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OSTROWSKI TYPE FRACTIONAL INTEGRAL INEQUALITIES FOR S-GODUNOVA-LEVIN FUNCTIONS VIA KATUGAMPOLA FRACTIONAL INTEGRALS

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ABSTRACT. In this paper, we give some fractional integral inequalities of Ostrowski type for s-Godunova-Levin functions via Katugampola fractional integrals. We also deduce some known Ostrowski type fractional integral inequalities for Riemann-Liouville fractional integrals.

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1. INTRODUCTION

In 1938 Ostrowski [1] proved an inequality stated in the following result (see also [2, p.468]).

Theorem 1.1. *Let $f : I \to \mathbb{R}$ where I is interval in $\mathbb{R}$, be a mapping differentiable in I° the interior of I and $a, b \in I^\circ$, $a < b$. If $\left|f^{'}(t)\right| \leq M$, for all $t \in [a, b]$, then we have*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}\right](b-a)M, x \in [a, b].$$

Ostrowski inequality gives bounds of integral average of a function f over an interval $[a, b]$ to its value $f(x)$ at point $x \in [a, b]$. Ostrowski and Ostrowski type inequalities have great importance in numerical analysis as they provide the error bound of many quaderature rules [3]. Therefore in recent years, so many

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such type of inequalities have been obtained and generalized (see [4, 5]).
As fractional calculus is a generalization of classical calculus concerned with operations of integration and differentiation of fractional order so in this research article we will use Katugampola fractional integrals to generalize the Ostrowski type inequalities given in [4].
In [6] Laurent give definition of Riemann-Liouville fractional integrals.

Definition 1.2. [6] Let $f \in L_1[a,b]$. The Riemann-Liouville fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, x < b,$$

respectively, where $\Gamma(\alpha) = \int_0^{\infty} e^{-u}u^{\alpha-1}du$. Here $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$. In case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Definition 1.3. J. Hadamard introduced the Hadamard fractional integral in [7], and is given by

$$I^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(log\frac{x}{\tau}\right)^{\alpha-1} f(\tau)\frac{d\tau}{\tau},$$

for $Re(\alpha) > 0$, $x > a \geq 0$.

Recently Katugampola generalized Riemann-Liouville and Hadamard fractional integrals into a single form called Katugampola fractional integrals.

Definition 1.4. [8] Let $[a,b]$ be a finite interval in $\mathbb{R}$. Then Katugampola fractional integrals of order $\alpha > 0$ for a real valued function f are defined by

$${}^{\rho}I^{\alpha}_{a+}f\left(x\right) = \frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)} \int_a^x t^{\rho-1}\left(x^{\rho}-t^{\rho}\right)^{\alpha-1} f\left(t\right)dt$$

and

$${}^{\rho}I^{\alpha}_{b-}f\left(x\right) = \frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)} \int_x^b t^{\rho-1}\left(t^{\rho}-x^{\rho}\right)^{\alpha-1} f\left(t\right)dt$$

with $a < x < b$ and $\rho > 0$.
Where $\Gamma\left(\alpha\right)$ is the Euler gamma function. For $\rho = 1$, Katugampola fractional integrals give Riemann-Liouville fractional integrals, while $\rho \to 0^+$ produces the Hadamard fractional integral. For its proof one can check [8].

The ρ-Gamma function [9] for any two positive numbers x, y denoted by ${}^{\rho}\Gamma(x,y)$, is defined by

$${}^{\rho}\Gamma(\alpha) = \int_0^{\infty} e^{-t^{\rho}}(t^{\rho})^{\alpha-\frac{1}{\rho}}dt.$$

We can have the following relation

$$^{\rho}\beta(x,y)=\frac{^{\rho}\Gamma(x)\ ^{\rho}\Gamma(y)}{^{\rho}\Gamma(x,y)}. \tag{1}$$

