Open Journal of Mathematical Analysis

PSR Press

Article

Invariant Sample Measures for 3D Stochastic Globally Modified Navier–Stokes Equations

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Received: 13 October 2026; Accepted: 29 October 2026; Published: 15 November 2026.

Abstract: This paper investigates invariant sample measures for the three-dimensional stochastic globally modified Navier–Stokes equations driven by additive white noise. Building on the pullback random attractor theory for this system, we establish continuity properties of the solution process with respect to the initial time. These properties are essential for the construction of invariant sample measures. By employing a generalized Banach limit, we construct a family of Borel probability measures supported on the pullback random attractor and prove that the measures satisfy the invariance property under the associated cocycle. The results extend the invariant sample measure framework from two-dimensional stochastic Navier–Stokes equations to the three-dimensional stochastic globally modified Navier–Stokes equations.

Keywords: pullback random attractor, invariant sample measure, generalized Banach limit, 3D globally modified Navier–Stokes equations, random dynamical system

MSC: 35B41, 76D06, 37L55.

1. Introduction

In the study of stochastic partial differential equations, invariant measures characterize the statistical equilibrium of a system [1–5]. For non-autonomous stochastic systems, because time-dependent external forces and noise act simultaneously, a single invariant measure is insufficient to describe sample-wise statistical behavior; instead, one needs a family of invariant sample measures depending on the sample ω. Zhao, Wang, and Caraballo [6] established sufficient conditions for constructing invariant sample measures from global random attractors for the two-dimensional stochastic Navier–Stokes equations, and proved a stochastic Liouville-type theorem; Kinra and Mohan [7] further extended this framework to the case of bi-spatial pullback random attractors. However, these works mainly concern two-dimensional equations or abstract dynamical systems, and results on invariant sample measures for three-dimensional fluid equations remain scarce.

The global well-posedness of the three-dimensional Navier–Stokes equations remains a well-known and challenging problem in mathematical fluid mechanics. To bypass this obstacle, Caraballo, Real, and Kloeden [8] introduced the globally modified Navier–Stokes equations, which admit global strong solutions in three dimensions by truncating the convective term while preserving the core structure of the original equations. In the stochastic non-autonomous setting, Hang and Nguyen [9] studied pullback random attractors for the three-dimensional globally modified Navier–Stokes equations (GMNSE) driven by additive white noise on unbounded domains, establishing existence, uniqueness and periodicity of the attractors by using the Ornstein–Uhlenbeck process and Ball's energy equation. In addition, Zhao and Yang [10], Caraballo et al. [11], Chen et al. [12] and Liu et al. [13] investigated pullback attractors and invariant measures for non-autonomous or stochastic GMNSE and tamed 3D Navier–Stokes equations, respectively. These works address several related notions, including invariant probability measures, families of invariant measures, and measure attractors. The present paper specifically concerns pathwise families of invariant sample measures for the additive-noise system considered below.

Building on the pullback random attractor results established by Hang and Nguyen [9], and the invariant sample measure framework developed by Zhao, Wang, and Caraballo [6] and Kinra and Mohan [7], this paper investigates the existence of invariant sample measures for the three-dimensional stochastic globally modified

Navier–Stokes equations driven by additive white noise. The studies of Chen and Yang [14] on invariant measures and stochastic Liouville-type theorems for non-autonomous stochastic reaction–diffusion equations, and of Wang [15] on the asymptotic behavior of non-autonomous fractional stochastic reaction–diffusion equations, also provide relevant methodological background. Unlike the existing works, which primarily address random attractors or invariant measures for other classes of stochastic systems, the present work focuses on constructing invariant sample measures for the three-dimensional stochastic globally modified Navier–Stokes equations. Specifically, using the pullback random attractor established in [9], we prove the continuity properties of the solution process required for the measure construction and employ a generalized Banach limit to construct a family of Borel probability measures supported on the pullback random attractor. We then establish the invariance of this family under the associated cocycle.

In this paper, we consider the three-dimensional stochastic globally modified Navier–Stokes equations defined on a domain $\mathcal{O} \subset \mathbb{R}^3$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + F_N(\|u\|)(u\cdot\nabla)u = f - \nabla p + g\dfrac{\mathrm{d}W(t)}{\mathrm{d}t}, \\ \nabla\cdot u = 0, \\ u(x,t) = 0, \quad x \in \partial\mathcal{O}, \\ u(x,\tau) = u_\tau(x). \end{cases} \tag{1}$$

Here, $\nu > 0$ is the viscosity coefficient, f is a non-autonomous external force, $u = u(x,t)$ is the velocity vector field, u_τ is the initial velocity field, and $p = p(x,t)$ is the scalar pressure field. Moreover, W is a two-sided real-valued Wiener process defined on a given probability space, and $g \in D(A)$, where A denotes the Stokes operator. The truncation function is defined by

$$F_N(r) = \min\left\{1, \frac{N}{r}\right\}, \quad r > 0,$$

where $N > 0$ is a prescribed threshold.

The remainder of this paper is organized as follows. §2 introduces the preliminary material, including the basic formulation of the Navier–Stokes equations, and reviews some important results established in the literature. §3 constructs invariant sample measures and proves their existence.

2. Preliminaries

Let $\mathcal{O} \subset \mathbb{R}^3$ be a domain satisfying the Poincaré inequality. More precisely, assume that there exists a constant $\lambda > 0$ such that

$$\lambda \int_{\mathcal{O}} |\varphi(x)|^2 \,\mathrm{d}x \le \int_{\mathcal{O}} |\nabla\varphi(x)|^2 \,\mathrm{d}x, \qquad \forall\, \varphi \in H_0^1(\mathcal{O}).$$

This is the Poincaré-type domain assumption used in [9]; it provides the domain condition needed for the energy estimates and the argument establishing pullback asymptotic compactness. Define

$$\mathcal{V} = \left\{u \in [C_0^\infty(\mathcal{O})]^3 : \nabla\cdot u = 0\right\}.$$

