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Some inequalities for linear canonical Fourier transform

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Abstract: The linear canonical Fourier transform, (LCFT), satisfies some uncertainty principales similar to classical Fourier transform. The aim of this paper is to prove a generalization of these principles for the LCFT.

Keywords: linear canonical Fourier transform, Heisenberg's inequality

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1. Introduction

In this section, we define some definitions, notations and results from harmonic analysis related the linear canonical Fourier transform. For this purpose, throughout this work, we fix an arbitrary matrix $N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, such that $b > 0$, in $SL(2, \mathbb{R})$.

Assume that $\mathrm{L}^p(\mathbb{R})$, $p \in [1, +\infty)$, as the space of all those real-valued measurable functions f on $\mathbb{R}$, such that

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{1/p}.$$

The LCFT of function $f \in \mathrm{L}^1(\mathbb{R})$ is defined by (see [1,2])

$$\mathcal{F}_N(f)(\lambda) = \frac{1}{\sqrt{2\pi ib}} \int_{\mathbb{R}} K_N(\lambda, x) f(x) dx, \quad b > 0,$$

where the kernel $K_N(\lambda, x)$ is given by

$$K_N(\lambda, x) = exp\left(i \left(\frac{a}{2b} x^2 - \frac{1}{b} x\lambda + \frac{d}{2b}\lambda \right) \right).$$

The inverse of the LCFT $\mathcal{F}_N^{-1}$ of $f \in \mathrm{L}^1(\mathbb{R})$ is given by (see [3,4])

$$f(x) = \sqrt{\frac{i}{2\pi b}} \int_{\mathbb{R}} exp\left(-i \left(\frac{a}{2b} x^2 - \frac{1}{b} x\lambda + \frac{d}{2b}\lambda \right) \right) \mathcal{F}_N(f)(\lambda) d\lambda.$$

Proposition 1 (Parseval's identity [5]). *Let f and g be two functions in $\mathrm{L}^2(\mathbb{R})$. Then $\langle f, g \rangle = \langle \mathcal{F}_N(f), \mathcal{F}_N(g) \rangle$ and for $f = g$, we have*

$$\|f\|_2 = \|\mathcal{F}_N(f)\|_2.$$

2. Relationship between the Fourier transform and LCFT

In this section, we describe relationship between the Fourier transform and the Linear canonical Fourier transform (LCFT)

$$\mathcal{F}_N(f)(\lambda) = \frac{1}{\sqrt{2\pi ib}} \int_{\mathbb{R}} K_N(\lambda, x) f(x) dx$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi ib}} \int_{\mathbb{R}} e^{i\left(\frac{a}{2b}x^2 - \frac{1}{b}x\lambda + \frac{d}{2b}\lambda\right)} f(x)dx \\
&= \frac{1}{\sqrt{2\pi ib}} e^{i\frac{d}{2b}\lambda} \int_{\mathbb{R}} e^{i\frac{a}{2b}x^2} f(x) e^{-ix\frac{\lambda}{b}} dx \\
&= \frac{1}{\sqrt{ib}} e^{i\frac{d}{2b}\lambda} \mathcal{F}(h)(\frac{\lambda}{b}).
\end{aligned}
$$

Here

$$h(x) = f(x)e^{i\frac{a}{2b}x^2}. \tag{1}$$

Therefore

$$\mathcal{F}_N(f)(\lambda) = \frac{1}{\sqrt{ib}} e^{i\frac{d}{2b}\lambda} \mathcal{F}(h)(\frac{\lambda}{b}), \tag{2}$$

where $\mathcal{F}(h)(\lambda)$ is the Fourier transform of $h \in \mathrm{L}^1(\mathbb{R})$ defined by

$$\mathcal{F}(h)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\lambda} h(x)dx.$$

3. Uncertainty principles for LCFT

In this section, we prove uncertainty principles related to the LCFT. the principles can be considered as an extension of uncertainty principles for the Fourier transform ([6–8]).