Definition 1.5. [10] A non-negative function $f:I\to\mathbb{R}$ is said to be p-function, if for any two points $x,y\in I$ and $t\in[0,1]$

$$f\left(tx+(1-t)y\right)\le f(x)+f(y).$$

Definition 1.6. [11] A function $f:I\to\mathbb{R}$ is said to be Godunova-Levin function, if for any two points $x,y\in I$ and $t\in(0,1)$

$$f\left(tx+(1-t)y\right)\le\frac{f(x)}{t}+\frac{f(y)}{1-t}.$$

Definition 1.7. [12] A function $f:I\to\mathbb{R}$ is said to be s-Godunova-Levin function of first kind, if $s\in(0,1]$, for all $x,y\in I$ and $t\in(0,1)$ then we have

$$f\left(tx+(1-t)y\right)\le\frac{f(x)}{t^s}+\frac{f(y)}{1-t^s}.$$

Definition 1.8. [13] A function $f:I\to\mathbb{R}$ is said to be s-Godunova-Levin function of second kind, if $s\in[0,1]$, for all $x,y\in I$ and $t\in(0,1)$ then we have

$$f\left(tx+(1-t)y\right)\le\frac{f(x)}{t^s}+\frac{f(y)}{(1-t)^s}.$$

We organize the paper in such a way that in the following section we prove some Ostrowski type fractional integral inequalities for s-Godunova-Levin functions of second kind via Katugampola fractional integrals. Also we will obtain some corollaries for p-functions and Godunova-Levin functions and deduce some known results of [4].

2. Ostrowski type fractional integral inequalities for mappings whose derivatives are s-Godunova-Levin of second kind via Katugampola fractional integrals

The following lemma (given and also proved in [9]) is very useful to obtain our results.

Lemma 2.1. *Let $f:[a^\rho,b^\rho]\to\mathbb{R}$ be a differentiable mapping on (a^ρ,b^ρ) with $a<b$ such that $f^{'}\in L_1[a,b]$, where $\rho>0$. Then we have the following equality*

$$\begin{aligned}&\left(\frac{(x^\rho-a^\rho)^\alpha+(b^\rho-x^\rho)^\alpha}{b-a}\right)f(x^\rho)-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)}\times\\&[^{\rho}I^{\alpha}_{x^-}f(a^\rho)+\ ^{\rho}I^{\alpha}_{x^+}f(b^\rho)]\\&=\frac{\rho(x^\rho-a^\rho)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha\rho+\rho-1}f^{'}(t^\rho x^\rho+(1-t^\rho)a^\rho)dt\\&-\frac{\rho(b^\rho-x^\rho)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha\rho+\rho-1}f^{'}(t^\rho x^\rho+(1-t^\rho)b^\rho)dt;\ x\in[a,b].\end{aligned} \tag{2}$$

Theorem 2.2. *Let $f : [a^\rho, b^\rho] \to \mathbb{R}$, $a, b \geq 0$, $a < b$ be a differentiable function on (a^ρ, b^ρ) and $f^{'} \in L_1[a,b]$. If $\left|f^{'}\right|$ is s-Godunova-Levin function of second kind and $\left|f^{'}(x^\rho)\right| \leq M$, $x \in [a,b]$, then the following inequality holds*

$$\left| \left(\frac{(x^\rho - a^\rho)^\alpha + (b^\rho - x^\rho)^\alpha}{b-a} \right) f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)} \times \right.$$
$$\left. [{}^\rho I^\alpha_{x^-} f(a^\rho) + {}^\rho I^\alpha_{x^+} f(b^\rho)] \right| \leq M \left[\frac{(x^\rho - a^\rho)^{\alpha+1} + (b^\rho - x^\rho)^{\alpha+1}}{b-a} \right] \times$$
$$\left[\frac{1}{\alpha + 1 - s} + \frac{{}^\rho\Gamma(\alpha+1)\ {}^\rho\Gamma(1-s)}{{}^\rho\Gamma(\alpha+2-s)} \right];\ x \in [a,b]. \tag{3}$$

Proof. Using Lemma 2.1 and the fact that $\left|f^{'}\right|$ is s-Godunova-Levin function of second kind, we have

