Open Journal of Mathematical Analysis

PSR Press

*Article*

# Some Contributions to Multiple-Variable Cyclic Inequalities

**Christophe Chesneau**

Department of Mathematics, University of Caen-Normandie, UFR des Sciences – Campus 2, Caen, France; christophe.chesneau@gmail.com

Received: 8 August 2026; Accepted: 26 October 2026; Published: 15 November 2026.

**Abstract:** In this paper, we propose two general theorems concerning inequalities in several variables, subject to flexible parametric constraints: the first presents a new sum-type inequality, while the second introduces its product-type counterpart. These results find natural applications in the study of cyclic inequalities. A wide range of examples is provided to illustrate the applicability of our approach, complemented by four additional theorems that further expand the analytical framework.

**Keywords:** cyclic inequalities, Nesbitt inequality, Jensen inequality, Young inequality

**MSC:** 26D15.

## 1. Introduction

Cyclic inequalities constitute a class of mathematical expressions characterized by invariance under cyclic permutations of their variables. These structures naturally arise in algebraic and functional analysis, where determining precise bounds requires convexity, majorization, and classical inequalities.

A famous example within this domain is the Nesbitt inequality. Its classical three-variable formulation is stated below. Let $x, y, z > 0$. Then we have

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2}.$$

Equality holds if and only if $x = y = z$. This result generalizes to an arbitrary number of variables, as stated below. Let $n \in \mathbb{N} \setminus \{0, 1\}$ and $x_1, x_2, \ldots, x_n > 0$. Then we have

$$\sum_{i=1}^{n} \frac{x_i}{\left(\sum_{j=1}^{n} x_j\right) - x_i} \geq \frac{n}{n-1}.$$

Equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

For background on the Nesbitt inequality and its subsequent developments, see the classical formulation attributed to A. M. Nesbitt [1], the general methodology presented by M. O. Drîmbe [2], and treatments of cyclic systems by V. Cîrtoaje [3]. Specific refinements and generalizations are developed by M. Bencze and C. Zhao [4], M. Bencze and O. T. Pop [5], and Fuhua Wei and Shanhe Wu [6], while related parametric families of cyclic inequalities are explored by C. Chesneau [7]. Despite their significance, results on multiple-variable cyclic inequalities incorporating several adjustable parameters and functions remain relatively scarce.

In this paper, we establish two general theorems concerning sum-type and product-type inequalities involving multiple variables.

The first provides a sum-type inequality, governed by two adjustable parameters and quantities that satisfy a global sum constraint involving the variables. The proof relies on refined algebraic decompositions and two specific applications of the Young (multiplicative) inequality (see [8]). Applications are presented, primarily within the context of cyclic inequalities.

The second serves as its natural product-type counterpart; it is derived using an integration technique.

To illustrate the versatility of these general inequalities, we deduce several propositions concerning well-defined structural configurations, including arithmetic, geometric, and harmonic means, as well as cyclic neighborhoods of elements.

Finally, we prove and discuss four complementary theorems that extend the parameter ranges, incorporate convex functions, determine a precise upper bound, and generalize this bound using concave functions.

The remainder of this paper is organized as follows: §2 details our two core theorems. §3 presents some applications. §4 investigates the four complementary theorems. Finally, §5 provides concluding remarks.

## 2. Main Contributions

### 2.1. A General Sum Inequality Theorem

The result below presents a multiple-variable sum-type inequality, governed by specific parameter constraints and conditions for equality.

**Theorem 1.** *Let $n \in \mathbb{N} \setminus \{0,1\}$, $x_1, x_2, \ldots, x_n > 0$, and $R_1, R_2, \ldots, R_n > 0$ such that*

$$\sum_{i=1}^{n} R_i \leq \sum_{i=1}^{n} x_i. \tag{1}$$

*Let $\lambda \geq 0$ and $p$ such that*

$$1 \leq p < 2 + \lambda.$$

*Then the following n-variable sum inequality holds:*

$$\sum_{i=1}^{n} \frac{x_i^p}{x_i^{p-1} + \lambda R_i^{p-1}} \geq \frac{1}{1+\lambda} \sum_{i=1}^{n} x_i.$$

*Equality holds if and only if $p = 1$, $\lambda = 0$, or $R_i = x_i$ for every $i = 1, 2, \ldots, n$.*

**Proof.** For $\lambda = 0$, we have

$$\sum_{i=1}^{n} \frac{x_i^p}{x_i^{p-1} + \lambda R_i^{p-1}} = \sum_{i=1}^{n} \frac{x_i^p}{x_i^{p-1}} = \sum_{i=1}^{n} x_i = \frac{1}{1+\lambda} \sum_{i=1}^{n} x_i.$$

Similarly, for $p = 1$, we obtain

$$\sum_{i=1}^{n} \frac{x_i^p}{x_i^{p-1} + \lambda R_i^{p-1}} = \sum_{i=1}^{n} \frac{x_i}{1+\lambda} = \frac{1}{1+\lambda} \sum_{i=1}^{n} x_i.$$

Thus, equality holds in both cases.

In what follows, we restrict our attention to the case $\lambda > 0$ and $1 < p < 2 + \lambda$.

A suitable decomposition of the ratio gives

$$\begin{aligned} \frac{x_i^p}{x_i^{p-1} + \lambda R_i^{p-1}} &= \frac{x_i \left( x_i^{p-1} + \lambda R_i^{p-1} \right) - \lambda x_i R_i^{p-1}}{x_i^{p-1} + \lambda R_i^{p-1}} \\ &= x_i - \lambda \frac{x_i R_i^{p-1}}{x_i^{p-1} + \lambda R_i^{p-1}}. \end{aligned}$$

We now estimate the denominator by applying the multiplicative form of the Young inequality.

For any $A, B, \alpha, \beta > 0$ with $\alpha + \beta = 1$, the Young inequality states that

$$\alpha A + \beta B \geq A^{\alpha} B^{\beta}. \tag{2}$$

Setting $A = x_i^{p-1}$, $B = R_i^{p-1}$, $\alpha = 1/(1+\lambda)$, and $\beta = \lambda/(1+\lambda)$, we get

$$\frac{1}{1+\lambda}x_i^{p-1} + \frac{\lambda}{1+\lambda}R_i^{p-1} \geq \left(x_i^{p-1}\right)^{1/(1+\lambda)} \left(R_i^{p-1}\right)^{\lambda/(1+\lambda)} = x_i^{(p-1)/(1+\lambda)} R_i^{\lambda(p-1)/(1+\lambda)},$$

so that

$$x_i^{p-1} + \lambda R_i^{p-1} \geq (1+\lambda) x_i^{(p-1)/(1+\lambda)} R_i^{\lambda(p-1)/(1+\lambda)}.$$

Therefore, we have

$$\begin{aligned} x_i - \lambda \frac{x_i R_i^{p-1}}{x_i^{p-1} + \lambda R_i^{p-1}} &\geq x_i - \lambda \frac{x_i R_i^{p-1}}{(1+\lambda) x_i^{(p-1)/(1+\lambda)} R_i^{\lambda(p-1)/(1+\lambda)}} \\ &= x_i - \frac{\lambda}{1+\lambda} x_i^{\gamma} R_i^{\tau}, \end{aligned}$$

where

$$\gamma = 1 - \frac{p-1}{1+\lambda} = \frac{2+\lambda-p}{1+\lambda},$$

and

$$\tau = (p-1) - \frac{\lambda(p-1)}{1+\lambda} = \frac{p-1}{1+\lambda}.$$

Since $1 < p < 2+\lambda$, we have $\gamma, \tau > 0$ with

$$\gamma + \tau = \frac{2+\lambda-p}{1+\lambda} + \frac{p-1}{1+\lambda} = \frac{1+\lambda}{1+\lambda} = 1.$$

Another application of the Young inequality in Eq. (2) with $A = x_i$, $B = R_i$, $\alpha = \gamma$ and $\beta = \tau$ yields

$$x_i^{\gamma} R_i^{\tau} \leq \gamma x_i + \tau R_i.$$

Combining the inequalities above, we obtain

$$\frac{x_i^p}{x_i^{p-1} + \lambda R_i^{p-1}} \geq x_i - \frac{\lambda}{1+\lambda}(\gamma x_i + \tau R_i).$$