Let H and V denote the closures of $\mathcal{V}$ in $[L^2(\mathcal{O})]^3$ and $[H_0^1(\mathcal{O})]^3$, respectively. We denote the norms in H and V by $|\cdot|$ and $\|\cdot\|$, respectively, and the duality pairing between V and V^* by $\langle\cdot,\cdot\rangle$. The Stokes operator $A : V \to V^*$ is defined by

$$\langle Au, v\rangle = ((u,v)), \qquad u, v \in V,$$

where $((\cdot,\cdot))$ denotes the inner product in V. The trilinear form $b : V \times V \times V \to \mathbb{R}$ is defined by

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \,\mathrm{d}x,$$

which satisfies $b(u,v,w) = -b(u,w,v)$ and $b(u,v,v) = 0$. Following [9], we assume that there exists a constant $k > 0$ such that

$$|b(u,g,u)| \le k|u|^2, \qquad u \in V.$$

This assumption is used to estimate the nonlinear term in the energy inequality. Throughout the paper, we assume that $g \in D(A)$ and that the parameter η satisfies

$$\eta > \frac{16k^2}{\pi(\nu\lambda - \alpha)^2}, \qquad \alpha \in (0, \nu\lambda).$$

Define the operator $B : V \times V \to V^*$. Then Eq. (1) can be written as

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu Au + F_N(\|u\|)Bu = f(x,t) + g(x)\frac{\mathrm{d}W(t)}{\mathrm{d}t}. \tag{2}$$

Here, $Bu = B(u,u) = P(u \cdot \nabla)u$, where P denotes the Leray projection. By the standard estimate for the trilinear form associated with B, we have

$$\|B(u,u)\|_{V^*} \le C|u|^{1/2}\|u\|^{3/2}, \qquad u \in V.$$

Since $F_N(r) = \min\{1, N/r\}$, we have $F_N(\|u\|) \le N/\|u\|$ whenever $\|u\| > 0$. Consequently,

$$\|F_N(\|u\|)B(u,u)\|_{V^*}^2 \le CN^2|u|\|u\|, \qquad u \in V.$$

The estimate is also valid when $u = 0$, since the nonlinear term then vanishes.

Let $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$ be the canonical metric dynamical system associated with a two-sided real-valued Wiener process, where $\Omega = C_0(\mathbb{R},\mathbb{R})$, $\mathcal{F}$ is the Borel σ-algebra on Ω, P is the two-sided Wiener measure, and the Wiener shift is defined by $(\vartheta_t\omega)(s) = \omega(t+s) - \omega(t)$ for $s,t \in \mathbb{R}$. Define $W(t,\omega) = \omega(t)$ for $t \in \mathbb{R}$. Consider the one-dimensional Ornstein–Uhlenbeck process

$$dz(\vartheta_t\omega) = -\eta z(\vartheta_t\omega)\,dt + dW(t),$$

whose stationary solution is given by

$$z(\omega) = -\eta\int_{-\infty}^{0} e^{\eta s}\omega(s)\,ds.$$

Here, $g \in D(A)$ is a deterministic spatial function, so the stochastic forcing $g\,dW(t)$ is driven by the scalar Wiener process $W(t)$. For almost every $\omega \in \Omega$, the stationary Ornstein–Uhlenbeck process satisfies

$$\lim_{|t|\to\infty}\frac{|z(\vartheta_t\omega)|}{|t|} = 0,$$

and, by the ergodicity of the process,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t |z(\vartheta_s\omega)|\,\mathrm{d}s = \lim_{t\to\infty}\frac{1}{t}\int_{-t}^0 |z(\vartheta_s\omega)|\,\mathrm{d}s = \frac{1}{\sqrt{\pi\eta}}.$$

By the assumption

$$\eta > \frac{16k^2}{\pi(\nu\lambda - \alpha)^2}, \qquad 0 < \alpha < \nu\lambda,$$

we have

$$\nu\lambda - \frac{4k}{\sqrt{\pi\eta}} > \alpha > 0.$$

Consequently, for almost every $\omega \in \Omega$,

$$\lim_{t\to\infty} \frac{1}{t} \int_{-t}^{0} \big(\nu\lambda - 4k|z(\vartheta_s\omega)|\big)\, \mathrm{d}s = \nu\lambda - \frac{4k}{\sqrt{\pi\eta}} > \alpha.$$

Equivalently,

$$\lim_{t\to\infty} \frac{1}{t} \int_{0}^{-t} \big(\nu\lambda - 4k|z(\vartheta_s\omega)|\big)\, \mathrm{d}s = -\left(\nu\lambda - \frac{4k}{\sqrt{\pi\eta}}\right) < -\alpha.$$

Thus, the exponential factor

$$\exp\left(\int_{0}^{-t} \big(\nu\lambda - 4k|z(\vartheta_r\omega)|\big)\, \mathrm{d}r\right),$$

decays exponentially as $t \to \infty$. This negative-time damping estimate is used in establishing the pullback estimates below.

Introduce the transformation

$$v(t,\tau,\omega) = u(t,\tau,\omega) - g(x)z(\vartheta_t\omega). \tag{3}$$

Then Eq. (2) is transformed into the following random-parameter equation:

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \nu Av + F_N\big(\|v + gz(\vartheta_t\omega)\|\big)B\big(v + gz(\vartheta_t\omega), v + gz(\vartheta_t\omega)\big) = f + \eta gz(\vartheta_t\omega) - \nu Agz(\vartheta_t\omega). \tag{4}$$

For subsequent analysis, we first impose an assumption on the non-autonomous external force. Fix a constant $\alpha \in (0, \nu\lambda)$ and assume that f satisfies

$$\int_{-\infty}^{\tau} e^{\alpha s}\|f(s,\cdot)\|_{V^*}^2\, \mathrm{d}s < \infty, \qquad \text{for all } \tau \in \mathbb{R}. \tag{5}$$

Furthermore, for every $c > 0$, assume that

$$\lim_{\tau\to-\infty} e^{c\tau} \int_{-\infty}^{0} e^{\alpha s}\|f(s+\tau,\cdot)\|_{V^*}^2\, \mathrm{d}s = 0. \tag{6}$$

Define the continuous cocycle $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H \to H$ by $\Phi(t,\tau,\omega,u_\tau) = u(t+\tau,\tau,\vartheta_{-\tau}\omega,u_\tau) = v(t+\tau,\tau,\vartheta_{-\tau}\omega,v_\tau) + gz(\vartheta_t\omega)$, where $v_\tau = u_\tau - gz(\omega)$.

