Open Journal of
Mathematical Analysis

PSR Press

Article

Rotation induced weighted composition semigroups on a weighted Dirichlet-type space

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Received: 31 March 2026; Accepted: 18 July 2026; Published: 10 August 2026.

Abstract: We analyze both the semigroup and spectral properties of a semigroup of weighted composition operators on the weighted Dirichlet-type space of the unit disc. These composition semigroups are induced by the rotation class of the automorphisms of the upper half plane.

Keywords: one-parameter semigroup, spectrum, resolvent, infinitesimal generator, similarity theory

MSC: 47D06, 47B33.

1. Introduction and preliminaries

Let $\mathbb{U} = \{\omega \in \mathbb{C} : \operatorname{Im}(\omega) > 0\}$ denote the upper half-plane of $\mathbb{C}$ where $\operatorname{Im}(\omega)$ is the imaginary part of the complex number ω. The function $\psi(z) = \frac{i(1+z)}{1-z}$ maps $\mathbb{D}$ conformally onto $\mathbb{U}$ with its inverse given by $\psi^{-1}(\omega) = \frac{\omega-i}{\omega+i}$. Let dm denote the (normalised) Lebesgue area measure on $\mathbb{D}$ while $d\mu$ the area Lebesgue measure on $\mathbb{U}$. The corresponding weighted measures on $\mathbb{D}$ and $\mathbb{U}$ are respectively given by $dm_\alpha(z) := (1-|z|^2)^\alpha dm(z)$ and $d\mu_\alpha(\omega) := \operatorname{Im}(\omega)^\alpha d\mu(\omega)$, for $\alpha > -1$. If Ω denotes either $\mathbb{D}$ or $\mathbb{U}$, then $\mathcal{H}(\Omega)$ shall denote the Fréchet space of analytic functions $f : \Omega \to \mathbb{C}$ endowed with the topology of uniform convergence on compact subsets of Ω while $\operatorname{Aut}(\Omega) \subset \mathcal{H}(\Omega)$ denotes the group of biholomorphic maps $f : \Omega \to \Omega$.

For $1 \leq p < \infty$, the Hardy spaces $H^p(\mathbb{D})$ and $H^p(\mathbb{U})$, consist of analytic functions $f \in \mathcal{H}(\mathbb{D})$, resp. $\mathcal{H}(\mathbb{U})$, such that

$$\|f\|_{H^p(\mathbb{D})} := \sup_{0\leq r<1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p} < \infty, \quad \text{resp.}$$

$$\|f\|_{H^p(\mathbb{U})} := \sup_{y>0} \left(\int_{-\infty}^{\infty} |f(x+iy)|^p \, dx \right)^{1/p} < \infty.$$

On the other hand, the weighted Bergman spaces are defined by $L^p_a(m_\alpha) = L^p(m_\alpha) \cap \mathcal{H}(\mathbb{D})$ and $L^p_a(\mu_\alpha) = L^p(\mu_\alpha) \cap \mathcal{H}(\mathbb{U})$. Also, consider the weighted Dirichlet-type spaces $\mathcal{D}^p(m_\alpha)$ and $\mathcal{D}^p(\mu_\alpha)$ consisting of analytic functions satisfying

$$\mathcal{D}^p(m_\alpha) := \left\{ f \in \mathcal{H}(\mathbb{D}) : |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p dm_\alpha(z) < \infty \right\}, \tag{1}$$

equipped with the norm

$$\|f\|_{\mathcal{D}^p(m_\alpha)} = \left(|f(0)|^p + \|f'\|^p_{L^p_a(m_\alpha)} \right)^{\frac{1}{p}}, \tag{2}$$

and

$$\mathcal{D}^p(\mu_\alpha) := \left\{ f \in \mathcal{H}(\mathbb{U}) : |f(i)|^p + \int_{\mathbb{U}} |f'(\omega)|^p d\mu_\alpha(\omega) < \infty \right\}, \tag{3}$$

equipped with the norm

$$\|f\|_{\mathcal{D}^p(\mu_\alpha)} = \left(|f(i)|^p + \|f'\|^p_{L^p_a(\mu_\alpha)} \right)^{\frac{1}{p}}. \tag{4}$$

We further define the subspaces $\mathcal{D}^p_\circ(m_\alpha)$ and $\mathcal{D}^p_\circ(\mu_\alpha)$ by

$$\mathcal{D}^p_\circ(m_\alpha) = \{f \in \mathcal{D}^p(m_\alpha) : f(0) = 0\}\,, \tag{5}$$

$$\mathcal{D}^p_\circ(\mu_\alpha) = \{f \in \mathcal{D}^p(\mu_\alpha) : f(i) = 0\}\,. \tag{6}$$

It is easy to see that $f \in \mathcal{D}^p_\circ(\cdot)$ if and only if $f' \in L^p_a(\cdot)$. In fact $\mathcal{D}^p_\circ(\cdot)$ is a Banach space with respect to the norm

$$\|f\|_{\mathcal{D}^p_\circ(\cdot)} = \|f'\|_{L^p_a(\cdot)}$$

for every $f \in \mathcal{D}^p_\circ(\cdot)$.

We call a family $(\varphi_t)_{t\geq 0}$ of self analytic maps on Ω a semigroup if it satisfies the following conditions:

(i) φ_0 is the identity map on Ω
(ii) $\varphi_t \circ \varphi_s = \varphi_{t+s}$ for all $t, s \geq 0$
(iii) $\varphi_t \to \varphi_0$ uniformly on compact subsets of Ω as $t \to 0$.

Every such (φ_t) induces a semigroup of composition operators (composition semigroup) C_t on $\mathcal{H}(\Omega)$ defined by

$$C_t f(z) = f(\varphi_t(z)), \quad f \in \mathcal{H}(\Omega).$$

For some appropriately chosen weight γ, φ_t shall induce a semigroup of weighted composition operators (weighted composition semigroup) S_t given by

$$S_t f(z) = \left(\varphi_t'(z)\right)^\gamma f(\varphi_t(z)), \quad f \in \mathcal{H}(\Omega).$$

We now briefly discuss the theory of spectra.

Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X, and let Γ denote its infinitesimal generator. We denote the spectrum, point spectrum and resolvent set of Γ by $\sigma(\Gamma)$, $\sigma_p(\Gamma)$ and $\rho(\Gamma)$, respectively. Let X and Y be Banach spaces and let U be an invertible operator in $\mathcal{L}(X, Y)$, the set of all bounded linear operators from X into Y, with $\mathcal{L}(X) := \mathcal{L}(X, X)$. Then $(T_t)_{t\geq 0} \subset \mathcal{L}(X)$ is a strongly continuous semigroup if and only if $S_t := UT_tU^{-1}$, $t \geq 0$, is a strongly continuous semigroup in $\mathcal{L}(Y)$. If $(T_t)_{t\geq 0}$ has the generator Γ, then $(S_t)_{t\geq 0}$ has generator $G = U\Gamma U^{-1}$ with domain

$$\mathrm{dom}(G) = U\mathrm{dom}(\Gamma) := \{y \in Y : Uy \in \mathrm{dom}(\Gamma)\}.$$

Furthermore, $\sigma_p(G) = \sigma_p(\Gamma)$ and $\sigma(G) = \sigma(\Gamma)$ and therefore if $\lambda \in \rho(\Gamma)$, then we have $R(\lambda, G) = UR(\lambda, \Gamma)U^{-1}$ and $\ker(\lambda - \Gamma) = U^{-1}(\ker(\lambda - G))$. We refer to [1] for a comprehensive theory of similar semigroups.

The study of composition semigroups on analytic spaces was initiated by Berkson and Porta [2] in 1978 who established strong continuity on Hardy space $H^p(\mathbb{D})$. This was later extended by Siskakis to the weighted Bergman space $L^p_a(m_\alpha)$ [3] and to the classical Dirichlet space [4]. Usually in the study of semigroups of operators, characterization of their infinitesimal generators whenever they are defined, is a question that naturally arises. Through the investigation of the spectral properties of these infinitesimal operators (usually unbounded) on analytic spaces, some classical integral operators were obtained. In particular, Cowen [5] and Siskakis [6] used the theory in their study of Cesàro and other averaging operator on Hardy and Bergman spaces.

The corresponding study of composition semigroups on analytic spaces of the upper half-plane had not been considered comprehensively until the work of Arvanitidis and Siskakis [7] in 2013 where they constructed a strongly continuous composition semigroup on Hardy spaces $H^p(\mathbb{U})$ and used it to study the properties of the Cesáro operator on $H^p(\mathbb{U})$. The study was immediately generalized by the current second author and three others in [8] in the setting of both $H^p(\mathbb{U})$ and weighted Bergman spaces $L^p_a(\mu_\alpha)$. In that work, [8], all the continuous automorphisms of $\mathbb{U}$ were classified according to the location of their fixed points into three distinct classes, namely; scaling, translation and the rotation groups. Each of these classes induces weighted composition semigroup $(S_t)_{t\geq 0}$ on $\mathcal{H}(\mathbb{U})$. Notably, the rotation group induces a weighted composition semigroup on $\mathcal{H}(\mathbb{D})$ of the form

$$T_t f(z) := e^{ict} f(e^{ikt} z), \quad c, k \in \mathbb{R}, k \neq 0, f \in \mathcal{H}(\mathbb{D}). \tag{7}$$

Extending the analysis of semigroup and spectral properties of composition semigroups to the Dirichlet-type space setting introduces complications that do not arise in the Hardy and Bergman settings. For instance, composition operators are not necessarily bounded on Dirichlet space setting, unlike on Hardy and Bergman spaces where they are always bounded. It is also shown in [9] that both the scaling and translation groups do not induce isometric weighted composition groups on the weighted classical Dirichlet-type space as is the case on Hardy and weighted Bergman spaces. Moreover, generator characterization on Dirichlet-type spaces have remained confined to the Hilbert space case $p = 2$ and to unweighted settings leaving the general p and weighted settings open.

The present study fills this gap for the rotation class. Working in the setting of the generalized weighted Dirichlet-type space $\mathcal{D}^p(m_\alpha)$, $\alpha > -1$ and $1 \leq p < \infty$, we establish that the rotation induced semigroup is a strongly continuous semigroup of isometries on $\mathcal{D}^p(m_\alpha)$ and show that the differentiation operator is an isometric isomorphism from $\mathcal{D}^p_\circ(m_\alpha)$ to $L^p_a(m_\alpha)$, yielding a similarity correspondence between the corresponding semigroups. This similarity relation, combined with the reflexivity of $\mathcal{D}^p_\circ(m_\alpha)$, allows for spectral analysis of the generator and its resolvent thereby adding new results to this area of study.

2. Weighted composition semigroup on $\mathcal{D}^p(m_\alpha)$

In this section, we consider the weighted composition semigroup $(T_t)_{t\geq 0}$ given by Eq. (7) on the weighted Dirichlet-type space $\mathcal{D}^p(m_\alpha)$. It can easily be verified that $(T_t)_{t\in\mathbb{R}}$ forms a group on $\mathcal{D}^p(m_\alpha)$. The following theorem characterizes the semigroup properties of the group $(T_t)_{t\in\mathbb{R}}$.

Theorem 1. *The group $(T_t)_{t\in\mathbb{R}}$ is a strongly continuous group of isometries on $\mathcal{D}^p(m_\alpha)$ with the infinitesimal generator Γ given by*

$$\Gamma f(z) = icf(z) + ikzf'(z),$$

and domain $dom(\Gamma) = \{f \in \mathcal{D}^p(m_\alpha) : zf' \in \mathcal{D}^p(m_\alpha)\}$.