Summing both sides and using Eq. (1) and $\gamma + \tau = 1$, we get

$$\begin{aligned} \sum_{i=1}^{n} \frac{x_i^p}{x_i^{p-1} + \lambda R_i^{p-1}} &\geq \sum_{i=1}^{n} \left( x_i - \frac{\lambda}{1+\lambda}(\gamma x_i + \tau R_i) \right) \\ &= \sum_{i=1}^{n} x_i - \frac{\lambda}{1+\lambda} \left( \gamma \sum_{i=1}^{n} x_i + \tau \sum_{i=1}^{n} R_i \right) \\ &\geq \sum_{i=1}^{n} x_i - \frac{\lambda}{1+\lambda} \left( \gamma \sum_{i=1}^{n} x_i + \tau \sum_{i=1}^{n} x_i \right) \\ &= \left( 1 - \frac{\lambda}{1+\lambda}(\gamma+\tau) \right) \sum_{i=1}^{n} x_i = \left( 1 - \frac{\lambda}{1+\lambda} \right) \sum_{i=1}^{n} x_i \\ &= \frac{1}{1+\lambda} \sum_{i=1}^{n} x_i. \end{aligned}$$

The desired inequality is proved.

Furthermore, equality holds if and only if equality is attained in Eq. (1) (that is, $\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} x_i$) and in each application of the Young inequality (i.e., when $A = B$ in the framework of Eq. (2)). Consequently, this forces $R_i = x_i$ for every $i = 1, 2, \ldots, n$. This completes the proof. □

**Remark 1.** The parameter restriction $1 \leq p < 2+\lambda$ comes from theoretical considerations. When $p = 1$ or $\lambda = 0$, the inequality reduces to an equality, showing that the nontrivial content of the theorem concerns the parameter range $1 < p < 2+\lambda$. The strict upper bound $2+\lambda$ guarantees that the exponent $\gamma = (2+\lambda-p)/(1+\lambda)$ remains strictly positive. This is critical for the validity of the second application of the Young inequality.

**Remark 2.** The global constraint in Eq. (1) acts as a mild condition rather than a strict term-by-term restriction. Thanks to this flexibility, Theorem 1 provides a unifying multiple-variable inequality for cyclic sums. To support this claim, Table 1 presents diverse examples satisfying this condition, including cyclic shifts, global and local means, geometric-harmonic means, and hybrid configurations. Each row specifies a possible choice of the auxiliary quantities and states the corresponding relation between their total and the sum of the original variables. In the logarithmic-mean row, the value at $x_{i+1} = x_i$ is defined by continuity as $R_i = x_i$. Thus, this entry remains well defined when adjacent variables coincide.

**Remark 3.** Note that the lower bound in Theorem 1 does not depend on $p$, which is a consequence of the condition $1 \leq p < 2+\lambda$. In future work, it would be of interest to examine the regime $p \geq 2+\lambda$ to determine how $p$ affects the lower bound.

**Remark 4.** For the special case $p = 2$, Theorem 1 can be derived more directly using the Titu lemma. Indeed, for $\lambda \geq 0$, applying this lemma and Eq. (1) yields

$$\begin{aligned}\sum_{i=1}^{n} \frac{x_i^2}{x_i+\lambda R_i} &\geq \frac{\left(\sum\limits_{i=1}^{n} x_i\right)^2}{\sum\limits_{i=1}^{n}(x_i+\lambda R_i)} = \frac{\left(\sum\limits_{i=1}^{n} x_i\right)^2}{\sum\limits_{i=1}^{n} x_i + \lambda \sum\limits_{i=1}^{n} R_i} \\ &\geq \frac{\left(\sum\limits_{i=1}^{n} x_i\right)^2}{\sum\limits_{i=1}^{n} x_i + \lambda \sum\limits_{i=1}^{n} x_i} = \frac{\left(\sum\limits_{i=1}^{n} x_i\right)^2}{(1+\lambda)\sum\limits_{i=1}^{n} x_i} = \frac{1}{1+\lambda}\sum_{i=1}^{n} x_i.\end{aligned}$$

This approach differs from the proof based on the Young inequality presented above. However, extending this technique to the case of a general exponent $p$, as it appears in Theorem 1, does not appear to be straightforward. Nevertheless, we do not rule out the possibility that this method, or others related to it, could serve as alternative proof strategies.

**Remark 5.** The parameter $\lambda$ can be specialized in various ways. For instance, applying Theorem 1 with $\lambda = k$ for $k \in \mathbb{N}$, and $1 \leq p < 2$, squaring both sides of the resulting inequality, and summing with respect to $k$ over $\mathbb{N}$ gives

$$\sum_{k=0}^{\infty}\left(\sum_{i=1}^{n}\frac{x_i^p}{x_i^{p-1}+kR_i^{p-1}}\right)^2 \geq \left(\sum_{k=0}^{\infty}\frac{1}{(1+k)^2}\right)\left(\sum_{i=1}^{n} x_i\right)^2 = \frac{\pi^2}{6}\left(\sum_{i=1}^{n} x_i\right)^2.$$

To the best of our knowledge, this general inequality has not previously been recorded in this form, and it contains the explicit constant factor $\pi^2/6$.

Similarly, an integration technique with respect to $\lambda$ is developed in §2.2.

**Remark 6.** We highlight a generalized Nesbitt-type inequality derived from Theorem 1. Choosing $R_1, R_2, \ldots, R_n$ as the Nesbitt-type means given in the fourth entry of Table 1 and setting $\lambda = (n-1)^{p-1}$ with $1 \leq p < 2+(n-1)^{p-1}$, we obtain

$$\sum_{i=1}^{n}\frac{x_i^p}{x_i^{p-1}+\left(\left(\sum\limits_{j=1}^{n} x_j\right)-x_i\right)^{p-1}} \geq \frac{1}{1+(n-1)^{p-1}}\sum_{i=1}^{n} x_i.$$

Further illustrative examples are detailed in §3 along with precise equality conditions.

**Table 1.** Examples of $R_1, R_2, \ldots, R_n$ satisfying Eq. (1). The cyclic conventions $x_0 = x_n$ and $x_{n+1} = x_1$ are assumed throughout the table.

| Classification | $R_i$ | Property |
|---|---|---|
| Forward cyclic shift | $R_i = x_{i+1}$ | $\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} x_i$ |
| Global arithmetic mean | $R_i = \frac{1}{n}\sum_{j=1}^{n} x_j$ | $\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} x_i$ |
| Symmetric adjacent mean | $R_i = \frac{x_{i-1}+x_{i+1}}{2}$ | $\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} x_i$ |
| Nesbitt-type mean | $R_i = \frac{1}{n-1}\left(\left(\sum_{j=1}^{n} x_j\right) - x_i\right)$ | $\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} x_i$ |
| Adjacent geometric mean | $R_i = \sqrt{x_i x_{i+1}}$ | $\sum_{i=1}^{n} R_i \leq \sum_{i=1}^{n} x_i$ |
| Second-order geometric mean | $R_i = \sqrt{x_{i-1} x_{i+1}}$ | $\sum_{i=1}^{n} R_i \leq \sum_{i=1}^{n} x_i$ |
| Three-term cyclic product | $R_i = (x_{i-1} x_i x_{i+1})^{1/3}$ | $\sum_{i=1}^{n} R_i \leq \sum_{i=1}^{n} x_i$ |
| Global geometric mean | $R_i = \left(\prod_{j=1}^{n} x_j\right)^{1/n}$ | $\sum_{i=1}^{n} R_i \leq \sum_{i=1}^{n} x_i$ |
| Global harmonic mean | $R_i = n\left(\sum_{j=1}^{n} \frac{1}{x_j}\right)^{-1}$ | $\sum_{i=1}^{n} R_i \leq \sum_{i=1}^{n} x_i$ |
| Adjacent harmonic mean | $R_i = \frac{2x_i x_{i+1}}{x_i + x_{i+1}}$ | $\sum_{i=1}^{n} R_i \leq \sum_{i=1}^{n} x_i$ |
| Harmonic-arithmetic hybrid | $R_i = \frac{x_i x_{i+1}}{x_{i-1} + x_{i+1}}$ | $\sum_{i=1}^{n} R_i \leq \sum_{i=1}^{n} x_i$ |
| Minimum-bound coupling | $R_i = \min(x_i, x_{i+1})$ | $\sum_{i=1}^{n} R_i \leq \sum_{i=1}^{n} x_i$ |
| Logarithmic mean coupling | $R_i = \frac{x_{i+1} - x_i}{\ln x_{i+1} - \ln x_i}$ | $\sum_{i=1}^{n} R_i \leq \sum_{i=1}^{n} x_i$ |
| Weighted power-mean hybrid | $R_i = x_{i+1}^{1/2}\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right)^{1/2}$ | $\sum_{i=1}^{n} R_i \leq \sum_{i=1}^{n} x_i$ |
| Heronian mean coupling | $R_i = \frac{x_i + \sqrt{x_i x_{i+1}} + x_{i+1}}{3}$ | $\sum_{i=1}^{n} R_i \leq \sum_{i=1}^{n} x_i$ |
| Normalized power sum ($0 < r < 1$) | $R_i = \frac{x_{i+1}^r}{\left((1/n)\sum_{j=1}^{n} x_j^r\right)^{(r-1)/r}}$ | $\sum_{i=1}^{n} R_i \leq \sum_{i=1}^{n} x_i$ |