3.1. Heisenberg's inequality

Theorem 1. *Let $f \in \mathrm{L}^2(\mathbb{R})$ be a complex function. If $\mathcal{F}_N(f)(\lambda) \in \mathrm{L}^2(\mathbb{R})$, we have*

$$\left(\int_{\mathbb{R}} x^2 |f(x)|^2 dx\right) \left(\int_{\mathbb{R}} \lambda^2 |\mathcal{F}_N(f)(\lambda)|^2 d\lambda\right) \geq \frac{b|b|}{4} \left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^2.$$

Proof. From the generalisation of the Heisenberg's inequality for the Fourier transform, we have

$$\left(\int_{\mathbb{R}} x^2 |f(x)|^2 dx\right) \left(\int_{\mathbb{R}} \lambda^2 |\mathcal{F}(f)(\lambda)|^2 d\lambda\right) \geq \frac{1}{4} \left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^2.$$

Since $h(x)$ defined by (1) belongs to $\mathrm{L}^2(\mathbb{R})$, then by remplacing f by h, we obtain

$$\left(\int_{\mathbb{R}} x^2 |h(x)|^2 dx\right) \left(\int_{\mathbb{R}} \lambda^2 |\mathcal{F}(h)(\lambda)|^2 d\lambda\right) \geq \frac{1}{4} \left(\int_{\mathbb{R}} |h(x)|^2 dx\right)^2,$$

subsituting λ by $\frac{\lambda}{b}$, we have

$$\left(\int_{\mathbb{R}} x^2 |h(x)|^2 dx\right) \left(\int_{\mathbb{R}} \left(\frac{\lambda}{b}\right)^2 |\mathcal{F}(h)(\frac{\lambda}{b})|^2 d(\frac{\lambda}{b})\right) \geq \frac{1}{4} \left(\int_{\mathbb{R}} |h(x)|^2 dx\right)^2.$$

From formula (2) we get

$$\left(\int_{\mathbb{R}} x^2 |f(x)e^{i\frac{a}{2b}x^2}|^2 dx\right) \left(\int_{\mathbb{R}} \frac{1}{b^3} \lambda^2 |\sqrt{ib}\mathcal{F}_N(f)(\lambda)|^2 d\lambda\right) \geq \frac{1}{4} \left(\int_{\mathbb{R}} |f(x)e^{i\frac{a}{2b}x^2}|^2 dx\right)^2.$$

Then

$$\left(\int_{\mathbb{R}} x^2 |f(x)|^2 dx\right) \left(\int_{\mathbb{R}} \lambda^2 |\mathcal{F}_N(f)(\lambda)|^2 d\lambda\right) \geq \frac{b|b|}{4} \left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^2,$$

and the proof is complete □

3.2. Hausdorff-Young inequality

Theorem 2. *Let* $1 \le p \le 2$ *and assume that* $f \in L^p(\mathbb{R})$. *Then*

$$\left(\int_{\mathbb{R}} |\mathcal{F}_N(f)(\lambda)|^q d\lambda\right)^{1/q} \le |\sqrt{ib}| b^{-\frac{1}{q}} p^{\frac{1}{2p}} q^{-\frac{1}{2q}} \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{1/p},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ *and* $b > 0$.

Proof. Applying the relation Hausdorff-Young inequality related to the Fourier transform

$$\left(\int_{\mathbb{R}} |\mathcal{F}(f)(\lambda)|^q d\lambda\right)^{1/q} \le p^{\frac{1}{2p}} q^{-\frac{1}{2q}} \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{1/p}, \forall f \in L^p(\mathbb{R}).$$

Replacing f by h defined by (1), we obtain

$$\left(\int_{\mathbb{R}} |\mathcal{F}(h)(\lambda)|^q d\lambda\right)^{1/q} \le p^{\frac{1}{2p}} q^{-\frac{1}{2q}} \left(\int_{\mathbb{R}} |h(x)|^p dx\right)^{1/p}.$$

Then

$$\left(\int_{\mathbb{R}} |\mathcal{F}(h)(\frac{\lambda}{b})|^q d(\frac{\lambda}{b})\right)^{1/q} \le p^{\frac{1}{2p}} q^{-\frac{1}{2q}} \left(\int_{\mathbb{R}} |h(x)|^p dx\right)^{1/p}.$$

From the formulas (1) and (2), we have

$$\frac{1}{b^{\frac{1}{q}}} \left(\int_{\mathbb{R}} |\sqrt{ib}\mathcal{F}_N(f)(\lambda)|^q d\lambda\right)^{1/q} \le p^{\frac{1}{2p}} q^{-\frac{1}{2q}} \left(\int_{\mathbb{R}} |f(x)e^{i\frac{a}{2b}x^2}|^p dx\right)^{1/p}.$$