We define the universe $\mathcal{D}$ as the collection of all families $D = D(\tau,\omega) : \tau \in \mathbb{R},\ \omega \in \Omega$ of nonempty bounded subsets of H satisfying

$$\lim_{t\to-\infty} e^{\gamma t}|D(\tau+t,\vartheta_t\omega)| = 0,$$

for every $\gamma > 0$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$.

We next present some fundamental properties of solutions to the three-dimensional stochastic globally modified Navier–Stokes equations, together with relevant results from random dynamical systems theory. These results are mainly drawn from [9] and provide a theoretical foundation for the subsequent construction of invariant sample measures.

Lemma 1. *Suppose that assumption (5) and the condition on g stated above are both satisfied. Then, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $v_\tau \in H$, Eq. (4) admits a unique solution*

$$v(\cdot,\tau,\omega,v_\tau) \in C([\tau,\infty);H) \cap L^2_{\mathrm{loc}}([\tau,\infty);V). \tag{7}$$

Moreover, the solution depends continuously on the initial datum v_τ and is measurable with respect to ω. It satisfies the energy inequality

$$\frac{d}{dt}|v(t,\tau,\omega,v_\tau)|^2 + \big(\nu\lambda - 4k|z(\vartheta_t\omega)|\big)\,|v(t,\tau,\omega,v_\tau)|^2 + \frac{\nu}{4}\|v(t,\tau,\omega,v_\tau)\|^2 \le \frac{4}{\nu}\|f(t,\cdot)\|_{V^*}^2 + \zeta(\vartheta_t\omega), \tag{8}$$

where

$$\zeta(\vartheta_t\omega) = 4\nu\|g\|^2|z(\vartheta_t\omega)|^2 + 4k|g|^2|z(\vartheta_t\omega)|^3 + \frac{4\eta^2}{\nu\lambda}|g|^2|z(\vartheta_t\omega)|^2.$$

The above conclusions follow from the well-posedness result and the energy estimate established in [9, Section 2.2 and Lemma 3.1].

Lemma 2. *Assume that condition (5) holds. Let v be a solution of Eq. (4). Fix $v_* \in H$, $\tau \in \mathbb{R}, t^* \in \mathbb{R}$, and $\omega \in \Omega$. Then there exist positive constants C_1 and C_2, independent of $t \in [t^* - 1, t^*]$, such that*

$$\sup_{r\in[t,t^*]} |v(r;t,\vartheta_{-\tau}\omega,v_*)|^2 \le C_1, \tag{9}$$

and

$$\int_t^{t^*} \left\| \frac{\mathrm{d}}{\mathrm{d}r} v(r;t,\vartheta_{-\tau}\omega,v_*) \right\|_{V^*}^2 \mathrm{d}r \le C_2, \tag{10}$$

for every $t \in [t^* - 1, t^*]$.

Proof. Fix $v_* \in H$, $\tau, t_* \in \mathbb{R}$, and $\omega \in \Omega$ such that the corresponding Ornstein–Uhlenbeck process has continuous sample paths. Let $t \in [t_* - 1, t_*]$ and set

$$v(r) = v(r;t,\vartheta_{-\tau}\omega,v_*), \qquad r \in [t, t_*].$$

By the energy inequality (8), we have

$$\frac{d}{dr}|v(r)|^2 + \big(\nu\lambda - 4k|z(\vartheta_{r-\tau}\omega)|\big)|v(r)|^2 + \frac{\nu}{4}\|v(r)\|^2 \le \frac{4}{\nu}\|f(r,\cdot)\|_{V^*}^2 + \zeta(\vartheta_{r-\tau}\omega).$$

Since $z(\vartheta_r\omega)$ has continuous sample paths, the constants

$$M_z := \sup_{r\in[t_*-1,t_*]} |z(\vartheta_{r-\tau}\omega)|, \qquad M_\zeta := \sup_{r\in[t_*-1,t_*]} \zeta(\vartheta_{r-\tau}\omega),$$

are finite. Consequently,

$$4k|z(\vartheta_{r-\tau}\omega)| - \nu\lambda \le 4kM_z =: M_0,$$

for every $r \in [t_* - 1, t_*]$. Discarding the nonnegative term involving the V-norm, we obtain

$$\frac{d}{dr}|v(r)|^2 \le M_0|v(r)|^2 + \frac{4}{\nu}\|f(r,\cdot)\|_{V^*}^2 + M_\zeta.$$

By Gronwall's inequality and the initial condition $v(t) = v_*$, it follows that

$$\begin{aligned} |v(r)|^2 &\le e^{M_0(r-t)}|v_*|^2 + \int_t^r e^{M_0(r-s)}\left(\frac{4}{\nu}\|f(s,\cdot)\|_{V^*}^2 + M_\zeta\right) ds \\ &\le e^{M_0}|v_*|^2 + e^{M_0}\frac{4}{\nu}\int_{t_*-1}^{t_*}\|f(s,\cdot)\|_{V^*}^2\, ds + e^{M_0}M_\zeta. \end{aligned} \tag{11}$$

Here we used $0 \le r - t \le 1$. By assumption (5),

$$\int_{t_*-1}^{t_*}\|f(s,\cdot)\|_{V^*}^2\, ds < \infty.$$

Thus there exists a constant $C_1 > 0$, depending on v_*, τ, t_*, ω, and the fixed parameters, but independent of $t \in [t_* - 1, t_*]$, such that

$$\sup_{r\in[t,t_*]} |v(r;t,\vartheta_{-\tau}\omega,v_*)|^2 \le C_1.$$

This proves (9).