Proof. To show that $(T_t)_{t\in\mathbb{R}}$ is an isometry on $\mathcal{D}^p(m_\alpha)$, we apply a change of variables argument:

$$\begin{aligned}\|T_t f\|^p_{\mathcal{D}^p(m_\alpha)} &= |T_t f(0)|^p + \int_{\mathbb{D}} |(T_t f)'(z)|^p dm_\alpha(z) \\ &= |e^{ict} f(0)|^p + \int_{\mathbb{D}} \left| e^{ict} e^{ikt} f'(e^{ikt} z) \right|^p dm_\alpha(z).\end{aligned}$$

But $|e^{ict}| = |e^{ikt}| = 1$. Changing variables, we let $\omega = e^{ikt} z$, then $dm_\alpha(z) = dm_\alpha(\omega)$. Therefore,

$$\begin{aligned}\|T_t f\|^p_{\mathcal{D}^p(m_\alpha)} &= |f(0)|^p + \int_{\mathbb{D}} |f'(\omega)|^p dm_\alpha(\omega) \\ &= \|f\|^p_{\mathcal{D}^p(m_\alpha)},\end{aligned}$$

as desired. Next, to prove strong continuity of $(T_t)_{t\geq 0}$ on $\mathcal{D}^p(m_\alpha)$, we show that $\lim_{t\to 0^+} \|T_t f - f\|_{\mathcal{D}^p(m_\alpha)} = 0$ for all $f \in \mathcal{D}^p(m_\alpha)$. That is to say that

$$|(T_t f - f)(0)|^p + \|T_t f - f\|^p_{\mathcal{D}^p_\circ(m_\alpha)} \to 0,$$

as $t \to 0$. This is equivalent to $|(T_t f - f)(0)| \to 0$ and $\|T_t f - f\|_{\mathcal{D}^p_\circ(m_\alpha)} \to 0$ as $t \to 0$. For the first part, we have that

$$|(T_t f - f)(0)| = |T_t f(0) - f(0)| = |e^{ict} f(e^{ikt} \cdot 0) - f(0)| = |e^{ict} - 1| \cdot |f(0)| \to 0,$$

as desired.

Now, to show the integral part, we recall that analytic polynomials are dense in $\mathcal{D}^p(m_\alpha)$ (See [10, Theorem 3.3]), for an arbitrary $f \in \mathcal{D}^p(m_\alpha)$, and any $\epsilon > 0$, there exists a polynomial $g \in P$ such that $\|f - g\|_{\mathcal{D}^p_\circ(m_\alpha)} < \frac{\epsilon}{3}$. Therefore,

$$\|T_t f - f\|_{\mathcal{D}^p_\circ(m_\alpha)} \leq \|T_t f - T_t g\|_{\mathcal{D}^p_\circ(m_\alpha)} + \|T_t g - g\|_{\mathcal{D}^p_\circ(m_\alpha)} + \|g - f\|_{\mathcal{D}^p_\circ(m_\alpha)} < \epsilon.$$

Since T_t is an isometry on $\mathcal{D}^p_\circ(m_\alpha)$,

$$\|T_t f - T_t g\| = \|f - g\| < \frac{\epsilon}{3}.$$

For polynomials g, we have

$$(T_t g)'(z) = e^{ict} e^{ikt} g'(e^{ikt} z),$$

and hence

$$(T_t(g))'(z) \to g'(z),$$

pointwise on $\mathbb{D}$ as $t \to 0$. Since g' is a polynomial, there exists $M > 0$ such that

$$|g'(z)| \leq M, \quad z \in \mathbb{D}.$$

Therefore,

$$|(T_t g)'(z) - g'(z)|^p \leq (2M)^p, \quad z \in \mathbb{D}.$$

Thus by Dominated Convergence Theorem,

$$\int_{\mathbb{D}} |(T_t g)'(z) - g'(z)|^p dm_\alpha(z) \to 0, \quad \text{as} \quad t \to 0.$$

It follows that

$$\|T_t g - g\|_{\mathcal{D}^p_\circ(m_\alpha)} \to 0,$$

and hence T_t is strongly continuous on $\mathcal{D}^p(m_\alpha)$.

For the infinitesimal generator, we have for each $f \in \mathcal{D}^p(m_\alpha)$,

$$\begin{aligned}\Gamma f(z) =& \lim_{t \to 0} \frac{e^{ict} f(e^{ikt} z) - f(z)}{t} \\ =& \frac{\partial}{\partial t}(e^{ict} f(e^{ikt} z))\Big|_{t=0} \\ =& icf(z) + ikzf'(z),\end{aligned}$$

which implies that the domain $\operatorname{dom}(\Gamma) \subseteq \{f \in \mathcal{D}^p(m_\alpha) : zf' \in \mathcal{D}^p(m_\alpha)\}$. Conversely, let $f \in \mathcal{D}^p(m_\alpha)$ such that $zf' \in \mathcal{D}^p(m_\alpha)$. Then for $f \in \mathcal{D}^p(m_\alpha)$, define $F(z) = icf(z) + ikzf'(z)$. By fundamental theorem of Calculus, we have,

$$\begin{aligned}\frac{T_t f(z) - f(z)}{t} =& \frac{1}{t} \int_0^t \partial_s (T_s f(z)) ds \\ =& \frac{1}{t} \int_0^t e^{ics} \left(icf(e^{iks} z) + ike^{iks} zf'(e^{iks} z) \right) ds \\ =& \frac{1}{t} \int_0^t T_s F(z) ds.\end{aligned}$$

Since $(T_s)_{s \geq 0}$ is strongly continuous, and $F \in \mathcal{D}^p(m_\alpha)$,

$$\left\| \frac{T_t f - f}{t} - F \right\| \leq \frac{1}{t} \int_0^t \|T_s F - F\| ds \to 0 \quad \text{as} \quad t \to 0^+.$$

Thus

$$\Gamma f(z) = \lim_{t \to 0^+} \frac{T_t f - f}{t} = i(cf + kzf'),$$

hence $f \in \operatorname{dom}(\Gamma)$. Consequently, $\operatorname{dom}(\Gamma) = \{f \in \mathcal{D}^p(m_\alpha) : zf' \in \mathcal{D}^p(m_\alpha)\}$ completing the proof. $\square$

As noted in the introduction, $f \in \mathcal{D}^p_\circ(m_\alpha)$ if and only if $f' \in L^p_a(m_\alpha)$. Now, for every $f \in \mathcal{D}^p_\circ(m_\alpha)$, $T_t f \in \mathcal{D}^p_\circ(m_\alpha)$ and so

$$\begin{aligned}(T_t f)'(z) =& \left(e^{ict} f(e^{ikt} z)\right)' \\ =& e^{ict} e^{ikt} f'(e^{ikt} z) \\ =& \varphi_t'(z) T_t f'(z) \in L^p_a(m_\alpha),\end{aligned}$$

where $\varphi_t(z) = e^{ikt} z$.