### 2.2. A General Product Inequality Theorem

Building on Theorem 1 and an integration technique, the result below presents a multiple-variable product-type inequality.

**Theorem 2.** *Let $n \in \mathbb{N} \setminus \{0, 1\}$, $x_1, x_2, \ldots, x_n > 0$, and $R_1, R_2, \ldots, R_n > 0$ such that*

$$\sum_{i=1}^{n} R_i \leq \sum_{i=1}^{n} x_i.$$

*Let $\theta > \eta \geq 0$ and $p$ such that*

$$1 \leq p < 2 + \eta.$$

*Then the following n-variable product inequality holds:*

$$\prod_{i=1}^{n} \left( \frac{x_i^{p-1} + \theta R_i^{p-1}}{x_i^{p-1} + \eta R_i^{p-1}} \right)^{x_i^p / R_i^{p-1}} \geq \left( \frac{1+\theta}{1+\eta} \right)^{\sum_{i=1}^{n} x_i}.$$

*Equality holds if and only if $p = 1$ or $R_i = x_i$ for every $i = 1, 2, \ldots, n$.*

**Proof.** Under the indicated conditions, it follows from Theorem 1 that

$$\sum_{i=1}^{n} \frac{x_i^p}{x_i^{p-1} + \lambda R_i^{p-1}} \geq \frac{1}{1+\lambda} \sum_{i=1}^{n} x_i.$$

Equality holds if and only if $p = 1$, $\lambda = 0$, or $R_i = x_i$ for every $i = 1, 2, \ldots, n$. Considering $\lambda$ as a continuous variable and integrating both sides with respect to $\lambda$ over the interval $(\eta, \theta)$, we get

$$\int_{\eta}^{\theta} \left( \sum_{i=1}^{n} \frac{x_i^p}{x_i^{p-1} + \lambda R_i^{p-1}} \right) d\lambda \geq \int_{\eta}^{\theta} \left( \frac{1}{1+\lambda} \sum_{i=1}^{n} x_i \right) d\lambda.$$

Using standard integration techniques and the elementary properties of natural logarithms, the left-hand side becomes

$$\begin{aligned} \int_{\eta}^{\theta} \left( \sum_{i=1}^{n} \frac{x_i^p}{x_i^{p-1} + \lambda R_i^{p-1}} \right) d\lambda &= \sum_{i=1}^{n} x_i^p \int_{\eta}^{\theta} \frac{1}{x_i^{p-1} + \lambda R_i^{p-1}} d\lambda \\ &= \sum_{i=1}^{n} \frac{x_i^p}{R_i^{p-1}} \left[ \ln(x_i^{p-1} + \lambda R_i^{p-1}) \right]_{\eta}^{\theta} = \sum_{i=1}^{n} \frac{x_i^p}{R_i^{p-1}} \ln \left( \frac{x_i^{p-1} + \theta R_i^{p-1}}{x_i^{p-1} + \eta R_i^{p-1}} \right) \\ &= \ln \left( \prod_{i=1}^{n} \left( \frac{x_i^{p-1} + \theta R_i^{p-1}}{x_i^{p-1} + \eta R_i^{p-1}} \right)^{x_i^p / R_i^{p-1}} \right). \end{aligned}$$

The right-hand side evaluates to

$$\begin{aligned} \int_{\eta}^{\theta} \left( \frac{1}{1+\lambda} \sum_{i=1}^{n} x_i \right) d\lambda &= \left( \sum_{i=1}^{n} x_i \right) \int_{\eta}^{\theta} \frac{1}{1+\lambda} d\lambda = \left( \sum_{i=1}^{n} x_i \right) [\ln(1+\lambda)]_{\eta}^{\theta} \\ &= \left( \sum_{i=1}^{n} x_i \right) \ln \left( \frac{1+\theta}{1+\eta} \right) = \ln \left( \left( \frac{1+\theta}{1+\eta} \right)^{\sum_{i=1}^{n} x_i} \right). \end{aligned}$$

Combining these expressions into the main inequality yields

$$\ln \left( \prod_{i=1}^{n} \left( \frac{x_i^{p-1} + \theta R_i^{p-1}}{x_i^{p-1} + \eta R_i^{p-1}} \right)^{x_i^p / R_i^{p-1}} \right) \geq \ln \left( \left( \frac{1+\theta}{1+\eta} \right)^{\sum_{i=1}^{n} x_i} \right).$$

Exponentiating both sides gives

$$\prod_{i=1}^{n} \left( \frac{x_i^{p-1} + \theta R_i^{p-1}}{x_i^{p-1} + \eta R_i^{p-1}} \right)^{x_i^p / R_i^{p-1}} \geq \left( \frac{1+\theta}{1+\eta} \right)^{\sum_{i=1}^{n} x_i}.$$

The desired inequality is proved.

Furthermore, equality holds if and only if equality is achieved in Theorem 1 for every $\lambda \in (\eta, \theta)$. The condition $\lambda = 0$ is excluded, leaving $p = 1$ or $R_i = x_i$ for every $i = 1, 2, \ldots, n$. This completes the proof. □

**Remark 7.** Setting $\eta = 0$ in Theorem 2 yields the following one-parameter product inequality:

$$\prod_{i=1}^{n} \left(1 + \theta \frac{R_i^{p-1}}{x_i^{p-1}}\right)^{x_i^p / R_i^{p-1}} \geq (1+\theta)^{\sum_{i=1}^{n} x_i}.$$

This special case isolates the interaction between the variables, $R_1, R_2, \ldots, R_n$, and the parameter $\theta$. It provides a direct multiplicative counterpart to Theorem 1.

**Remark 8.** In the context of Theorem 2, taking the reciprocal of both sides of the resulting inequality yields the following upper bound:

$$\prod_{i=1}^{n} \left(\frac{x_i^{p-1} + \eta R_i^{p-1}}{x_i^{p-1} + \theta R_i^{p-1}}\right)^{x_i^p / R_i^{p-1}} \leq \left(\frac{1+\eta}{1+\theta}\right)^{\sum_{i=1}^{n} x_i}.$$

Numerous inequalities, including generalized cyclic variants, can be established via Theorems 1 and 2. This is illustrated in the subsequent section through a series of propositions.

## 3. Propositions

### 3.1. Applications of Theorem 1

The result below presents an application of Theorem 1, emphasizing the presence of an arithmetic mean in the denominator of each summand.