It remains to prove (10). Integrating the energy inequality (8) over the interval $[t, t_*]$ and using (9), we obtain

$$\int_t^{t_*} \|v(r; t, \vartheta_{-\tau}\omega, v_*)\|^2 \, \mathrm{d}r \le C,$$

where $C > 0$ is independent of $t \in [t_* - 1, t_*]$. By the nonlinear estimate

$$\|F_N(\|w\|)B(w,w)\|_{V^*}^2 \le CN^2|w|\|w\|,$$

we have

$$\begin{aligned}
&\int_t^{t_*} \big\|F_N(\|v(r) + gz(\vartheta_{r-\tau}\omega)\|)B(v(r) + gz(\vartheta_{r-\tau}\omega), v(r) + gz(\vartheta_{r-\tau}\omega))\big\|_{V^*}^2 \, \mathrm{d}r \\
&\le CN^2 \sup_{r\in[t,t_*]} |v(r) + gz(\vartheta_{r-\tau}\omega)|(t_* - t)^{1/2}\Big(\int_t^{t_*} \|v(r) + gz(\vartheta_{r-\tau}\omega)\|^2 \, \mathrm{d}r\Big)^{1/2} \\
&\le C.
\end{aligned}$$

Here, the last inequality follows from (9), the preceding integral estimate, and the boundedness of $z(\vartheta_{r-\tau}\omega)$ on $[t_* - 1, t_*]$. It follows from Eq. (4) that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}v(r) = \; &-\nu Av(r) - F_N(\|v(r) + gz(\vartheta_{r-\tau}\omega)\|)B(v(r) + gz(\vartheta_{r-\tau}\omega), v(r) + gz(\vartheta_{r-\tau}\omega)) \\
&+ f(r,\cdot) + \eta gz(\vartheta_{r-\tau}\omega) - \nu Agz(\vartheta_{r-\tau}\omega).
\end{aligned}$$

Since $A : V \to V^*$ is bounded and $g \in D(A)$, it follows that

$$\begin{aligned}
\Big\|\frac{\mathrm{d}}{\mathrm{d}r}v(r)\Big\|_{V^*}^2 \le C\big(&\|v(r)\|^2 + \|F_N(\|v(r) + gz(\vartheta_{r-\tau}\omega)\|)B(v(r) + gz(\vartheta_{r-\tau}\omega), v(r) + gz(\vartheta_{r-\tau}\omega))\|_{V^*}^2 \\
&+ \|f(r,\cdot)\|_{V^*}^2 + |z(\vartheta_{r-\tau}\omega)|^2\big).
\end{aligned}$$

Condition (5) implies that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V^*)$. Integrating this inequality over $[t, t_*]$ and applying the estimates established above therefore yields

$$\int_t^{t_*} \Big\|\frac{\mathrm{d}}{\mathrm{d}r}v(r; t, \vartheta_{-\tau}\omega, v_*)\Big\|_{V^*}^2 \, \mathrm{d}r \le C_2,$$

where $C_2 > 0$ is independent of $t \in [t_* - 1, t_*]$. This proves (10). □

Lemma 3. *Suppose that assumptions (5) and (6) hold, that $g \in D(A)$ satisfies the condition stated above, and that$\eta > \frac{16k^2}{\pi(\nu\lambda - \alpha)^2}$.Then the continuous cocycle Φ generated by problem (2) possesses a unique $\mathcal{D}$-pullback attractor*

$$\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R},\ \omega \in \Omega\},$$

in H. Moreover,

$$\mathcal{A}(\tau,\omega) = \Omega(K,\tau,\omega),$$

where

$$K(\tau,\omega) = \{u \in H : |u|^2 \le R(\tau,\omega)\},$$

and

$$R(\tau,\omega) = M\Big[1 + |z(\omega)|^2 + \int_{-\infty}^0 \exp\Big(\int_0^s \big(\nu\lambda - 4k|z(\vartheta_r\omega)|\big)\, dr\Big)\big(|z(\vartheta_s\omega)|^2 + |z(\vartheta_s\omega)|^3 + \|f(s+\tau,\cdot)\|_{V^*}^2\big)\, ds\Big],$$

with $M > 0$ independent of τ and ω. Here, $\Omega(K,\tau,\omega)$ denotes the pullback ω-limit set of K.

By [9, Lemma 3.2], under assumptions (5) and (6), K is a closed measurable $\mathcal{D}$-pullback absorbing set and $K \in \mathcal{D}$. Moreover, [9, Lemma 3.5 and Theorem 3.6] establish the pullback asymptotic compactness and the existence of the $\mathcal{D}$-pullback attractor, respectively.

3. Invariant Sample Measures

A family of Borel probability measures $\{\mu_{\tau,\omega}\}_{(\tau,\omega)\in\mathbb{R}\times\Omega}$ is called an invariant sample measure for the cocycle Φ if, for every Borel set $E \subset H$, the following identity holds:

$$\mu_{\tau,\omega}(E) = \mu_{\tau-t,\vartheta_{-t}\omega}\left(\Phi^{-1}(t,\tau-t,\vartheta_{-t}\omega,\cdot)E\right), \quad \forall\, t \geq 0,\ \omega \in \Omega.$$

In the following, we establish the continuity of solutions to equation (4) with respect to the initial time. This result plays an important role in the construction of invariant sample measures and in the study of the regularity of random attractors.

Lemma 4. *Fix $v_* \in H$, $\tau \in \mathbb{R}, t^* \in \mathbb{R}$, and $\omega \in \Omega$ such that the corresponding Ornstein–Uhlenbeck process has continuous sample paths. Suppose that assumptions (5) and (6) hold. Then, for every $\varepsilon > 0$, there exists a constant $\delta_0 = \delta_0(\varepsilon, t^*, v_*, \omega, \eta, \nu) > 0$ such that*

$$|v(t^*;t,\vartheta_{-\tau}\omega,v_*) - v_*|^2 < \varepsilon, \qquad \forall\, t \in (t^*-\delta_0, t^*). \tag{12}$$

Proof. Fix $v_* \in H$, $\tau, t_* \in \mathbb{R}$, and $\omega \in \Omega$ in a set of full probability. Suppose that assumptions (5) and (6) hold. Let $\varepsilon > 0$. For any $t \in (t_*-1, t_*)$, we have

$$\begin{aligned}|v(t_*;t,\vartheta_{-\tau}\omega,v_*) - v_*|^2 &= \big(v(t_*;t,\vartheta_{-\tau}\omega,v_*) - v_*, v(t_*;t,\vartheta_{-\tau}\omega,v_*) - v_*\big)\\ &= \int_t^{t_*} \frac{d}{dr}|v(r;t,\vartheta_{-\tau}\omega,v_*)|^2\,dr - 2\big(v(t_*;t,\vartheta_{-\tau}\omega,v_*) - v_*, v_*\big).\end{aligned} \tag{13}$$