Now, define $S_t : L^p_a(m_\alpha) \to L^p_a(m_\alpha)$ by

$$S_t := \varphi_t' T_t. \tag{8}$$

Then clearly $(S_t)_{t\geq 0}$ defines a semigroup on $L^p_a(m_\alpha)$. Indeed, $S_0 = e^0 T_0 = I$ and for $s, t \in \mathbb{R}$, $S_{t+s} = \varphi'_{t+s} T_{t+s} = e^{ik(t+s)} T_{t+s} = e^{ikt} e^{iks} T_t T_s = \varphi_t' T_t \varphi_s' T_s = S_t S_s$, as desired.

Before we detail the properties of the weighted composition semigroup $(S_t)_{t\geq 0}$, recall from [8, Theorem 5.1] that the multiplication operator $M_z f(z) := z f(z)$ is bounded on $L^p_a(m_\alpha)$, $1 \leq p < \infty$ and $\alpha > -1$, with range $\operatorname{ran}(M_z) = \{f \in L^p_a(m_\alpha) : f(0) = 0\}$. The operator $Qf(z) := \frac{f(z)-f(0)}{z}$ is the left inverse of M_z. Moreover, $L^p_a(m_\alpha)$ can be decomposed as $L^p_a(m_\alpha) = \operatorname{ran}(M_z^m) \oplus \operatorname{span}\{z^n : n \in \mathbb{Z}_+, n < m\}$ for every $m \in \mathbb{N}$, and $P_m = M_z^m Q^m$ is the projection of $L^p_a(m_\alpha)$ onto $\operatorname{ran}(M_z^m)$ with kernel $\operatorname{span}\{z^n : n \in \mathbb{Z}_+, n < m\}$.

Theorem 2. *Let $(S_t)_{t\geq 0}$ be the composition semigroup defined on $L^p_a(m_\alpha)$ by Eq. (8) and let G be its infinitesimal generator. Then*

1. *$(S_t)_{t\geq 0}$ is a strongly continuous semigroup of isometries on $L^p_a(m_\alpha)$.*
2. *$Gf(z) = i[(c+k)f(z) + kzf'(z)]$ with domain $dom(G) = \{f \in L^p_a(m_\alpha) : zf' \in L^p_a(m_\alpha)\}$*
3. *$\sigma(G) = \sigma_p(G) = \{i(c + (n+1)k) : n \in \mathbb{Z}_+\}$, and for each $n \in \mathbb{Z}_+$, $\ker(i(c + (n+1)k) - G) = span(z^n)$.*
4. *Let $\lambda \in \rho(G)$, and let $m \in \mathbb{Z}_+$ such that $m + \operatorname{Re}\left(\frac{c+k+\lambda i}{k}\right) > 0$. Then $\operatorname{ran}(M_z^m)$ is $R(\lambda, G)$-invariant. Moreover, for every $h \in ran(M_z^m)$,*

$$R(\lambda, G)h(z) = \frac{i}{k} z^{-(\frac{c+k+\lambda i}{k})} \int_0^z \omega^{\frac{c+k+\lambda i}{k} - 1} h(\omega) d\omega = \frac{i}{k} z^m \int_0^1 t^{m + \frac{c+k+\lambda i}{k} - 1} Q^m h(tz) dt.$$

Proof. The semigroup $(S_t)_{t\geq 0}$ acts on $L^p_a(m_\alpha)$ by

$$S_t f(z) = e^{ikt} e^{ict} f(e^{ikt} z) = e^{i(c+k)t} f(e^{ikt} z), \quad \forall f \in L^p_a(m_\alpha).$$

To prove isometry, we observe that

$$\|S_t f\|^p_{L^p_a(m_\alpha)} = \int_{\mathbb{D}} |S_t f(z)|^p dm_\alpha(z) = \int_{\mathbb{D}} |e^{i(c+k)t} f(e^{ikt} z)|^p dm_\alpha(z).$$

Since $\left|e^{i(c+k)t}\right| = 1$, and the weighted measure dm_α is rotation invariant, a change of variables $\omega = e^{ikt} z$ gives

$$\|S_t f\|^p_{L^p_a(m_\alpha)} = \int_{\mathbb{D}} |f(e^{ikt} z)|^p dm_\alpha(z) = \int_{\mathbb{D}} |f(\omega)|^p dm_\alpha(\omega) = \|f\|^p_{L^p_a(m_\alpha)}.$$

Hence $(S_t)_{t\geq 0}$ is an isometry. Moreover, $S_{-t} f(z) = e^{-i(c+k)t} f(e^{-ikt} z)$ satisfies $S_t S_{-t} = S_{-t} S_t = I$, and so S_t is surjective. Strong continuity follows from [8, Theorem 2.3].