Hence

$$\left(\int_{\mathbb{R}} |\mathcal{F}_N(f)(\lambda)|^q d\lambda\right)^{1/q} \le |\sqrt{ib}| b^{-\frac{1}{q}} p^{\frac{1}{2p}} q^{-\frac{1}{2q}} \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{1/p}.$$

Thus proof is completed. □

3.3. Matolcsi-Szucs inequality

Theorem 3. *Let* $f \in L^{p_1}(\mathbb{R}) \cap L^{p_2}(\mathbb{R})$ *such that* $1 < p_2 \le p_2 \le 2$, *then*

$$\|\mathcal{F}_N(f)\|_{q_2} \le |\sqrt{ib}| b^{-\frac{1}{q_1}} p_1^{\frac{1}{2p_1}} q_1^{-\frac{1}{2q_1}} |supp\mathcal{F}_N(f)|^{\frac{q_1-q_2}{q_1q_2}} |supp(f)|^{\frac{p_2-p_1}{p_1p_2}} \|f\|_{p_2},$$

where $\frac{1}{p_1} + \frac{1}{q_1} = 1$, $\frac{1}{p_2} + \frac{1}{q_2} = 1$ *and* $b > 0$.

Proof. Applying Hausdorff inequality for the LCFT and Hölder's inequality, we obtain

$$\begin{aligned}\|\mathcal{F}_N(f)\|_{q_2} \le& |supp\mathcal{F}_N(f)|^{\frac{q_1-q_2}{q_1q_2}} \|\mathcal{F}_N(f)\|_{q_2}\\ \le& |supp\mathcal{F}_N(f)|^{\frac{q_1-q_2}{q_1q_2}} |\sqrt{ib}| b^{-\frac{1}{q_1}} p_1^{\frac{1}{2p_1}} q_1^{-\frac{1}{2q_1}} \|f\|_{p_1},\end{aligned}$$

and

$$\begin{aligned}\|f\|_{p_1} =& \|\chi_{supp(f)} f\|_{p_1}\\ \le& \left(\int_R |\chi_{supp(f)}| dx\right)^{\frac{p_2-p_1}{p_1p_2}} \left(\int_{\mathbb{R}} |f(x)|^{\frac{p_1p_2}{p_1}} dx\right)^{\frac{p_1}{p_1p_2}}\\ =& |supp(f)|^{\frac{p_2-p_1}{p_1p_2}} \|f\|_{p_2}.\end{aligned}$$

Then
$$\|\mathcal{F}_N(f)\|_{q_2} \leq |\sqrt{ib}|b^{-\frac{1}{q_1}} p_1^{\frac{1}{2p_1}} q_1^{-\frac{1}{2q_1}} |supp\mathcal{F}_N(f)|^{\frac{q_1-q_2}{q_1q_2}} |supp(f)|^{\frac{p_2-p_1}{p_1p_2}} \|f\|_{p_2}.$$

Thus proof is completed. □

4. Some inequalities for the LCFT

In [4], Soltani and Ghazwani proved the L^p uncertainty inequalities for the Fourier transform. In this section, we prove the generalization of thse results for the LCFT. The authors proved the following theorems(see [4]).

Theorem 4 (Nash-type inequality). *Les $s > 0$. If $1 < p \leq 2$, $q = \frac{p}{p-1}$ and $f \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, then*

$$\|\mathcal{F}(f)\|_q \leq K(s,p,q)\|f\|_1^{\frac{qs}{d+qs}} \||y|^s\mathcal{F}(f)\|_q^{\frac{d}{d+qs}},$$

where

$$K(s,p,q) = \frac{\left[(\frac{qs}{d})^{\frac{d}{d+qs}} + (\frac{d}{qs})^{\frac{qs}{d+qs}} \right]^{1/q}}{\left[2^{\frac{d}{2}}\Gamma(\frac{d}{2}+1) \right]^{\frac{s}{d+qs}}},$$