We estimate the two terms on the right-hand side of (13) separately. By the energy inequality (8), we obtain

$$\begin{aligned}&\frac{d}{dr}|v(r;t,\vartheta_{-\tau}\omega,v_*)|^2 + (\nu\lambda - 4k|z(\vartheta_{r-\tau}\omega)|)|v(r;t,\vartheta_{-\tau}\omega,v_*)|^2\\ &\quad + \frac{\nu}{4}\|v(r;t,\vartheta_{-\tau}\omega,v_*)\|_V^2 \leq \frac{4}{\nu}\|f(r,\cdot)\|_{V^*}^2 + \zeta(\vartheta_{r-\tau}\omega).\end{aligned}$$

Since $z(\vartheta_{r-\tau}\omega)$ is continuous on the compact interval $[t_*-1, t_*]$, it is bounded on this interval. We therefore define

$$C_z := \sup_{r\in[t_*-1,t_*]} |z(\vartheta_{r-\tau}\omega)| < \infty.$$

Consequently, for every $r \in [t_*-1, t_*]$,

$$4k|z(\vartheta_{r-\tau}\omega)| - \nu\lambda \leq 4kC_z.$$

Set $C_0 = 1 + 4kC_z > 0$. Then

$$4k|z(\vartheta_{r-\tau}\omega)| - \nu\lambda \leq 4kC_z < C_0, \qquad r \in [t_*-1, t_*].$$

Thus, by the energy inequality (8), after discarding the nonnegative dissipation term, we obtain

$$\begin{aligned}\int_t^{t_*} \frac{d}{dr}|v(r;t,\vartheta_{-\tau}\omega,v_*)|^2\,dr &\leq \int_t^{t_*} (4k|z(\vartheta_{r-\tau}\omega)| - \nu\lambda)|v(r;t,\vartheta_{-\tau}\omega,v_*)|^2\,dr\\ &\quad + \frac{4}{\nu}\int_t^{t_*} \|f(r,\cdot)\|_{V^*}^2\,dr + \int_t^{t_*} \zeta(\vartheta_{r-\tau}\omega)\,dr.\end{aligned} \tag{14}$$

By Lemma 2, there exists a constant $C_1 > 0$, independent of $t \in (t_* - 1, t_*)$, that bounds the squared solution norm. Combining this uniform estimate with the definition of C_0, we obtain

$$\int_t^{t_*} (4k|z(\vartheta_{r-\tau}\omega)| - \nu\lambda)|v(r;t,\vartheta_{-\tau}\omega,v_*)|^2\,dr \le C_0 \int_t^{t_*} |v(r;t,\vartheta_{-\tau}\omega,v_*)|^2\,dr \le C_0C_1(t_* - t).$$

Therefore, (14) implies that

$$\int_t^{t_*} \frac{d}{dr}|v(r;t,\vartheta_{-\tau}\omega,v_*)|^2\,dr \le C_0C_1(t_* - t) + \frac{4}{\nu}\int_t^{t_*} \|f(r,\cdot)\|_{V^*}^2\,dr + \int_t^{t_*} \zeta(\vartheta_{r-\tau}\omega)\,dr. \tag{15}$$

For $\varepsilon > 0$, choose $\delta_1 = \min\left\{1, \frac{\varepsilon}{6C_0C_1}\right\}$. Then, whenever $t \in (t_* - \delta_1, t_*)$, we have

$$C_0C_1(t_* - t) < \frac{\varepsilon}{6}.$$

Since $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V^*)$, the function $r \mapsto \|f(r,\cdot)\|_{V^*}^2$ is locally integrable. By the absolute continuity of the Lebesgue integral, there exists $\delta_2 \in (0,1]$ such that

$$\frac{4}{\nu}\int_{t_*-\delta_2}^{t_*} \|f(r,\cdot)\|_{V^*}^2\,dr < \frac{\varepsilon}{6}.$$

Hence, for every $t \in (t_* - \delta_2, t_*)$,

$$\frac{4}{\nu}\int_t^{t_*} \|f(r,\cdot)\|_{V^*}^2\,dr < \frac{\varepsilon}{6}.$$

Moreover, $\zeta(\vartheta_{r-\tau}\omega)$ is nonnegative and locally integrable. Thus, there exists $\delta_3 \in (0,1]$ such that

$$\int_{t_*-\delta_3}^{t_*} \zeta(\vartheta_{r-\tau}\omega)\,dr < \frac{\varepsilon}{6}.$$

It follows that, whenever $t \in (t_* - \delta_3, t_*)$,

$$\int_t^{t_*} \zeta(\vartheta_{r-\tau}\omega)\,dr < \frac{\varepsilon}{6}.$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Combining the preceding estimates with (15), we obtain

$$\int_t^{t_*} \frac{d}{dr}|v(r;t,\vartheta_{-\tau}\omega,v_*)|^2\,dr < \frac{\varepsilon}{2}, \qquad t \in (t_* - \delta, t_*).$$

For the second term on the right-hand side of (13), let $\varepsilon > 0$ be arbitrary. Since V is dense in H, we can choose $\widetilde{v} \in V \setminus \{0\}$ such that $|v_* - \widetilde{v}| < \frac{\varepsilon}{8(1+\sqrt{C_1}+|v_*|)}$. It follows that

$$\begin{aligned}
2|\big(v(t_*;t,\vartheta_{-\tau}\omega,v_*) - v_*, v_*\big)| &\le 2|\big(v(t_*;t,\vartheta_{-\tau}\omega,v_*) - v_*, v_* - \widetilde{v}\big)| + 2|\big(v(t_*;t,\vartheta_{-\tau}\omega,v_*) - v_*, \widetilde{v}\big)| \\
&\le 2|v(t_*;t,\vartheta_{-\tau}\omega,v_*) - v_*||v_* - \widetilde{v}| + 2|\big(v(t_*;t,\vartheta_{-\tau}\omega,v_*) - v_*, \widetilde{v}\big)| \\
&\le 2(\sqrt{C_1} + |v_*|)|v_* - \widetilde{v}| + 2|\big(v(t_*;t,\vartheta_{-\tau}\omega,v_*) - v_*, \widetilde{v}\big)| \\
&< \frac{\varepsilon}{4} + 2|\big(v(t_*;t,\vartheta_{-\tau}\omega,v_*) - v_*, \widetilde{v}\big)|.
\end{aligned}$$