We mimic the proof of [8, Theorem 5.1] for the remaining claims. We start by computing the infinitesimal generator using the definition.

$$\begin{aligned}Gf(z) =& \lim_{t\to 0} \frac{e^{i(c+k)t} f(e^{ikt} z) - f(z)}{t} \\ =& \frac{\partial}{\partial t}(e^{i(c+k)t} f(e^{ikt} z))\bigg|_{t=0}\end{aligned}$$

$$=i[(c+k)f(z)+kzf'(z)],$$

which implies that the domain $\mathrm{dom}(G) \subseteq \{f \in L_a^p(m_\alpha) : zf' \in L_a^p(m_\alpha)\}$. Conversely, let $f \in L_a^p(m_\alpha)$ such that $zf' \in L_a^p(m_\alpha)$. For $f \in L_a^p(m_\alpha)$, define $F(z) = i[(c+k)f(z)+kzf'(z)]$. By fundamental theorem of Calculus, we have,

$$\begin{aligned}\frac{S_t f(z) - f(z)}{t} &= \frac{1}{t}\int_0^t \partial_s(S_s f(z))ds \\ &= \frac{1}{t}\int_0^t S_s F(z) ds.\end{aligned}$$

Since $(S_s)_{s\geq 0}$ is strongly continuous, and $F \in L_a^p(m_\alpha)$,

$$\left\|\frac{S_t f - f}{t} - F\right\| \leq \frac{1}{t}\int_0^t \|S_s F - F\| ds \to 0 \quad \text{as} \quad t \to 0^+.$$

Thus

$$Gf(z) = \lim_{t\to 0^+} \frac{S_t f - f}{t} = i[(c+k)f(z)+kzf'(z)],$$

hence $f \in \mathrm{dom}(G)$. Consequently, $\mathrm{dom}(G) = \{f \in L_a^p(m_\alpha) : zf' \in L_a^p(m_\alpha)\}$ as desired.

We now determine the $\sigma_p(G)$ and show that $\sigma_p(G) = \sigma(G)$. Using the eigenvalue equation, $\lambda f = Gf$, we see that

$$\lambda f(z) = i[(c+k)f(z)+kzf'(z)].$$

Rearranging this gives

$$f'(z) = \frac{\lambda - i(c+k)}{ik}\frac{f(z)}{z}.$$

Solving this differential equation by use of the integrating factor gives us the solution as

$$f(z) = Cz^{-\frac{c+k+\lambda i}{k}},$$

for some constant $C \in \mathbb{C}$. Since f must be analytic on $\mathbb{D}$, it follows that $-\frac{c+k+\lambda i}{k} = n$ for some $n \in \mathbb{Z}_+$. Thus

$$\lambda = i[c+(n+1)k], \quad n \in \mathbb{Z}_+,$$

and so,

$$\sigma_p(G) = \{i(c+(n+1)k) : n \in \mathbb{Z}_+\}.$$

Moreover, for $\lambda_n = i[c+(n+1)k]$, the corresponding eigenfunctions are $f(z) = Cz^n$, and hence $\ker(i(c+(n+1)k) - G) = \mathrm{span}(z^n)$.

Since (S_t) is a surjective isometry, then $\sigma(S_t) \subseteq \partial\mathbb{D}$. By spectral mapping theorem for strongly continuous semigroups [1, Chapter 2], we have that $e^{t\sigma(G)} \subseteq \sigma(S_t) \subseteq \partial\mathbb{D}$ and so $\sigma(G) \subseteq i\mathbb{R}$. We now construct the bounded resolvent operator $R(\lambda, G)$ for all $\lambda \notin \sigma_p(G)$. This will simultaneously establish that $\sigma_p(G) = \sigma(G)$. For $\lambda \notin \sigma_p(G)$, we let $h \in L_a^p(m_\alpha)$ be arbitrary. Then the equation $(\lambda - G)f(z) = h(z)$ is equivalent to

$$f'(z) + \frac{c+k+\lambda i}{kz}f(z) = \frac{i}{kz}h(z),$$

or

$$(z^{\frac{c+k+\lambda i}{k}}f(z))' = \frac{i}{k}z^{\frac{c+k+\lambda i}{k}-1}h(z).$$

Integrating from 0 to z;

$$z^{\frac{c+k+\lambda i}{k}}f(z) = \frac{i}{k}\int_0^z \omega^{\frac{c+k+\lambda i}{k}-1}h(\omega)d\omega,$$

which solves to

$$f(z) = z^{-(\frac{c+k+\lambda i}{k})}\frac{i}{k}\int_0^z \omega^{\frac{c+k+\lambda i}{k}-1}h(\omega)d\omega. \tag{9}$$

We now show that (9) defines the resolvent operator

$$R(\lambda, G)h(z) = z^{-(\frac{c+k+\lambda i}{k})} \frac{i}{k} \int_0^z \omega^{\frac{c+k+\lambda i}{k}-1} h(\omega) d\omega, \tag{10}$$

by first differentiating the Eq. (10) to give

$$z(R(\lambda, G)h)'(z) = -\left(\frac{c+k+\lambda i}{k}\right) R(\lambda, G)h(z) + \frac{i}{k} h(z),$$

and since both terms on the right hand side belong to $L_a^p(m_\alpha)$, $z(R(\lambda, G)h)' \in L_a^p(m_\alpha)$. Therefore, $R(\lambda, G)h \in \mathrm{dom}(G)$, hence is the resolvent operator. To confirm that it is bounded, we choose $m \in \mathbb{Z}_+$ such that $m + \mathrm{Re}\left(\frac{c+k+\lambda i}{k}\right) > 0$, and define $h = M_z^m Q^m h = z^m Q^m h$, $h \in \mathrm{ran}(M_z^m)$. Substituting in Eq. (10)

$$R(\lambda, G)h(z) = \frac{i}{k} z^{-(\frac{c+k+\lambda i}{k})} \int_0^1 \omega^{\frac{c+k+\lambda i}{k}+m-1} (Q^m h)(\omega) d\omega.$$

By change of variables, we let $\omega = tz$, then $d\omega = zdt$ and obtain

$$R(\lambda, G)h(z) = \frac{i}{k} z^m \int_0^1 t^{\frac{c+k+\lambda i}{k}+m-1} (Q^m h)(tz) dt,$$

which can also be expressed as

$$R(\lambda, G)h(z) = \frac{i}{k} M_z^m \int_0^1 t^{\frac{c+k+\lambda i}{k}+m-1} (Q^m h)(t\cdot) dt.$$

By Minkowski's integral inequality,

$$\|R(\lambda, G)h\| \leq \frac{\|M_z^m\|}{|k|} \int_0^1 t^{\mathrm{Re}(\frac{c+k+\lambda i}{k})+m-1} \|(Q^m h)(t\cdot)\| dt. \tag{11}$$