Theorem 5 (Clarkson-type inequality). *Let $s > 0$. If $1 < p \leq 2$, $q = \frac{p}{p-1}$ and $f \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, then*

$$\|f\|_1 \leq D(s,p,q)\|f\|_p^{\frac{qs}{d+qs}} \||x|^s f\|_1^{\frac{d}{d+qs}},$$

where

$$D(s,p,q) = \frac{(\frac{qs}{d})^{\frac{d}{d+qs}} + (\frac{d}{qs})^{\frac{qs}{d+qs}}}{\left[2^{\frac{d}{2}}\Gamma(\frac{d}{2}+1) \right]^{\frac{s}{d+qs}}}.$$

Theorem 6 (Nash-type inequality for LCFT). *Let $s > 0$. If $1 < p \leq 2$, $q = \frac{p}{p-1}$ and $f \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$, then*

$$\|\mathcal{F}_N(f)\|_q \leq K(s,p,q)\|f\|_1^{\frac{qs}{1+qs}} \||\lambda|^s\mathcal{F}_N(f)\|_q^{\frac{1}{1+qs}},$$

where

$$K(s,p,q) = \left(\frac{1}{|\sqrt{ib}|} \right)^{\frac{s}{1+qs}} \frac{\left[(qs)^{\frac{1}{1+qs}} + (\frac{1}{qs})^{\frac{qs}{1+qs}} \right]^{1/q}}{\left[\sqrt{2}\Gamma(\frac{3}{2}) \right]^{\frac{s}{1+qs}}}.$$

Proof. Let $f \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$, $1 < p \leq 2$ and $s > 0$. Since h defined by formula (1) belongs to $L^1(\mathbb{R}) \cap L^p(\mathbb{R})$, then by replacing f by h in the formula of theorem 4, we obtain

$$\|\mathcal{F}(h)\|_q \leq K(s,p,q)\|f\|_1^{\frac{qs}{1+qs}} \||\lambda|^s\mathcal{F}(h)\|_q^{\frac{1}{1+qs}},$$

i.e

$$\left(\int_{\mathbb{R}} |\mathcal{F}(h)(\lambda)|^q d\lambda \right)^{\frac{1}{q}} \leq K(s,p,q) \left(\int_{\mathbb{R}} |h(x)|dx \right)^{\frac{qs}{1+qs}} \left(\int_{\mathbb{R}} |\lambda|^s |\mathcal{F}(h)(\lambda)|^q d\lambda \right)^{\frac{1}{q(1+qs)}}.$$

Substituting with λ with $\frac{\lambda}{b}$.

$$\left(\int_{\mathbb{R}} |\mathcal{F}(h)(\frac{\lambda}{b})|^q d(\frac{\lambda}{b}) \right)^{\frac{1}{q}} \leq K(s,p,q) \left(\int_{\mathbb{R}} |h(x)|dx \right)^{\frac{qs}{1+qs}} \left(\int_{\mathbb{R}} \frac{|\lambda|^s}{b^s} |\mathcal{F}(h)(\frac{\lambda}{b})|^q d(\frac{\lambda}{b}) \right)^{\frac{1}{q(1+qs)}}.$$

From formula (2) we get

$$\frac{1}{b^{\frac{1}{q}}}\left(\int_{\mathbb{R}}|\sqrt{ib}e^{-i\frac{d}{2b}\lambda}\mathcal{F}_N(f)(\lambda)|^q d\lambda\right)^{\frac{1}{q}} \leq \frac{1}{b^{\frac{1}{q}}}K(s,p,q)\left(\int_{\mathbb{R}}|f(x)e^{i\frac{a}{2b}x^2}|dx\right)^{\frac{qs}{1+qs}}\left(\int_{\mathbb{R}}|\lambda|^s\sqrt{ib}\mathcal{F}_N(f)(\lambda)|^q d\lambda\right)^{\frac{1}{q(1+qs)}}.$$

Then

$$\left(\int_{\mathbb{R}}|\mathcal{F}_N(f)(\lambda)|^q d\lambda\right)^{\frac{1}{q}} \leq |\sqrt{ib}|^{\frac{-qs}{1+qs}}K(s,p,q)\left(\int_{\mathbb{R}}|f(x)|dx\right)^{\frac{qs}{1+qs}}\left(\int_{\mathbb{R}}|\lambda|^s\mathcal{F}_N(f)(\lambda)|^q d\lambda\right)^{\frac{1}{q(1+qs)}}.$$