Since $\widetilde{v} \in V$, we have

$$\big(v(t_*;t,\vartheta_{-\tau}\omega,v_*) - v_*, \widetilde{v}\big) = \int_t^{t_*} \left\langle \frac{d}{dr}v(r;t,\vartheta_{-\tau}\omega,v_*), \widetilde{v} \right\rangle_{V^*,V} dr.$$

By the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}2|\big(v(t_*;t,\vartheta_{-\tau}\omega,v_*)-v_*,\widetilde{v}\big)| &\le 2\|\widetilde{v}\|_V\int_t^{t_*}\|\frac{d}{dr}v(r;t,\vartheta_{-\tau}\omega,v_*)\|_{V^*}\,dr\\ &\le 2\|\widetilde{v}\|_V(t_*-t)^{1/2}\Big(\int_t^{t_*}\|\frac{d}{dr}v(r;t,\vartheta_{-\tau}\omega,v_*)\|_{V^*}^2\,dr\Big)^{1/2}\\ &\le 2\|\widetilde{v}\|_V C_2^{1/2}(t_*-t)^{1/2},\end{aligned}$$

where C_2 is a uniform bound for the time-derivative estimate on the interval under consideration. Choose $\delta'=\min\left\{1,\frac{\varepsilon^2}{64\|\widetilde{v}\|_V^2C_2}\right\}$. Then, whenever $t_*-\delta'<t<t_*$, we obtain

$$2\left|\big(v(t_*;t,\vartheta_{-\tau}\omega,v_*),\widetilde{v}\big)-\big(v_*,\widetilde{v}\big)\right|<\frac{\varepsilon}{4}.$$

Consequently,

$$2\left|\big(v(t_*;t,\vartheta_{-\tau}\omega,v_*)-v_*,v_*\big)\right|<\frac{\varepsilon}{2}.$$

Combining this estimate with the estimate for the first term on the right-hand side of (13), we conclude that, for

$$t_*-\delta_0<t<t_*,\qquad \delta_0=\min\{\delta,\delta'\},$$

the following estimate holds:

$$\begin{aligned}|v(t_*;t,\vartheta_{-\tau}\omega,v_*)-v_*|^2 &\le |\int_t^{t_*}\frac{d}{dr}|v(r;t,\vartheta_{-\tau}\omega,v_*)|^2\,dr|+2|\big(v(t_*;t,\vartheta_{-\tau}\omega,v_*)-v_*,v_*\big)|\\ &<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

This completes the proof of Lemma 4. □

Lemma 5. *Suppose that assumptions (5) and (6) hold. For any given $u_*\in H$, $\tau\in\mathbb{R}$, and $\omega\in\Omega$ belonging to a set of full probability, the mapping $t\longmapsto\Phi(t,\tau-t,\vartheta_{-t}\omega,u_*)$ is continuous on $[0,\infty)$.*

Proof. Fix $u_*\in H$, $\tau\in\mathbb{R}$, and $t_*\in[0,+\infty)$. Let $\omega\in\Omega$ be such that the corresponding Ornstein–Uhlenbeck process has continuous sample paths. For $t\ge 0$, set

$$v_t=u_*-gz(\vartheta_{-t}\omega),\quad v_{t_*}=u_*-gz(\vartheta_{-t_*}\omega).$$

By the definition of Φ and the Ornstein–Uhlenbeck transformation, for every $t\in(\max\{0,t_*-1\},t_*+1)$, we have

$$\Phi(t,\tau-t,\vartheta_{-t}\omega,u_*)=v(\tau,\tau-t,\vartheta_{-\tau}\omega,v_t)+gz(\omega).$$

Consequently, by the triangle inequality,

$$\begin{aligned}|\Phi(t,\tau-t,\vartheta_{-t}\omega,u_*)-\Phi(t_*,\tau-t_*,\vartheta_{-t_*}\omega,u_*)|^2 &\le |v(\tau,\tau-t,\vartheta_{-\tau}\omega,v_t)-v(\tau,\tau-t_*,\vartheta_{-\tau}\omega,v_t)|^2\\ &\quad+|v(\tau,\tau-t_*,\vartheta_{-\tau}\omega,v_t)-v(\tau,\tau-t_*,\vartheta_{-\tau}\omega,v_{t_*})|^2\\ &:=\mathscr{M}_1(t)+\mathscr{M}_2(t).\end{aligned}$$

It remains to prove that $\mathscr{M}_1(t)\to 0$. If $t_*<t<t_*+1$, the process property gives

$$v(\tau;\tau-t,\vartheta_{-\tau}\omega,v_t)=v\big(\tau;\tau-t_*,\vartheta_{-\tau}\omega,v(\tau-t_*;\tau-t,\vartheta_{-\tau}\omega,v_t)\big).$$

By Lemma 4,

$$\lim_{t\to t_*^+}v(\tau-t_*;\tau-t,\vartheta_{-\tau}\omega,v_t)=v_t.$$

Thus, by continuous dependence on the initial data, $\lim_{t\to t_*^+}\mathscr{M}_1(t)=0$.