Since Q^m is bounded,

$$\|(Q^m h)(t\cdot)\| \leq \|Q^m h\| \leq \|Q^m\| \|h\|,$$

and therefore

$$\|R(\lambda, G)h\| \leq \frac{\|M_z^m\| \|Q^m\|}{|k|} \left(\int_0^1 t^{\mathrm{Re}(\frac{c+k+\lambda i}{k})+m-1} dt\right) \|h\|.$$

But $\mathrm{Re}(\frac{c+k+\lambda i}{k}) > -m$, so

$$\int_0^1 t^{\mathrm{Re}(\frac{c+k+\lambda i}{k})+m-1} dt = \frac{1}{\mathrm{Re}(\frac{c+k+\lambda i}{k}) + m} < \infty.$$

Consequently,

$$\|R(\lambda, G)h\| \leq C_\lambda \|h\|,$$

where $C_\lambda = \frac{\|M_z^m\| \|Q^m\|}{|k|(\mathrm{Re}(\frac{c+k+\lambda i}{k})+m)}$. Thus $R(\lambda, G)$ is bounded on $\mathrm{ran}(M_z^m)$. Since $P_m = M_z^m Q^m$ is a bounded projection, $L_a^p(m_\alpha) = \ker(P_m) \oplus \mathrm{ran}(M_z^m)$. For arbitrary $h \in L_a^p(m_\alpha)$, decompose $h = \sum_{n<m} a_n z^n + P_m h(z)$ and apply linearity to see that

$$R(\lambda, G)h(z) = \sum_{n \in \mathbb{Z}_+, n<m} a_n \frac{i z^n}{c + \lambda i + (n+1)k} + \frac{i}{k} z^m \int_0^1 t^{m+\frac{c+k+\lambda i}{k}} Q^m h(tz) dt.$$

The first sum is bounded since $\lambda \notin \sigma_p(G)$ ensures $c + \lambda i + (n+1)k \neq 0$ $\forall n < m$. The second term is bounded following the Minkowski's integral inequality in (11). Therefore both parts are bounded, so $R(\lambda, G) \in \mathcal{L}(L_\alpha^p(\mathbb{D}))$. Since $R(\lambda, G)$ is bounded for all $\lambda \in i\mathbb{R}$, $\lambda \notin \sigma_p(G)$, it follows that

$$\sigma(G) \subseteq \sigma_p(G).$$

Consequently,
$$\sigma(G) = \sigma_p(G) = \{i(c + (n+1)k) : n \in \mathbb{Z}_+\}.$$
This completes the proof. □

Now, we consider the subspace $\mathcal{D}^p_\circ(m_\alpha) = \{f \in \mathcal{D}^p(m_\alpha) : f(0) = 0\}$ of $\mathcal{D}^p(m_\alpha)$. Then the differentiation operator D given by $Df = f'$ for all $f \in \mathcal{D}^p_\circ(m_\alpha)$ is an isometric isomorphism from $\mathcal{D}^p_\circ(m_\alpha)$ onto $L^p_a(m_\alpha)$ whose inverse D^{-1} is the integration operator $D^{-1}g(z) = \int_0^z g(\xi)\,d\xi$. We readily see that
$$\|Df\|_{L^p_a(m_\alpha)} = \|f'\|_{L^p_a(m_\alpha)} = \|f\|_{\mathcal{D}^p_\circ(m_\alpha)}.$$

Therefore the actions of the maps D, D^{-1} and S_t can be summarized as follows:
$$\mathcal{D}^p_\circ(m_\alpha) \xrightarrow{D} L^p_a(m_\alpha) \xrightarrow{S_t} L^p_a(m_\alpha) \xrightarrow{D^{-1}} \mathcal{D}^p_\circ(m_\alpha).$$

Thus, it follows that $T_t = D^{-1}S_tD$, and so $S_t = DT_tD^{-1}$ implying that $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are similar semigroups. We observe that if we let $g \in L^p_a(m_\alpha)$, then $f = D^{-1}g$ belongs to $\mathcal{D}^p_\circ(m_\alpha)$ and hence
$$DT_tD^{-1}g = DT_tf = (T_tf)' = \varphi_t'T_tf' = S_tf' = S_tDf = S_tg,$$
as claimed.

By applying the theory of similar semigroups highlighted in the introduction, we detail both the semigroup and spectral properties of the semigroup $(T_t)_{t\geq 0}$ on the weighted Dirichlet-type space $\mathcal{D}^p_\circ(m_\alpha)$ in the following theorem.

Theorem 3. *Let $(T_t)_{t\geq 0} \subseteq \mathcal{L}(\mathcal{D}^p_\circ(m_\alpha))$ be the composition semigroup in Eq. (7) and Γ be its infinitesimal generator. Then the following hold:*

1. *$\Gamma h(z) = i\,(ch(z) + kzh'(z))$ with $dom(\Gamma) = \{h \in \mathcal{D}^p_\circ(m_\alpha) : h' \in dom(G)\}$.*
2. *$\sigma_p(\Gamma) = \sigma(\Gamma) = \{i(c + (n+1)k) : n \in \mathbb{Z}_+\}$, and for each $n \in \mathbb{Z}_+$, $\ker(i(c + (n+1)k) - \Gamma) = span(z^{n+1})$.*
3. *Let $\lambda \in \rho(\Gamma)$. If $h \in \mathcal{D}^p_\circ(m_\alpha)$, then*
$$R(\lambda, \Gamma)h(z) = \frac{i}{k} \cdot z^{1-\frac{c+k+\lambda i}{k}} \int_0^z \omega^{\frac{c+k+\lambda i}{k}-2} h(\omega)\,d\omega.$$
4. *The spectrum $\sigma(R(\lambda, \Gamma))$ is a set of discrete points lying on the circle centred at $\frac{1}{2\,\mathrm{Re}(\lambda)}$ with radius $\frac{1}{|2\,\mathrm{Re}(\lambda)|}$, moreover 0, is the only accumulation point of the spectrum.*