Thus we have

$$\|\mathcal{F}_N(f)\|_q \leq |\sqrt{ib}|^{\frac{-qs}{1+qs}}K(s,p,q)\|f\|_1^{\frac{qs}{1+qs}}\||\lambda|^s\mathcal{F}_N(f)(\lambda)\|_q^{\frac{1}{1+qs}},$$

this completes the proof of theorem. □

By combining the Nash-type inequality for LCFT and Clarkson-type inequality, we obtain the following uncertainty of Heisenberg-type

Theorem 7. *Let $\alpha,\beta>0$. If $1<p\leq 2$, $q=\frac{p}{p-1}$ and $f\in L^1(\mathbb{R})\cap L^p(\mathbb{R})$, then*

(i)

$$\|f\|_1^{\frac{1}{1+q\beta}}\|\mathcal{F}_N(f)\|_q\|f\|_p^{\frac{-q\alpha}{1+q\alpha}} \leq C(p,q)\|x^\alpha f\|_1^{\frac{1}{1+q\alpha}}\|\lambda^\beta\mathcal{F}_N(f)\|_q^{\frac{1}{1+q\beta}},$$

where $C(p,q)=D(\alpha,p,q)K(\beta,p,q)\left(\frac{b}{|\sqrt{ib}|}\right)^{\frac{\beta}{1+q\beta}}$,

(ii)

$$\|f\|_p^{\frac{-q\alpha}{1+q\alpha}}\|\mathcal{F}_N(f)\|_q^{\frac{1+q\beta}{q\beta}} \leq D(\alpha,p,q)K(\beta,p,q)^{\frac{1+q\beta}{q\beta}}\|x^\alpha f\|_1^{\frac{1}{1+q\alpha}}\|\lambda^\beta\mathcal{F}_N(f)\|_q^{\frac{1}{q\beta}}.$$

5. Hardy's theoremù and Miyach's theorem for the LCFT

In 1933, Hardy [8] established the following theorem

Theorem 8. *If $|f(x)|\leq Ce^{-\alpha x^2}$ and $|\mathcal{F}(f)(\lambda)|\leq Ce^{-\beta\lambda^2}$, for some positive numbers α,β and C then $f=0$ whenever $\alpha\beta>\frac{1}{4}$. If $\alpha\beta=\frac{1}{4}$ the function f is a constant multiple of $e^{-\alpha x^2}$.*

Another generalization of Hardy's theorem is given by Miyach [9]. In 1997, he proved the following theorem

Theorem 9. *Let f be an integrable function on $\mathbb{R}$ such that*

$$e^{\alpha x^2}f(x)\in L^1(\mathbb{R})+L^\infty(\mathbb{R}).$$

Further, assume that

$$\int_{\mathbb{R}}\log^+\left(\frac{|e^{\beta\lambda^2}\mathcal{F}(f)(\lambda)|}{C}\right)d\lambda<+\infty,$$

for some positive numbers α,β and C. If $\alpha\beta=\frac{1}{4}$, then f is a constant multiple of $e^{-\alpha x^2}$.

The aim of this section is establish Hardy's and Miyach's theorems for the LCFT.

Theorem 10. *Let f be a mesurable function on $\mathbb{R}$ such that*

$$|f(x)|\leq Ce^{-\alpha b^2x^2} \ \ and \ \ |\mathcal{F}(f)(\lambda)|\leq Ce^{-\beta\lambda^2},$$

for some constants $a,b,C>0$. Then

(i) *If $\alpha\beta>\frac{1}{4b^2}$, then $f=0$*

(ii) *If $\alpha\beta=\frac{1}{4b^2}$; then $f=cte\times e^{(-\alpha+i\frac{a}{2b})x^2}$.*

Proof. From the formulas (1) and(2), we obtain

$$|h(x)e^{-i\frac{a}{2b}x^2}| \leq Ce^{-\alpha b^2x^2} \;\; and \;\; |e^{-\frac{d}{2b}\lambda}\mathcal{F}(h)(\frac{\lambda}{b}| \leq Ce^{-\beta\lambda^2},$$

i.e

$$|h(x)| \leq Ce^{-\alpha x^2 b^2} \;\; and \;\; |\mathcal{F}(f)(\delta)| \leq Ce^{-\beta b^2 \delta}, \; (\delta = \frac{\lambda}{b}).$$