**Proposition 1.** *Let $n \in \mathbb{N} \setminus \{0, 1\}$, $x_1, x_2, \ldots, x_n > 0$, $\lambda \geq 0$, and $p$ such that*

$$1 \leq p < 2 + \lambda.$$

*Then the following n-variable sum inequality holds:*

$$\sum_{i=1}^{n} \frac{x_i^p}{x_i^{p-1} + \lambda \left((1/n) \sum_{j=1}^{n} x_j\right)^{p-1}} \geq \frac{1}{1+\lambda} \sum_{i=1}^{n} x_i.$$

*Equality holds if and only if $p = 1$, $\lambda = 0$, or $x_1 = x_2 = \ldots = x_n$.*

**Proof.** The proof follows from Theorem 1 applied with

$$R_i = \frac{1}{n} \sum_{j=1}^{n} x_j,$$

for every $i = 1, 2, \ldots, n$. To support this claim, we must verify the inequality in Eq. (1). It is sufficient to observe that

$$\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} \left(\frac{1}{n} \sum_{j=1}^{n} x_j\right) = \frac{n}{n} \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i.$$

Furthermore, Theorem 1 ensures that equality holds if and only if $p = 1$, $\lambda = 0$, or $R_i = x_i$ for every $i = 1, 2, \ldots, n$. The latter condition is equivalent to $x_1 = x_2 = \cdots = x_n$, representing the configuration where the variables coincide with their arithmetic mean. This concludes the proof. □

The result below features powers of cyclically adjacent variables in the denominator.

**Proposition 2.** *Let $n \in \mathbb{N} \setminus \{0,1\}$, $x_1, x_2, \ldots, x_n > 0$ with the cyclic convention $x_{n+1} = x_1$, $\lambda \geq 0$, and $p$ such that*

$$1 \leq p < 2 + \lambda.$$

*Then the following n-variable cyclic sum inequality holds:*

$$\sum_{i=1}^{n} \frac{x_i^p}{x_i^{p-1} + \lambda x_{i+1}^{p-1}} \geq \frac{1}{1+\lambda} \sum_{i=1}^{n} x_i.$$

*Equality holds if and only if $\lambda = 0$, $p = 1$, or if $x_1 = x_2 = \ldots = x_n$.*

**Proof.** The proof follows from Theorem 1 applied with

$$R_i = x_{i+1}$$

for every $i = 1, 2, \ldots, n$. To validate this application, we must verify the inequality in Eq. (1). It is sufficient to notice that, by the cyclic convention,

$$\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} x_{i+1} = \sum_{i=1}^{n} x_i.$$

Furthermore, Theorem 1 ensures that equality holds if and only if $p = 1$, $\lambda = 0$, or $R_i = x_i$ for every $i = 1, 2, \ldots, n$. The latter condition is equivalent to $x_i = x_{i+1}$ for every $i = 1, 2, \ldots, n$, so $x_1 = x_2 = \cdots = x_n$ by the cyclic convention. This concludes the proof. □

The result below is another application of Theorem 1. It features adjacent variable products in the denominator.

**Proposition 3.** *Let $n \in \mathbb{N} \setminus \{0,1\}$, $x_1, x_2, \ldots, x_n > 0$ with the cyclic conventions $x_0 = x_n$ and $x_{n+1} = x_1$, $\lambda \geq 0$, and $p$ such that*

$$1 \leq p < 2 + \lambda.$$

*Then the following n-variable cyclic sum inequality holds:*

$$\sum_{i=1}^{n} \frac{x_i^p}{x_i^{p-1} + \lambda x_{i-1}^{(p-1)/2} x_{i+1}^{(p-1)/2}} \geq \frac{1}{1+\lambda} \sum_{i=1}^{n} x_i.$$

*Equality holds if and only if $p = 1$, $\lambda = 0$, or $x_1 = x_2 = \ldots = x_n$.*

**Proof.** The proof follows from Theorem 1 applied with

$$R_i = \sqrt{x_{i-1} x_{i+1}},$$

for every $i = 1, 2, \ldots, n$. To support this claim, we must verify the inequality in Eq. (1). Using the classical arithmetic mean-geometric mean inequality with two terms, and $\sum_{i=1}^{n} x_{i-1} = \sum_{i=1}^{n} x_{i+1} = \sum_{i=1}^{n} x_i$ by the cyclic conventions, we get

$$\begin{aligned} \sum_{i=1}^{n} R_i = \sum_{i=1}^{n} \sqrt{x_{i-1} x_{i+1}} &\leq \sum_{i=1}^{n} \left( \frac{1}{2} (x_{i-1} + x_{i+1}) \right) \\ &= \frac{1}{2} \left( \sum_{i=1}^{n} x_{i-1} + \sum_{i=1}^{n} x_{i+1} \right) = \sum_{i=1}^{n} x_i. \end{aligned}$$

The desired inequality is proved.

Furthermore, Theorem 1 ensures that equality holds if and only if $p = 1$, $\lambda = 0$, or $R_i = x_i$ for every $i = 1, 2, \ldots, n$. The latter condition is equivalent to

$$x_i = \sqrt{x_{i-1}x_{i+1}},$$

for every $i = 1, 2, \ldots, n$. Squaring both sides and taking natural logarithms yields $2 \ln x_i = \ln x_{i-1} + \ln x_{i+1}$. Setting $y_i = \ln x_i$, we get

$$y_i = \frac{1}{2}(y_{i-1} + y_{i+1}),$$

for any $i = 1, 2, \ldots, n$. Let $M = \max(y_1, y_2, \ldots, y_n)$ and define the non-empty set $K = \{k \in \{1, \ldots, n\} \mid y_k = M\}$. For any $k \in K$, we have

$$M = y_k = \frac{1}{2}(y_{k-1} + y_{k+1}) \leq \frac{1}{2}(M + M) = M.$$

Equality forces $y_{k-1} = y_{k+1} = M$, which implies that both $k - 1$ and $k + 1$ belong to $K$. By cyclic propagation, $K$ contains all indices $\{1, 2, \ldots, n\}$. Thus, we have $y_1 = y_2 = \cdots = y_n = M$, and exponentiating gives $x_1 = x_2 = \cdots = x_n$. This concludes the proof. □

Using the notion of $m$-element cyclic neighborhood of an index, the result below applies Theorem 1 to incorporate a product-based term within the denominator.

**Proposition 4.** *Let $n \in \mathbb{N} \setminus \{0, 1\}$, $m \in \{1, 2, \ldots, n\}$, $x_1, x_2, \ldots, x_n > 0$ with indices taken modulo $n$, $\lambda \geq 0$, and $p$ such that*

$$1 \leq p < 2 + \lambda.$$

*For any $i = 1, 2, \ldots, n$, we define the $m$-element cyclic neighborhood of $i$ by*

$$\Omega_i = \{i + 1, i + 2, \ldots, i + m\},$$

*with indices taken modulo $n$.*

*Then the following $n$-variable cyclic sum inequality holds:*

$$\sum_{i=1}^{n} \frac{x_i^p}{x_i^{p-1} + \lambda \left(\prod_{j \in \Omega_i} x_j\right)^{(p-1)/m}} \geq \frac{1}{1 + \lambda} \sum_{i=1}^{n} x_i.$$

*Equality holds if and only if $p = 1$, $\lambda = 0$, or $x_1 = x_2 = \ldots = x_n$.*

**Proof.** The proof follows from Theorem 1 applied with

$$R_i = \left(\prod_{j \in \Omega_i} x_j\right)^{1/m},$$

for every $i = 1, 2, \ldots, n$. To validate this application, we must verify the inequality in Eq. (1). By the arithmetic mean-geometric mean inequality with $m$ terms,

$$\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} \left(\prod_{j \in \Omega_i} x_j\right)^{1/m} \leq \sum_{i=1}^{n} \left(\frac{1}{m} \sum_{j \in \Omega_i} x_j\right) = \frac{m}{m} \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i.$$

The desired inequality is proved.

Theorem 1 ensures that equality holds if and only if $p = 1$, $\lambda = 0$, or $R_i = x_i$ for every $i = 1, 2, \ldots, n$. The latter condition is equivalent to

$$x_i = \left(\prod_{j \in \Omega_i} x_j\right)^{1/m},$$

for every $i = 1, 2, \ldots, n$. Taking the natural logarithm of both sides, setting $y_i = \ln x_i$, and using the definition of $\Omega_i$, we get

$$y_i = \frac{1}{m} \sum_{j=1}^{m} y_{i+j},$$

for every $i = 1, 2, \ldots, n$. Let $M = \max(y_1, y_2, \ldots, y_n)$ and $K = \{i \in \{1, 2, \ldots, n\} \mid y_i = M\}$, which is non-empty by definition.

