An important problem is to identify broad classes of mappings for which the orbit conditions (O1) and (O2) follow automatically from intrinsic properties of the operator, such as differentiability, componentwise contractive estimates, or projective contraction principles. Another interesting direction is the extension of the orbit-controlled framework to higher-dimensional coupled systems, multivalued mappings, and operators acting on more general metric or Banach space structures.

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