By Theorem 8, we conclude that: If $\alpha\beta > \frac{1}{4b^2}$ then $h(x) = 0$ therefore $f = 0$ and if $\alpha\beta = \frac{1}{4b^2}$ then $h(x) = cte \times e^{-\alpha x^2}$, i.e $f(x) = cte \times e^{(-\alpha - i\frac{a}{2b})x^2}$. □

Now, we generalize Miyach's theorem for LCFT

Theorem 11. *Let f be an integrable function on $\mathbb{R}$ such that*

$$e^{(\alpha + i\frac{a}{2b}x^2)x^2} f \in \mathrm{L}^1(\mathbb{R}) + \mathrm{L}^\infty(\mathbb{R}).$$

Further, assume that

$$\int_{\mathbb{R}} \log^+\left(\frac{|e^{\beta\lambda^2}\mathcal{F}_N(f)(b\lambda)|}{C}\right) d\lambda < +\infty,$$

for some positive constants α, β and C . If $\alpha\beta = \frac{1}{4}$, then $f(x) = cte \times e^{(-\alpha - i\frac{a}{2b})x^2}$.

Proof. By (1) and (2), we have

$$e^{\alpha x^2} h \in \mathrm{L}^1(\mathbb{R}) + \mathrm{L}^\infty(\mathbb{R}),$$

and

$$\int_{\mathbb{R}} \log^+\left(\frac{|e^{\beta\lambda^2}\mathcal{F}(h)(\lambda)|}{C}\right) d\lambda < +\infty.$$

Moreover, by using the classical Miyach's Theorem 9, we obtain $h(x) = cte \times e^{-\alpha x^2}$ Thus, f(x) is a constant multiple of $e^{(-\alpha - i\frac{a}{2b})x^2}$. Theorem 11 is proved. □

6. Beurling's theorem for the LCFT

In this section, we obtain an analog of the classical Beurling theorem on the properties of the Fourier transform. Let us present a precise formulation of this theorem.

Theorem 12. *Let $M \geq 0$. Assume that $f \in \mathrm{L}^2(\mathbb{R})$ satisfying*

$$\int_{\mathbb{R}}\int_{R} \frac{|f(x)||\mathcal{F}(f)(\lambda)|}{(1 + |x| + |\lambda|)^M} e^{|x||\lambda|} dx d\lambda < +\infty,$$

then $f(x) = P(x)e^{-ax^2}$, where P is a polynomial of de degree $< \frac{M-1}{2}$ and $a > 0$.

Our main result is as follows.

Theorem 13. *Let $M > 0$. Assumme that $f \in \mathrm{L}^2(\mathbb{R})$ satisfying*

$$\int_{\mathbb{R}}\int_{R} \frac{|f(x)||\mathcal{F}_N(f)(b\lambda)|}{(1 + |x| + |\lambda|)^M} e^{|x||\lambda|} dx d\lambda < +\infty,$$

then $f(x) = P(x)e^{(-\alpha - i\frac{a}{2b})x^2}$, where P is a polynomial of de degree $< \frac{M-1}{2}$ and $\alpha > 0$.

Proof. It follows from (1) and (2) that

$$\int_{\mathbb{R}}\int_{R} \frac{|h(x)e^{-i\frac{a}{2b}x^2}||e^{i\frac{d}{2b}}\mathcal{F}(f)(\lambda)|}{(1 + |x| + |\lambda|)^M} e^{|x||\lambda|} dx d\lambda = \int_{\mathbb{R}}\int_{R} \frac{|h(x)||\mathcal{F}(f)(\lambda)|}{(1 + |x| + |\lambda|)^M} e^{|x||\lambda|} dx d\lambda < +\infty.$$

According to Theorem 12, we concthat

$$h(x) = P(x)e^{-\alpha x^2},$$

P is a polynomial of de degree $< \frac{M-1}{2}$ and $\alpha > 0$. On the other hand, by (1) we obtain

$$f(x) = P(x)e^{(-\alpha - i\frac{a}{2b})x^2},$$

and the theorem is proved. $\square$

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