For any $k \in K$, we have

$$M = \frac{1}{m} \sum_{j=1}^{m} y_{k+j} \leq \frac{1}{m} \sum_{j=1}^{m} M = M.$$

Since the arithmetic mean equals the upper bound $M$, every term in the sum must achieve this maximum. Thus, we have $y_{k+1} = y_{k+2} = \cdots = y_{k+m} = M$, which guarantees that $k + 1 \in K$. By cyclic propagation, $K$ contains all indices $\{1, 2, \ldots, n\}$. Consequently, $y_1 = y_2 = \cdots = y_n = M$, which is equivalent to $x_1 = x_2 = \cdots = x_n$. This concludes the proof. □

The following two choices of the neighborhood size are particularly useful:

- for $m = n - 1$, we have $\Omega_i = \{1, 2, \ldots, n\} \setminus \{i\}$,
- for $m = n$, we have $\Omega_i = \{1, 2, \ldots, n\}$.

In this last case, we have $\left(\prod_{j \in \Omega_i} x_j\right)^{1/m} = \left(\prod_{j=1}^{n} x_j\right)^{1/n}$, corresponding to the standard geometric mean. Finally, the result below incorporates a product-sum structural term within the denominator.

**Proposition 5.** *Let $n \in \mathbb{N} \setminus \{0, 1\}$, $m \in \{1, 2, \ldots, n\}$, $x_1, x_2, \ldots, x_n > 0$ with indices taken modulo $n$, $\lambda \geq 0$, and $p$ such that*

$$1 \leq p < 2 + \lambda.$$

*Let $\kappa_1, \kappa_2, \ldots, \kappa_n \geq 0$ and $\sigma_1, \sigma_2, \ldots, \sigma_n \geq 0$ such that, for every $i = 1, 2, \ldots, n$,*

$$\kappa_i + \sigma_i = 1.$$

*For any $i = 1, 2, \ldots, n$, we define the $m$-element cyclic neighborhood of $i$ by*

$$\Omega_i = \{i + 1, i + 2, \ldots, i + m\},$$

*with indices taken modulo $n$.*

*Then the following $n$-variable cyclic sum inequality holds:*

$$\sum_{i=1}^{n} \frac{x_i^p}{x_i^{p-1} + \lambda \left(\prod_{j \in \Omega_i} x_j\right)^{\kappa_i (p-1)/m} \left((1/m) \sum\limits_{j \in \Omega_i} x_j\right)^{\sigma_i (p-1)}} \geq \frac{1}{1 + \lambda} \sum_{i=1}^{n} x_i.$$

*Equality holds if and only if $p = 1$, $\lambda = 0$, or $x_1 = x_2 = \ldots = x_n$.*

**Proof.** The proof follows from Theorem 1 applied with

$$R_i = \left(\prod_{j \in \Omega_i} x_j\right)^{\kappa_i/m} \left(\frac{1}{m} \sum_{j \in \Omega_i} x_j\right)^{\sigma_i},$$

for every $i = 1, 2, \ldots, n$. To support this claim, we must verify the inequality in Eq. (1). By the arithmetic mean-geometric mean inequality with $m$ terms and $\kappa_i + \sigma_i = 1$, we have

$$\begin{aligned}\sum_{i=1}^{n} R_i &= \sum_{i=1}^{n} \left( \left( \prod_{j\in\Omega_i} x_j \right)^{1/m} \right)^{\kappa_i} \left( \frac{1}{m} \sum_{j\in\Omega_i} x_j \right)^{\sigma_i} \\ &\leq \sum_{i=1}^{n} \left( \frac{1}{m} \sum_{j\in\Omega_i} x_j \right)^{\kappa_i} \left( \frac{1}{m} \sum_{j\in\Omega_i} x_j \right)^{\sigma_i} = \sum_{i=1}^{n} \left( \frac{1}{m} \sum_{j\in\Omega_i} x_j \right)^{\kappa_i+\sigma_i} \\ &= \sum_{i=1}^{n} \left( \frac{1}{m} \sum_{j\in\Omega_i} x_j \right) = \frac{m}{m} \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i.\end{aligned}$$

The desired inequality is proved.

Furthermore, Theorem 1 ensures that equality holds if and only if $p = 1$, $\lambda = 0$, or $R_i = x_i$ for every $i = 1, 2, \ldots, n$. The latter condition is equivalent to

$$x_i = \left( \prod_{j\in\Omega_i} x_j \right)^{\kappa_i/m} \left( \frac{1}{m} \sum_{j\in\Omega_i} x_j \right)^{\sigma_i},$$

for every $i = 1, 2, \ldots, n$. Taking the natural logarithm of both sides, setting $y_i = \ln x_i$ and using the definition of $\Omega_i$, we get

$$y_i = \frac{\kappa_i}{m} \sum_{j=1}^{m} y_{i+j} + \sigma_i \ln \left( \frac{1}{m} \sum_{j=1}^{m} e^{y_{i+j}} \right),$$

for every $i = 1, 2, \ldots, n$. Let $M = \max(y_1, y_2, \ldots, y_n)$ and define the non-empty index set $K = \{k \in \{1, 2, \ldots, n\} \mid y_k = M\}$. For any $k \in K$, using $\kappa_k + \sigma_k = 1$, we have

$$\begin{aligned}M = y_k &= \frac{\kappa_k}{m} \sum_{j=1}^{m} y_{k+j} + \sigma_k \ln \left( \frac{1}{m} \sum_{j=1}^{m} e^{y_{k+j}} \right) \leq \frac{\kappa_k}{m} \sum_{j=1}^{m} M + \sigma_k \ln \left( \frac{1}{m} \sum_{j=1}^{m} e^{M} \right) \\ &= \kappa_k M + \sigma_k M = (\kappa_k + \sigma_k) M = M.\end{aligned}$$

Because the left- and right-hand sides are equal to $M$, equality must hold throughout. This forces every term in the sums to achieve the maximum, yielding $y_{k+1} = y_{k+2} = \cdots = y_{k+m} = M$. Thus, we have $k + 1 \in K$. By cyclic propagation, $K$ contains all indices $\{1, 2, \ldots, n\}$, which implies $y_1 = y_2 = \cdots = y_n = M$, and consequently $x_1 = x_2 = \cdots = x_n$. This completes the proof. □

### 3.2. Applications of Theorem 2

As a product counterpart to Proposition 1, the result below examines an arithmetic-mean-based product inequality.

**Proposition 6.** *Let $n \in \mathbb{N} \setminus \{0, 1\}$, $x_1, x_2, \ldots, x_n > 0$, $\theta > \eta \geq 0$, and $p$ such that*

$$1 \leq p < 2 + \eta.$$

*Then the following n-variable product inequality holds:*

$$\prod_{i=1}^{n} \left( \frac{x_i^{p-1} + \theta \left( (1/n) \sum_{j=1}^{n} x_j \right)^{p-1}}{x_i^{p-1} + \eta \left( (1/n) \sum_{j=1}^{n} x_j \right)^{p-1}} \right)^{x_i^p / \left( (1/n) \sum_{j=1}^{n} x_j \right)^{p-1}} \geq \left( \frac{1+\theta}{1+\eta} \right)^{\sum_{i=1}^{n} x_i}.$$

*Equality holds if and only if $p = 1$ or $x_1 = x_2 = \ldots = x_n$.*

**Proof.** The result follows from Theorem 2 by setting

$$R_i = \frac{1}{n}\sum_{j=1}^{n} x_j,$$

for every $i = 1, 2, \ldots, n$, combined with the main lines of the proof of Proposition 1. □

The result below provides a cyclic product variant involving adjacent variables.