If $t_* - 1 < t < t_*$, by the process property,

$$v(\tau; \tau - t_*, \vartheta_{-\tau}\omega, v_t) = v\big(\tau; \tau - t, \vartheta_{-\tau}\omega, v(\tau - t; \tau - t_*, \vartheta_{-\tau}\omega, v_t)\big).$$

The continuity of the solution with respect to time implies

$$\lim_{t \to t_*^-} v(\tau - t; \tau - t_*, \vartheta_{-\tau}\omega, v_t) = v_t.$$

Consequently, continuous dependence on the initial data gives $\lim_{t \to t_*^-} \mathscr{M}_1(t) = 0$.
For any $t \in (\max\{0, t_* - 1\},\ t_* + 1)$, we have

$$|v_t - v_{t_*}|^2 \le \big|g_{\mathcal{Z}}(\vartheta_{-t}w) - g_{\mathcal{Z}}(\vartheta_{-t_*}w)\big|^2 \to 0, \quad \text{as } t \to t_*.$$

Using the continuity of $v(\tau, \tau - t_*, \vartheta_{-t}w, \cdot\,)$ with respect to the initial value, we deduce that

$$\mathscr{M}_2(t) = \big|v(\tau, \tau - t_*, \vartheta_{-t}w, v_t) - v(\tau, \tau - t_*, \vartheta_{-t}w, v_{t_*})\big|^2 \to 0 \quad \text{as } t \to t_*.$$

Therefore,

$$\lim_{t \to t_*} \big|\Phi(t, \tau - t, \vartheta_{-t}\omega, u_*) - \Phi(t_*, \tau - t_*, \vartheta_{-t_*}\omega, u_*)\big| = 0.$$

Hence $t \mapsto \Phi(t, \tau - t, \vartheta_{-t}\omega, u_*)$ is continuous on $[0, \infty)$. The proof is complete. □

We now introduce the concept of a generalized Banach limit, denoted by $\operatorname{LIM}_{t \to +\infty}$. It is a continuous linear functional on the space of all bounded real-valued functions on $\mathbb{R}$ satisfying the following properties:

(i) If $f(t) \ge 0$ for all $t \in \mathbb{R}^+$, then $\operatorname{LIM}_{t \to +\infty} f(t) \ge 0$.
(ii) If the usual limit $\lim_{t \to +\infty} f(t)$ exists, then $\operatorname{LIM}_{t \to +\infty} f(t) = \lim_{t \to +\infty} f(t)$.
(iii) For every $s > 0$, $\operatorname{LIM}_{t \to +\infty} f(t + s) = \operatorname{LIM}_{t \to +\infty} f(t)$.

We shall use generalized Banach limits to construct invariant sample measures for the three-dimensional globally modified Navier–Stokes equations. Further properties of generalized Banach limits can be found in [16].

The family of measures constructed below is understood in the pathwise sense. For each fixed (τ, ω), the generalized Banach limit yields a Borel probability measure $\mu_{\tau,\omega}$ satisfying the stated properties. We do not claim that $\omega \mapsto \mu_{\tau,\omega}(E)$ is measurable for every Borel set $E \subset H$.

Theorem 1. *Assume that conditions (5) and (6) hold, and let $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R},\ \omega \in \Omega\}$ be the pullback random attractor established in Lemma 3 for the continuous cocycle generated by problem $(*)$. Then, for any generalized Banach limit $\operatorname{LIM}_{t \to +\infty}$ and every $u_* \in H$, there exists a family of Borel probability measures $\{\mu_{\tau,\omega}\}_{(\tau,\omega) \in \mathbb{R} \times \Omega}$ with the following properties:*

(i) *The measure $\mu_{\tau,\omega}$ is supported on $\mathcal{A}(\tau, \omega)$, that is, $\mu_{\tau,\omega}\big(\mathcal{A}(\tau, \omega)\big) = 1$.*
(ii) *For every $\xi \in C(H)$ for which the corresponding time average is bounded, we have*

$$\operatorname{LIM}_{t \to +\infty} \frac{1}{t} \int_0^t \xi\big(\Phi(r, \tau - r, \vartheta_{-r}\omega, u_*)\big)\, \mathrm{d}r = \int_{\mathcal{A}(\tau,\omega)} \xi(u)\, \mathrm{d}\mu_{\tau,\omega}(u) = \int_H \xi(u)\, \mathrm{d}\mu_{\tau,\omega}(u).$$

(iii) *The family $\{\mu_{\tau,\omega}\}$ is invariant under the cocycle.*

Proof. Fix $u_* \in H$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$ belonging to the invariant set of full probability on which the pullback attraction property holds. For each $\xi \in C(H)$, define

$$L_{\tau,\omega,u_*}(\xi) = \operatorname{LIM}_{t \to +\infty} \frac{1}{t} \int_0^t \xi\big(\Phi(r, \tau - r, \vartheta_{-r}\omega, u_*)\big)\, \mathrm{d}r. \tag{16}$$

By Lemma 5, the mapping $r \longmapsto \Phi(r, \tau - r, \vartheta_{-r}\omega, u_*)$ is continuous on $[0, \infty)$. Consequently, for every $\xi \in C(H)$, the mapping $r \longmapsto \xi\big(\Phi(r, \tau - r, \vartheta_{-r}\omega, u_*)\big)$ is also continuous. Moreover, the pullback attraction property implies that

$$\operatorname{dist}_H\big(\Phi(r, \tau - r, \vartheta_{-r}\omega, u_*), \mathcal{A}(\tau, \omega)\big) \longrightarrow 0 \quad \text{as } r \to +\infty.$$

Since $\mathcal{A}(\tau, \omega)$ is compact, the orbit

$$\{\Phi(r, \tau - r, \vartheta_{-r}\omega, u_*) : r \geq 0\},$$

is relatively compact in H. Indeed, its restriction to every bounded time interval has compact image by continuity, while its tail is arbitrarily close to the compact set $\mathcal{A}(\tau, \omega)$. Thus, its closure

$$K_{\tau,\omega,u_*} = \overline{\{\Phi(r, \tau - r, \vartheta_{-r}\omega, u_*) : r \geq 0\}},$$

is compact in H. It follows that the integrand in (16) is bounded on $[0, \infty)$ for every $\xi \in C(H)$. Hence $L_{\tau,\omega,u_*}(\xi)$ is well defined. Clearly, L_{τ,ω,u_*} is linear. Furthermore, if $\xi \geq 0$ on H, then the integrand in (16) is nonnegative. The positivity of the generalized Banach limit therefore yields $L_{\tau,\omega,u_*}(\xi) \geq 0$. Moreover,

$$L_{\tau,\omega,u_*}(1) = \mathrm{LIM}_{t\to+\infty} \frac{1}{t} \int_0^t 1 \, \mathrm{d}r = \mathrm{LIM}_{t\to+\infty} 1 = 1.$$