Proof. By the Similarity theory of strongly continuous groups, if G is the infinitesimal generator of $(S_t)_{t\geq 0}$, then the corresponding infinitesimal generator of $(T_t)_{t\geq 0}$ is given by
$$\Gamma = D^{-1}GD,$$
with domain
$$\mathrm{dom}(\Gamma) = D^{-1}\mathrm{dom}(G).$$

Now, suppose $f' \in L^p_a(m_\alpha)$. Then $f \in \mathrm{dom}(G)$ and setting $h = D^{-1}f$, then $h \in \mathrm{dom}(\Gamma)$ with $f = Dh$. Then we have
$$\begin{aligned}\Gamma h(z) =& D^{-1}GDh(z)\\ =& D^{-1}\Big(i((c+k)h'(z) + kzh''(z)\Big)\\ =& i\left((c+k)h(z) + k(zh'(z) - h(z))\right)\\ =& i(ch(z) + kzh'(z)),\end{aligned}$$

as desired. On the other hand,

$$h \in \mathrm{dom}(\Gamma) \Leftrightarrow Dh \in \mathrm{dom}(G) \Leftrightarrow h' \in \mathrm{dom}(G).$$

Therefore, the $\mathrm{dom}(\Gamma) = \{h \in \mathcal{D}^p_\circ(m_\alpha) : h' \in \mathrm{dom}(G)\}$. This proves Claim 1. Since S_t and T_t are similar, it follows from Theorem 2 that

$$\sigma_p(\Gamma) = \sigma(\Gamma) = \{i((n+1)k + c) : n \in \mathbb{Z}_+\}.$$

Since $\ker(i((n+1)k+c) - G) = \mathrm{span}(z^n)$, we obtain $\ker(i((n+1)k+c) - \Gamma) = D^{-1}\mathrm{span}(z^n)$. But $D^{-1}(z^n) = \frac{z^{n+1}}{n+1}$. Hence

$$\ker(i((n+1)k+c) - \Gamma) = \mathrm{span}(z^{n+1}), \quad \forall n \in \mathbb{Z}_+.$$

For the third claim, we recall that for similar semigroups, $\rho(\Gamma) = \rho(G)$, and

$$R(\lambda, \Gamma)h(z) = D^{-1}R(\lambda, G)Dh(z) = D^{-1}R(\lambda, G)h'(z).$$

Therefore,

$$\begin{aligned}
R(\lambda,\Gamma)h(z) &= D^{-1}\left(\frac{i}{k}z^{-(\frac{c+k+\lambda i}{k})}\int_0^z \omega^{\frac{c+k+\lambda i}{k}-1}Dh(\omega)d\omega\right)\\
&= \int_0^z \frac{i}{k}u^{-(\frac{c+k+\lambda i}{k})}\left(\int_0^u \omega^{\frac{c+k+\lambda i}{k}-1}h'(\omega)d\omega\right)du\\
&= \frac{i}{k(-(\frac{c+k+\lambda i}{k})+1)}\left[\left(h(z) - (\frac{c+k+\lambda i}{k}-1)z^{-(\frac{c+k+\lambda i}{k})+1}\int_0^z \omega^{\frac{c+k+\lambda i}{k}-2}h(\omega)\,d\omega\right) - h(z)\right]\\
&= \frac{i}{k}z^{1-\frac{c+k+\lambda i}{k}}\int_0^z \omega^{\frac{c+k+\lambda i}{k}-2}h(\omega)\,d\omega.
\end{aligned}$$

Next we now apply spectral mapping theorem for resolvents to obtain the spectrum of the resolvent operator. For all $\lambda \in \rho(\Gamma)$ with $\mathrm{Re}(\lambda) \neq 0$, we have that

$$\begin{aligned}
\sigma(R(\lambda,\Gamma)) &= \left\{\frac{1}{\lambda - z} : z \in \sigma(\Gamma)\right\} \cup \{0\}\\
&= \left\{\frac{1}{\lambda - i(c+(n+1)k)} : n \in \mathbb{Z}_+\right\} \cup \{0\}.
\end{aligned}$$

If we let $\lambda = a + ib$ with $a = \mathrm{Re}\,\lambda \neq 0$ and define

$$\omega_n = \frac{1}{\lambda - i(c+(n+1)k)}, \quad n \in \mathbb{Z}_+.$$

Fix $n \in \mathbb{Z}_+$ and let $\omega = \omega_n$, then we have

$$\omega_n = \frac{1}{a + i(b - c - (n+1)k)}. \tag{12}$$

Setting $y_n = b - c - (n+1)k$, we obtain

$$\omega = \frac{1}{a+iy_n} = \frac{a - iy_n}{a^2 + y_n^2}.$$

Hence

$$\mathrm{Re}(\omega) = \frac{a}{a^2+y_n^2}, \quad \text{and} \quad |\omega|^2 = \frac{1}{a^2+y_n^2},$$

and so, $\mathrm{Re}(\omega) = a|\omega|^2$. If $\omega = x + iy$, then $x = a(x^2+y^2)$, or equivalently, $x^2 + y^2 - \frac{x}{a} = 0$. Completing the square,

$$(x - \frac{1}{2a})^2 + y^2 = \frac{1}{4a^2}.$$

Therefore
$$\left|\omega - \frac{1}{2a}\right| = \frac{1}{2|a|}.$$
Since $a = \operatorname{Re}(\lambda)$, and n was arbitrary, it follows that
$$\left|\omega_n - \frac{1}{2\operatorname{Re}(\lambda)}\right| = \frac{1}{2|\operatorname{Re}(\lambda)|},$$
for every $n \in \mathbb{Z}_+$. This implies that the spectrum is a set of discrete points that lie on the circle
$$\left\{\omega \in \mathbb{C} : \left|\omega - \frac{1}{2\operatorname{Re}(\lambda)}\right| = \frac{1}{2|\operatorname{Re}(\lambda)|}\right\}, \tag{13}$$
together with the point 0, which is its only accumulation point. □