**Proposition 7.** *Let $n \in \mathbb{N} \setminus \{0,1\}$, $x_1, x_2, \ldots, x_n > 0$ with the cyclic convention $x_{n+1} = x_1$, $\theta > \eta \geq 0$, and $p$ such that*

$$1 \leq p < 2 + \eta.$$

*Then the following n-variable cyclic product inequality holds:*

$$\prod_{i=1}^{n}\left(\frac{x_i^{p-1} + \theta x_{i+1}^{p-1}}{x_i^{p-1} + \eta x_{i+1}^{p-1}}\right)^{x_i^p / x_{i+1}^{p-1}} \geq \left(\frac{1+\theta}{1+\eta}\right)^{\sum_{i=1}^{n} x_i}.$$

*Equality holds if and only if $p = 1$ or $x_1 = x_2 = \ldots = x_n$.*

**Proof.** The proof is a consequence of Theorem 2 by setting

$$R_i = x_{i+1},$$

for every $i = 1, 2, \ldots, n$, and mirroring the steps in the proof of Proposition 2. □

Extending this structure to adjacent variable products, the result below presents a product variant of Proposition 3.

**Proposition 8.** *Let $n \in \mathbb{N} \setminus \{0,1\}$, $x_1, x_2, \ldots, x_n > 0$ with the cyclic conventions $x_0 = x_n$ and $x_{n+1} = x_1$, $\theta > \eta \geq 0$, and $p$ such that*

$$1 \leq p < 2 + \eta.$$

*Then the following n-variable cyclic product inequality holds:*

$$\prod_{i=1}^{n}\left(\frac{x_i^{p-1} + \theta x_{i-1}^{(p-1)/2} x_{i+1}^{(p-1)/2}}{x_i^{p-1} + \eta x_{i-1}^{(p-1)/2} x_{i+1}^{(p-1)/2}}\right)^{x_i^p / \left(x_{i-1}^{(p-1)/2} x_{i+1}^{(p-1)/2}\right)} \geq \left(\frac{1+\theta}{1+\eta}\right)^{\sum_{i=1}^{n} x_i}.$$

*Equality holds if and only if $p = 1$ or $x_1 = x_2 = \ldots = x_n$.*

**Proof.** The result follows from Theorem 2 by setting

$$R_i = x_{i-1}^{1/2} x_{i+1}^{1/2},$$

for every $i = 1, 2, \ldots, n$, and adapting the argument used in the proof of Proposition 3. □

Finally, by using a product-based term, the result below gives a product inequality generalizing Proposition 4.

**Proposition 9.** *Let $n \in \mathbb{N} \setminus \{0,1\}$, $m \in \{1, 2, \ldots, n\}$, $x_1, x_2, \ldots, x_n > 0$ with indices taken modulo $n$, $\theta > \eta \geq 0$, and $p$ such that*

$$1 \leq p < 2 + \eta.$$

*For any $i = 1, 2, \ldots, n$, we define the m-element cyclic neighborhood of $i$ by*

$$\Omega_i = \{i+1, i+2, \ldots, i+m\},$$

*with indices taken modulo $n$.*

*Then the following $n$-variable cyclic product inequality holds:*

$$\prod_{i=1}^{n}\left(\frac{x_i^{p-1}+\theta\left(\prod_{j\in\Omega_i}x_j\right)^{(p-1)/m}}{x_i^{p-1}+\eta\left(\prod_{j\in\Omega_i}x_j\right)^{(p-1)/m}}\right)^{x_i^p/\left(\prod_{j\in\Omega_i}x_j\right)^{(p-1)/m}}\geq\left(\frac{1+\theta}{1+\eta}\right)^{\sum_{i=1}^{n}x_i}.$$

*Equality holds if and only if $p=1$ or $x_1=x_2=\ldots=x_n$.*

**Proof.** The proof is established by applying Theorem 2 with

$$R_i=\left(\prod_{j\in\Omega_i}x_j\right)^{1/m},$$

for every $i=1,2,\ldots,n$, in conjunction with the steps in the proof of Proposition 4. □

**Remark 9.** A similar result also emerges by combining the frameworks of Theorem 2 and Proposition 5.

## 4. Complementary Theorems

In this section, we present four complementary results that complement Theorem 1.

### 4.1. A Two-Parameter Sum Inequality Theorem

The result below generalizes Theorem 1 by incorporating an additional parameter.

**Theorem 3.** *Let $n\in\mathbb{N}\setminus\{0,1\}$, $x_1,x_2,\ldots,x_n>0$, and $R_1,R_2,\ldots,R_n>0$ such that*

$$\sum_{i=1}^{n}R_i\leq\sum_{i=1}^{n}x_i. \tag{3}$$

*Let $\lambda,\mu\geq 0$ with $\lambda+\mu>0$, and $p$ satisfy one of the following conditions:*

(i) *$\mu>0$ and*

$$1\leq p<2+\frac{\lambda}{\mu},$$

(ii) *$\mu=0$ and $p\geq 1$.*

*Then the following $n$-variable sum inequality holds:*

$$\sum_{i=1}^{n}\frac{x_i^p}{\mu x_i^{p-1}+\lambda R_i^{p-1}}\geq\frac{1}{\mu+\lambda}\sum_{i=1}^{n}x_i.$$

*Equality holds if and only if $p=1$, $R_i=x_i$ for every $i=1,2,\ldots,n$, or ($\lambda=0$ only under condition (i)).*

**Proof.** Under condition (i), applying Theorem 1 with $\lambda/\mu$ in place of $\lambda$ yields

$$\begin{aligned}\sum_{i=1}^{n}\frac{x_i^p}{\mu x_i^{p-1}+\lambda R_i^{p-1}}&=\frac{1}{\mu}\sum_{i=1}^{n}\frac{x_i^p}{x_i^{p-1}+(\lambda/\mu)R_i^{p-1}}\\&\geq\frac{1}{\mu}\times\frac{1}{1+\lambda/\mu}\sum_{i=1}^{n}x_i=\frac{1}{\mu+\lambda}\sum_{i=1}^{n}x_i.\end{aligned}$$

Furthermore, equality holds if and only if $p=1$, $\lambda/\mu=0$ so $\lambda=0$, or $R_i=x_i$ for every $i=1,2,\ldots,n$. The desired result is obtained.

Under condition (ii), the Radon inequality and Eq. (3) give

$$\sum_{i=1}^{n} \frac{x_i^p}{\mu x_i^{p-1} + \lambda R_i^{p-1}} = \frac{1}{\lambda} \sum_{i=1}^{n} \frac{x_i^p}{R_i^{p-1}} \geq \frac{1}{\lambda} \frac{\left(\sum_{i=1}^{n} x_i\right)^p}{\left(\sum_{i=1}^{n} R_i\right)^{p-1}}$$
$$\geq \frac{1}{\lambda} \frac{\left(\sum_{i=1}^{n} x_i\right)^p}{\left(\sum_{i=1}^{n} x_i\right)^{p-1}} = \frac{1}{\lambda} \sum_{i=1}^{n} x_i = \frac{1}{\mu + \lambda} \sum_{i=1}^{n} x_i.$$

Furthermore, equality holds if and only if equality is attained in Eq. (3) (that is, $\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} x_i$) and in the application of the Radon inequality, that is, $p = 1$ or $x_1/R_1 = x_2/R_2 = \cdots = x_n/R_n$. This latter condition combined with $\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} x_i$ uniquely implies $R_i = x_i$ for every $i = 1, 2, \ldots, n$. This concludes the proof. □

Setting $\mu = 1$, which satisfies condition (i), Theorem 3 reduces to Theorem 1.

**Remark 10.** In Theorem 3, choosing $\mu = 0$, $p = 2$, $R_1, R_2, \ldots, R_n$ as the Nesbitt-type means given in the fourth entry of Table 1 and setting $\lambda = n - 1$, we obtain

$$\sum_{i=1}^{n} \frac{x_i^2}{\left(\sum_{j=1}^{n} x_j\right) - x_i} \geq \frac{1}{n-1} \sum_{i=1}^{n} x_i.$$

Notice that, for every $i = 1, 2, \ldots, n$,

$$\frac{x_i^2}{\left(\sum_{j=1}^{n} x_j\right) - x_i} = \frac{x_i \left(x_i - \sum_{j=1}^{n} x_j\right) + x_i \sum_{j=1}^{n} x_j}{\left(\sum_{j=1}^{n} x_j\right) - x_i} = \left(\sum_{j=1}^{n} x_j\right) \frac{x_i}{\left(\sum_{j=1}^{n} x_j\right) - x_i} - x_i.$$

From this identity, we derive

$$\sum_{i=1}^{n} \frac{x_i^2}{\left(\sum_{j=1}^{n} x_j\right) - x_i} = \left(\sum_{j=1}^{n} x_j\right) \left( \sum_{i=1}^{n} \frac{x_i}{\left(\sum_{j=1}^{n} x_j\right) - x_i} - 1 \right).$$

Combining this identity with the preceding inequality yields, after suitable algebraic manipulations,

$$\sum_{i=1}^{n} \frac{x_i}{\left(\sum_{j=1}^{n} x_j\right) - x_i} \geq \frac{n}{n-1}.$$

This corresponds to the $n$-variable Nesbitt inequality.