Restricting L_{τ,ω,u_*} to the continuous functions on the compact set K_{τ,ω,u_*}, we obtain a positive linear functional of norm one. By the Riesz representation theorem, there exists a unique Borel probability measure $\mu_{\tau,\omega}$ on K_{τ,ω,u_*} such that

$$L_{\tau,\omega,u_*}(\xi) = \int_H \xi(u) \, \mathrm{d}\mu_{\tau,\omega}(u), \qquad \forall\, \xi \in C(H). \tag{17}$$

Here, the measure on K_{τ,ω,u_*} is regarded as a Borel probability measure on H through the natural inclusion $K_{\tau,\omega,u_*} \hookrightarrow H$. This proves property (ii).

We next show that $\mu_{\tau,\omega}$ is supported on $\mathcal{A}(\tau, \omega)$. Let $F \subset H \setminus \mathcal{A}(\tau, \omega)$ be closed. Since $\mathcal{A}(\tau, \omega)$ is compact and F is closed, we have

$$\varepsilon_0 := \operatorname{dist}_H(F, \mathcal{A}(\tau, \omega)) > 0,$$

provided that F is nonempty. The case $F = \varnothing$ is trivial. Choose $\xi_0 \in C(H)$ such that

$$0 \leq \xi_0 \leq 1, \qquad \xi_0|_F = 1, \qquad \xi_0|_{B(\mathcal{A}(\tau,\omega),\varepsilon_0/2)} = 0.$$

By the pullback attraction property, there exists $T > 0$ such that

$$\Phi(r, \tau - r, \vartheta_{-r}\omega, u_*) \in B(\mathcal{A}(\tau, \omega), \varepsilon_0/2) \qquad \text{for all } r \geq T.$$

Consequently, $\xi_0\big(\Phi(r, \tau - r, \vartheta_{-r}\omega, u_*)\big) = 0$, for all $r \geq T$. Therefore, for every $t > T$,

$$\frac{1}{t} \int_0^t \xi_0\big(\Phi(r, \tau - r, \vartheta_{-r}\omega, u_*)\big) \, \mathrm{d}r = \frac{1}{t} \int_0^T \xi_0\big(\Phi(r, \tau - r, \vartheta_{-r}\omega, u_*)\big) \, \mathrm{d}r \leq \frac{T}{t} \longrightarrow 0 \quad \text{as } t \to +\infty.$$

Since a generalized Banach limit agrees with the ordinary limit whenever the latter exists, we obtain

$$L_{\tau,\omega,u_*}(\xi_0) = 0.$$

It follows that

$$\mu_{\tau,\omega}(F) \leq \int_H \xi_0(u) \, \mathrm{d}\mu_{\tau,\omega}(u) = L_{\tau,\omega,u_*}(\xi_0) = 0.$$

Thus, every closed subset of $H \setminus \mathcal{A}(\tau, \omega)$ has zero $\mu_{\tau,\omega}$-measure. Since H is a Polish space and $\mu_{\tau,\omega}$ is a Borel probability measure, it is inner regular. As $\mathcal{A}(\tau, \omega)^c$ is open, we obtain

$$\mu_{\tau,\omega}\big(\mathcal{A}(\tau, \omega)^c\big) = \sup\big\{\mu_{\tau,\omega}(F) : F \subset \mathcal{A}(\tau, \omega)^c, \ F \text{ closed}\big\} = 0.$$

Hence

$$\mu_{\tau,\omega}(\mathcal{A}(\tau,\omega)) = 1,$$

which proves property (i).

Finally, we establish the invariance of the family $\{\mu_{\tau,\omega}\}$. Let $\xi \in C(H)$ and $s \geq 0$. By (17) and the cocycle property of Φ, we have

$$\begin{aligned}\int_H \xi\big(\Phi(s,\tau,\omega,u)\big)\,\mathrm{d}\mu_{\tau,\omega}(u) &= L_{\tau,\omega,u_*}\big(\xi\circ\Phi(s,\tau,\omega,\cdot)\big)\\ &= \mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^t \xi\big(\Phi(s,\tau,\omega,\Phi(r,\tau-r,\vartheta_{-r}\omega,u_*))\big)\,\mathrm{d}r\\ &= \mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^t \xi\big(\Phi(s+r,\tau-r,\vartheta_{-r}\omega,u_*)\big)\,\mathrm{d}r.\end{aligned}$$

Making the change of variables $r' = r + s$ gives

$$\begin{aligned}\mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_s^{t+s} \xi\big(\Phi(r',\tau+s-r',\vartheta_{s-r'}\omega,u_*)\big)\,\mathrm{d}r' &= \mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^t \xi\big(\Phi(r,\tau+s-r,\vartheta_{s-r}\omega,u_*)\big)\,\mathrm{d}r\\ &= L_{\tau+s,\vartheta_s\omega,u_*}(\xi) = \int_H \xi(u)\,\mathrm{d}\mu_{\tau+s,\vartheta_s\omega}(u),\end{aligned}$$

where we have used the translation invariance of the generalized Banach limit and the identity $\vartheta_{-r}(\vartheta_s\omega) = \vartheta_{s-r}\omega$. Since $\xi \in C(H)$ is arbitrary, the preceding identity yields

$$\Phi(s,\tau,\omega,\cdot)_*\mu_{\tau,\omega} = \mu_{\tau+s,\vartheta_s\omega}.$$

Equivalently, for every Borel set $E \subset H$,

$$\mu_{\tau,\omega}\big(\Phi(s,\tau,\omega,\cdot)^{-1}E\big) = \mu_{\tau+s,\vartheta_s\omega}(E).$$

This proves property (iii) and completes the proof. □

Conflicts of Interest: The author declares no conflicts of interest.

Data Availability: No datasets were used in this theoretical study.

Funding Information: This research received no external funding.

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