Remark 1. We verify the resolvent formula on monomials. For $h(z) = z^{n+1}$ with $n \in \mathbb{Z}_+$, we compute
$$\begin{aligned} R(\lambda,\Gamma)z^{n+1} =& \frac{i}{k} \cdot z^{1-\frac{c+k+\lambda i}{k}} \int_0^z \omega^{\frac{c+k+\lambda i}{k}-2} \cdot \omega^{n+1}\, d\omega \\ =& \frac{i}{k} z^{1-\frac{c+k+\lambda i}{k}} \int_0^z \omega^{\frac{c+k+\lambda i}{k}+n-1}\, d\omega \\ =& \frac{i}{k} z^{1-\frac{c+k+\lambda i}{k}} \cdot \frac{z^{\frac{c+k+\lambda i}{k}+n}}{\frac{c+k+\lambda i}{k}+n} \\ =& \frac{i}{k(\frac{c+k+\lambda i}{k}+n)} z^{n+1}. \end{aligned}$$
Hence
$$R(\lambda,\Gamma)z^{n+1} = \frac{1}{\lambda - i(c+(n+1)k)} z^{n+1},$$
which confirms consistency between the integral formula and spectral computations in Theorem 3.

2.1. Adjoint and duality

For $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, the duality $(\mathcal{D}^p_\circ(m_\alpha))^* \approx \mathcal{D}^q_\circ(m_\alpha)$ holds under conjugate linear integral pairing given by: For $f \in \mathcal{D}^p_\circ(m_\alpha)$, $g \in \mathcal{D}^q_\circ(m_\alpha)$,
$$\langle f, g\rangle = \int_{\mathbb{D}} f'(z)\overline{g'(z)} dm_\alpha(z).$$

Consider our composition semigroup T_t on $\mathcal{D}^p_\circ(m_\alpha)$ so that T_t^* acts on $\mathcal{D}^q_\circ(m_\alpha)$. Then for every $f \in \mathcal{D}^p_\circ(m_\alpha)$, $g \in \mathcal{D}^q_\circ(m_\alpha)$, we have by a change of variables argument,
$$\begin{aligned} \langle T_t f, g\rangle &= \int_{\mathbb{D}} (T_t f)'(z)\overline{g'(z)}\, dm_\alpha(z) \\ &= \int_{\mathbb{D}} e^{ict} e^{ikt} f'(e^{ikt} z)\overline{g'(z)}(1-|z|^2)^\alpha dm(z) \\ &= \int_{\mathbb{D}} e^{ict} e^{ikt} f'(\omega)\overline{g'(\omega e^{-ikt})}(1-|\omega|^2)^\alpha dm(\omega) \\ &= \int_{\mathbb{D}} f'(\omega)\overline{e^{-ict}e^{-ikt} g'(\omega e^{-ikt})}\, dm_\alpha(\omega) \\ &= \int_{\mathbb{D}} f'(\omega)\overline{T_{-t} g(\omega)}\, dm_\alpha(\omega) \\ &= \langle f, T_{-t} g\rangle. \end{aligned}$$

Therefore $T_t^* = T_{-t}$. We can now give a consequence of the Theorem 3 above.

Corollary 1. *For $1 < p < \infty$, if $\lambda \in \rho(\Gamma)$ and $R_\lambda := R(\lambda, \Gamma) \in \mathcal{L}(\mathcal{D}^p_\circ(m_\alpha))$, then*

$$R^*_\lambda = -R_{-\overline{\lambda}}, \quad R^*_\lambda \in \mathcal{L}(\mathcal{D}^q_\circ(m_\alpha))$$

Proof. Since $\mathcal{D}^p_\circ(m_\alpha)$ is reflexive for $1 < p < \infty$, the adjoint group $(T^*_t)_{t\in\mathbb{R}}$ is a strongly continuous group. The generator of (T^*_t) is Γ^*, while the generator of (T_{-t}) is $-\Gamma$. Therefore, $\Gamma^* = -\Gamma$. Let $\lambda \in \rho(\Gamma)$, then

$$R^*_\lambda := R(\lambda, \Gamma)^* = R(\overline{\lambda}, \Gamma^*) = R(\overline{\lambda}, -\Gamma) = -R(-\overline{\lambda}, \Gamma) := -R_{-\overline{\lambda}}.$$

This completes the proof. □

References

[1] Engel, K.-J., & Nagel, R. (2006). *A Short Course on Operator Semigroups*. Springer.

[2] Berkson, E., & Porta, H. (1978). Semigroups of analytic functions and composition operators. *Michigan Mathematical Journal, 25*, 101–115.

[3] Siskakis, A. G. (1987). Semigroups of composition operators in Bergman spaces. *Bulletin of the Australian Mathematical Society, 35*(3), 397–406.

[4] Siskakis, A. G. (1996). Semigroups of composition operators on the Dirichlet space. *Results in Mathematics, 30*(1–2), 165–173.

[5] Cowen, C. C. (1984). Subnormality of the Cesàro operator and a semigroup of composition operators. *Indiana University Mathematics Journal, 33*(2), 305–318.

[6] Siskakis, A. G. (1987). Composition semigroups and the Cesàro operator on H^p. *Journal of the London Mathematical Society, 36*(1), 153–164.

[7] Arvanitidis, A. G., & Siskakis, A. G. (2013). Cesàro operators on the Hardy spaces of the half-plane. *Canadian Mathematical Bulletin, 56*(2), 229–240.

[8] Ballamoole, S., Bonyo, J. O., Miller, T. L., & Miller, V. G. (2016). Cesàro-like operators on the Hardy and Bergman spaces of the half plane. *Complex Analysis and Operator Theory, 10*(1), 187–203.

[9] Agwang, M. O., & Bonyo, J. O. (2020). Spectra of composition groups on the weighted Dirichlet space of the upper half-plane. *Acta Mathematica Scientia, 40*(6), 1739–1752.

[10] Abkar, A. (2022). Density of polynomials in certain weighted Dirichlet-type spaces. *Journal of Mathematical Extension, 16*(1), 1–11.