### 4.2. Functional Generalization

The result below provides a functional generalization of Theorem 1 by incorporating a convex and increasing function.

**Theorem 4.** *Let $n \in \mathbb{N} \setminus \{0,1\}$, $x_1, x_2, \ldots, x_n > 0$, and $R_1, R_2, \ldots, R_n > 0$ such that*

$$\sum_{i=1}^{n} R_i \leq \sum_{i=1}^{n} x_i.$$

*Let $\lambda \geq 0$ and $p$ such that*

$$1 \leq p < 2 + \lambda.$$

*Let $f : [0,1] \to \mathbb{R}$ be a convex and increasing function. Then the following $n$-variable sum inequality holds:*

$$\sum_{i=1}^{n} x_i f\left(\frac{x_i^{p-1}}{x_i^{p-1} + \lambda R_i^{p-1}}\right) \geq \iota \sum_{i=1}^{n} x_i,$$

*where*

$$\iota = f\left(\frac{1}{1+\lambda}\right).$$

*If $f$ is strictly convex and strictly increasing, equality holds if and only if $p = 1$, $\lambda = 0$, or $R_i = x_i$ for every $i = 1, 2, \ldots, n$.*

**Proof.** The Jensen inequality applied to the convex function $f$ gives

$$\begin{aligned}
\sum_{i=1}^{n} x_i f\left(\frac{x_i^{p-1}}{x_i^{p-1} + \lambda R_i^{p-1}}\right) &= \left(\sum_{j=1}^{n} x_j\right) \sum_{i=1}^{n} \left(\frac{x_i}{\sum\limits_{j=1}^{n} x_j}\right) f\left(\frac{x_i^{p-1}}{x_i^{p-1} + \lambda R_i^{p-1}}\right) \\
&\geq \left(\sum_{j=1}^{n} x_j\right) f\left(\frac{1}{\sum\limits_{j=1}^{n} x_j} \sum_{i=1}^{n} x_i \frac{x_i^{p-1}}{x_i^{p-1} + \lambda R_i^{p-1}}\right) \\
&= \left(\sum_{j=1}^{n} x_j\right) f\left(\frac{1}{\sum\limits_{j=1}^{n} x_j} \sum_{i=1}^{n} \frac{x_i^{p}}{x_i^{p-1} + \lambda R_i^{p-1}}\right).
\end{aligned}$$

Under the indicated conditions, using Theorem 1 and the fact that $f$ is increasing, we obtain

$$\begin{aligned}
f\left(\frac{1}{\sum\limits_{j=1}^{n} x_j} \sum_{i=1}^{n} \frac{x_i^{p}}{x_i^{p-1} + \lambda R_i^{p-1}}\right) &\geq f\left(\frac{1}{\sum\limits_{j=1}^{n} x_j} \times \frac{1}{1+\lambda} \sum_{i=1}^{n} x_i\right) \\
&= f\left(\frac{1}{1+\lambda}\right) = \iota.
\end{aligned}$$

Combining the inequalities above, we get

$$\sum_{i=1}^{n} x_i f\left(\frac{x_i^{p-1}}{x_i^{p-1} + \lambda R_i^{p-1}}\right) \geq \iota \sum_{i=1}^{n} x_i.$$

The desired inequality is proved.

If $f$ is strictly convex and strictly increasing, equality holds if and only if equality holds in both the application of the Jensen inequality and Theorem 1. Equality in the Jensen inequality requires

$$\frac{x_i^{p-1}}{x_i^{p-1} + \lambda R_i^{p-1}} = c,$$

for some constant $c$ and every $i = 1, 2, \ldots, n$. Combining this with the equality conditions from Theorem 1 fixes $c = 1/(1+\lambda)$. Substituting this back gives $(1+\lambda)x_i^{p-1} = x_i^{p-1} + \lambda R_i^{p-1}$, so $\lambda x_i^{p-1} = \lambda R_i^{p-1}$. This identity is satisfied if and only if $p = 1$, $\lambda = 0$, or $R_i = x_i$ for every $i = 1, 2, \ldots, n$. This matches the equality conditions of Theorem 1. This concludes the proof. □

**Remark 11.** Examples of convex and increasing functions $f : [0,1] \to \mathbb{R}$ include $f(x) = x^q$ for $q \geq 1$, $f(x) = r^x$ for $r > 1$, $f(x) = 1/(2-x)$, $f(x) = x/(2-x)$, $f(x) = x\ln(1+x)$, $f(x) = \tan(x)$, and $f(x) = x\arctan(x)$. For instance, Theorem 4 applied with $f(x) = x\ln(1+x)$ yields

$$\sum_{i=1}^{n} \frac{x_i^p}{x_i^{p-1} + \lambda R_i^{p-1}} \ln\left(\frac{2x_i^{p-1} + \lambda R_i^{p-1}}{x_i^{p-1} + \lambda R_i^{p-1}}\right) \geq \frac{1}{1+\lambda} \ln\left(\frac{2+\lambda}{1+\lambda}\right) \sum_{i=1}^{n} x_i,$$

which is equivalent to the following product inequality:

$$\prod_{i=1}^{n} \left(\frac{2x_i^{p-1} + \lambda R_i^{p-1}}{x_i^{p-1} + \lambda R_i^{p-1}}\right)^{x_i^p / \left(x_i^{p-1} + \lambda R_i^{p-1}\right)} \geq \left(\frac{2+\lambda}{1+\lambda}\right)^{(1/(1+\lambda)) \sum_{i=1}^{n} x_i}.$$

### 4.3. An Upper Bound and Generalization

The result below complements Theorem 1 by providing a precise upper bound for a modified sum.

**Theorem 5.** *Let $n \in \mathbb{N} \setminus \{0,1\}$, $x_1, x_2, \ldots, x_n > 0$, and $R_1, R_2, \ldots, R_n > 0$ such that*

$$\sum_{i=1}^{n} R_i \leq \sum_{i=1}^{n} x_i.$$

*Let $\lambda \geq 0$, and $p$ such that*

$$1 \leq p < 2 + \lambda.$$

*Then the following n-variable sum inequality holds:*

$$\sum_{i=1}^{n} \frac{x_i R_i^{p-1}}{x_i^{p-1} + \lambda R_i^{p-1}} \leq \frac{1}{1+\lambda} \sum_{i=1}^{n} x_i.$$

*Equality holds if and only if $p = 1$, $\lambda = 0$, or $R_i = x_i$ for every $i = 1, 2, \ldots, n$.*

**Proof.** Notice that, for every $i = 1, 2, \ldots, n$,

$$\frac{x_i^p}{x_i^{p-1} + \lambda R_i^{p-1}} + \lambda \frac{x_i R_i^{p-1}}{x_i^{p-1} + \lambda R_i^{p-1}} = \frac{x_i\left(x_i^{p-1} + \lambda R_i^{p-1}\right)}{x_i^{p-1} + \lambda R_i^{p-1}} = x_i.$$

Summing both sides, we get

$$\sum_{i=1}^{n} \frac{x_i^p}{x_i^{p-1} + \lambda R_i^{p-1}} + \lambda \sum_{i=1}^{n} \frac{x_i R_i^{p-1}}{x_i^{p-1} + \lambda R_i^{p-1}} = \sum_{i=1}^{n} x_i.$$

Rearranging this identity, we derive

$$\lambda \sum_{i=1}^{n} \frac{x_i R_i^{p-1}}{x_i^{p-1} + \lambda R_i^{p-1}} = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{x_i^p}{x_i^{p-1} + \lambda R_i^{p-1}}.$$

Under the indicated conditions, applying Theorem 1, we obtain

$$\lambda \sum_{i=1}^{n} \frac{x_i R_i^{p-1}}{x_i^{p-1} + \lambda R_i^{p-1}} \leq \sum_{i=1}^{n} x_i - \frac{1}{1+\lambda} \sum_{i=1}^{n} x_i = \frac{\lambda}{1+\lambda} \sum_{i=1}^{n} x_i.$$

For $\lambda > 0$, dividing both sides by $\lambda$ yields

$$\sum_{i=1}^{n} \frac{x_i R_i^{p-1}}{x_i^{p-1} + \lambda R_i^{p-1}} \leq \frac{1}{1+\lambda} \sum_{i=1}^{n} x_i.$$

The desired inequality is obtained.

Furthermore, equality holds if and only if equality is achieved in Theorem 1, that is, $p = 1$, $\lambda = 0$, or $R_i = x_i$ for every $i = 1, 2, \ldots, n$. This completes the proof. □

Derived from the approach in Theorem 4, the result below provides a functional generalization of Theorem 5 by incorporating a concave and increasing function.

**Theorem 6.** *Let $n \in \mathbb{N} \setminus \{0, 1\}$, $x_1, x_2, \ldots, x_n > 0$, and $R_1, R_2, \ldots, R_n > 0$ such that*

$$\sum_{i=1}^{n} R_i \leq \sum_{i=1}^{n} x_i.$$

*Let $\lambda \geq 0$ and $p$ such that*

$$1 \leq p < 2 + \lambda.$$

*Let $f : [0, 1] \to \mathbb{R}$ be a concave and increasing function. Then the following n-variable sum inequality holds:*

$$\sum_{i=1}^{n} x_i f\left(\frac{R_i^{p-1}}{x_i^{p-1} + \lambda R_i^{p-1}}\right) \leq \iota \sum_{i=1}^{n} x_i,$$

*where*

$$\iota = f\left(\frac{1}{1+\lambda}\right).$$

*If $f$ is strictly concave and strictly increasing, equality holds if and only if $p = 1$, $\lambda = 0$, or $R_i = x_i$ for every $i = 1, 2, \ldots, n$.*

**Proof.** The Jensen inequality applied to the concave function $f$ gives

$$\begin{aligned} \sum_{i=1}^{n} x_i f\left(\frac{R_i^{p-1}}{x_i^{p-1} + \lambda R_i^{p-1}}\right) &= \left(\sum_{j=1}^{n} x_j\right) \sum_{i=1}^{n} \left(\frac{x_i}{\sum_{j=1}^{n} x_j}\right) f\left(\frac{R_i^{p-1}}{x_i^{p-1} + \lambda R_i^{p-1}}\right) \\ &\leq \left(\sum_{j=1}^{n} x_j\right) f\left(\frac{1}{\sum_{j=1}^{n} x_j} \sum_{i=1}^{n} \frac{x_i R_i^{p-1}}{x_i^{p-1} + \lambda R_i^{p-1}}\right). \end{aligned}$$

Under the indicated conditions, using Theorem 5 and the fact that $f$ is increasing, we obtain

$$f\left(\frac{1}{\sum_{j=1}^{n} x_j} \sum_{i=1}^{n} \frac{x_i R_i^{p-1}}{x_i^{p-1} + \lambda R_i^{p-1}}\right) \leq f\left(\frac{1}{\sum_{j=1}^{n} x_j} \times \frac{1}{1+\lambda} \sum_{i=1}^{n} x_i\right) = f\left(\frac{1}{1+\lambda}\right) = \iota.$$

Combining the inequalities above, we get

$$\sum_{i=1}^{n} x_i f\left(\frac{R_i^{p-1}}{x_i^{p-1}+\lambda R_i^{p-1}}\right) \leq \iota \sum_{i=1}^{n} x_i.$$

The desired inequality is proved.

If $f$ is strictly concave and strictly increasing, equality holds if and only if equality is attained in both the application of the Jensen inequality and Theorem 5. Equality in the Jensen inequality requires

$$\frac{R_i^{p-1}}{x_i^{p-1}+\lambda R_i^{p-1}} = c,$$

for some constant $c$ and every $i = 1, 2, \ldots, n$. Combining this with the equality conditions from Theorem 5 fixes $c = 1/(1+\lambda)$. Substituting this back gives $(1+\lambda)R_i^{p-1} = x_i^{p-1} + \lambda R_i^{p-1}$, so $\lambda R_i^{p-1} = \lambda x_i^{p-1}$. This identity is satisfied if and only if $p = 1$, $\lambda = 0$, or $R_i = x_i$ for every $i = 1, 2, \ldots, n$. This matches the equality conditions of Theorem 5. This ends the proof. □

**Remark 12.** Examples of concave and increasing functions $f : [0,1] \to \mathbb{R}$ include $f(x) = x^q$ for $0 < q \leq 1$, $f(x) = 1 - r^{-x}$ for $r > 1$, $f(x) = x/(1+x)$, $f(x) = \ln(1+x)$, $f(x) = \arctan(x)$, and $f(x) = \sin(x)$. For instance, Theorem 6 applied with $f(x) = \ln(1+x)$ yields

$$\sum_{i=1}^{n} x_i \ln\left(\frac{x_i^{p-1}+(1+\lambda)R_i^{p-1}}{x_i^{p-1}+\lambda R_i^{p-1}}\right) \leq \ln\left(\frac{2+\lambda}{1+\lambda}\right) \sum_{i=1}^{n} x_i,$$

which is equivalent to the following product inequality:

$$\prod_{i=1}^{n} \left(\frac{x_i^{p-1}+(1+\lambda)R_i^{p-1}}{x_i^{p-1}+\lambda R_i^{p-1}}\right)^{x_i} \leq \left(\frac{2+\lambda}{1+\lambda}\right)^{\sum_{i=1}^{n} x_i}.$$

Further new inequalities, including broader cyclic variants, can be stated via these theorems.

## 5. Conclusion

In this paper, we have developed a comprehensive analytical framework for multiple-variable inequalities. Two general theorems, one governing sum-type and the other product-type inequalities, are proved. By combining refined algebraic decompositions, applications of the Young inequality, and integral techniques, our approach enables the treatment of a wide range of parameters and global sum conditions. These results translate into propositions concerning arithmetic, geometric, and harmonic means, both local and global, as well as cyclic structures involving multiple elements and advanced hybrid configurations. Furthermore, additional extensions, incorporating auxiliary parameters, generalizations to convex functions via the Jensen inequality, and the derivation of a precise upper bound, demonstrate the versatility of the developed methodology. Future research will explore similar sum-type and product-type inequalities under the regime $p \geq 2 + \lambda$, weighted versions, and broader applications in mathematical analysis.

**Conflicts of Interest:** The author declares no conflicts of interest.

**Data Availability:** No datasets were used in this study.

**Funding Information:** This research received no external funding.

## References

[1] Nesbitt, A. M. (1903). Problem 15114. *Educational Times, Second Series, 3*, 37–38.

[2] Drîmbe, M. O. (2003). *Inegalităţi: Idei şi Metode* [Inequalities: Ideas and methods]. Editura GIL.

[3] Cîrtoaje, V. (2018). *Mathematical Inequalities: Vol. 3. Cyclic and Noncyclic Inequalities*. LAP Lambert Academic Publishing.

[4] Bencze, M., & Zhao, C. (2008). A refinement of Nesbitt's inequality. *Octogon Mathematical Magazine, 16*(1A), 275–276.

[5] Bencze, M., & Pop, O. T. (2011). Generalizations and refinements for Nesbitt's inequality. *Journal of Mathematical Inequalities*, *5*(1), 13–20.

[6] Wei, F., & Wu, S. (2009). Generalizations and analogues of the Nesbitt's inequality. *Octogon Mathematical Magazine*, *17*(1), 215–220.

[7] Chesneau, C. (2026). On three one-parameter families of cyclic inequalities. *Journal of Computer Science and Applied Mathematics*, *8*(2), 207–212.

[8] Mitrinović, D. S. (1970). *Analytic Inequalities*. Springer-Verlag.



© 2026 by the authors; licensee PSRP, Lahore, Pakistan. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC-BY) license (http://creativecommons.org/licenses/by/4.0/).