$$\left| \left(\frac{(x^\rho - a^\rho)^\alpha + (b^\rho - x^\rho)^\alpha}{b-a} \right) f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)} \times \right.$$
$$\left. [{}^\rho I^\alpha_{x^-} f(a^\rho) + {}^\rho I^\alpha_{x^+} f(b^\rho)] \right|$$
$$\leq \frac{\rho(x^\rho - a^\rho)^{\alpha+1}}{b-a} \int_0^1 t^{\alpha\rho+\rho-1} \left|f^{'}(t^\rho x^\rho + (1-t^\rho)a^\rho)\right| dt$$
$$+ \frac{\rho(b^\rho - x^\rho)^{\alpha+1}}{b-a} \int_0^1 t^{\alpha\rho+\rho-1} \left|f^{'}(t^\rho x^\rho + (1-t^\rho)b^\rho)\right| dt$$
$$\leq \frac{\rho(x^\rho - a^\rho)^{\alpha+1}}{b-a} \int_0^1 \left[\frac{t^{\alpha\rho+\rho-1}}{(t^\rho)^s} \left|f^{'}(x^\rho)\right| + \frac{t^{\alpha\rho+\rho-1}}{(1-t^\rho)^s} \left|f^{'}(a^\rho)\right| \right] dt$$
$$+ \frac{\rho(b^\rho - x^\rho)^{\alpha+1}}{b-a} \int_0^1 \left[\frac{t^{\alpha\rho+\rho-1}}{(t^\rho)^s} \left|f^{'}(x^\rho)\right| + \frac{t^{\alpha\rho+\rho-1}}{(1-t^\rho)^s} \left|f^{'}(b^\rho)\right| \right] dt$$
$$\leq \frac{M\rho(x^\rho - a^\rho)^{\alpha+1}}{b-a} \int_0^1 \left[\frac{t^{\alpha\rho+\rho-1}}{(t^\rho)^s} + \frac{t^{\alpha\rho+\rho-1}}{(1-t^\rho)^s} \right] dt$$
$$+ \frac{M\rho(b^\rho - x^\rho)^{\alpha+1}}{b-a} \int_0^1 \left[\frac{t^{\alpha\rho+\rho-1}}{(t^\rho)^s} + \frac{t^{\alpha\rho+\rho-1}}{(1-t^\rho)^s} \right] dt$$
$$= M\rho \left[\frac{(x^\rho - a^\rho)^{\alpha+1} + (b^\rho - x^\rho)^{\alpha+1}}{2(b-a)} \right] \times$$
$$\int_0^1 \left[t^{\alpha\rho - \rho s + \rho - 1} + t^{\alpha\rho+\rho-1}(1-t^\rho)^{-s} \right] dt.$$
$$= M\rho \left[\frac{(x^\rho - a^\rho)^{\alpha+1} + (b^\rho - x^\rho)^{\alpha+1}}{2(b-a)} \right] \times$$
$$\left[\frac{1}{\rho(\alpha+1-s)} + \frac{{}^\rho\Gamma(\alpha+1)\ {}^\rho\Gamma(1-s)}{\rho\ {}^\rho\Gamma(\alpha+2-s)} \right]$$
$$= M \left[\frac{(x^\rho - a^\rho)^{\alpha+1} + (b^\rho - x^\rho)^{\alpha+1}}{b-a} \right] \times$$

$$\left[\frac{1}{\alpha+1-s}+\frac{{}^{\rho}\Gamma(\alpha+1)\ {}^{\rho}\Gamma(1-s)}{{}^{\rho}\Gamma(\alpha+2-s)}\right].$$

Here we use (1). The proof is completed. □

Remark 2.3. *(i) If we put $\rho = 1$ in (3), then we get* [4, Theorem 3.1].
(ii) If we put $\rho = 1$ and $\alpha = 1$ in (3), then we get [4, Corollary 3.1].

Corollary 2.4. *In Theorem 2.2 , if we take $s = 0$, which means that $\left|f^{'}\right|$ is p-function, then (3) becomes the following inequality*

$$\Bigg|\left(\frac{(x^{\rho}-a^{\rho})^{\alpha}+(b^{\rho}-x^{\rho})^{\alpha}}{b-a}\right)f(x^{\rho})-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)}\times$$
$$\left[{}^{\rho}I^{\alpha}_{x^-}f(a^{\rho})+\ {}^{\rho}I^{\alpha}_{x^+}f(b^{\rho})\right]\Bigg|$$
$$\leq\frac{2M}{\alpha+1}\left[\frac{(x^{\rho}-a^{\rho})^{\alpha+1}+(b^{\rho}-x^{\rho})^{\alpha+1}}{b-a}\right];\ x\in[a,b].$$

Corollary 2.5. *In Theorem 2.2, if we take $s = 1$, which means that $\left|f^{'}\right|$ is Godunova-Levin function, then (3) becomes the following inequality*

$$\Bigg|\left(\frac{(x^{\rho}-a^{\rho})^{\alpha}+(b^{\rho}-x^{\rho})^{\alpha}}{b-a}\right)f(x^{\rho})-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)}\times$$
$$\left[{}^{\rho}I^{\alpha}_{x^-}f(a^{\rho})+\ {}^{\rho}I^{\alpha}_{x^+}f(b^{\rho})\right]\Bigg|$$
$$\leq\frac{M(\alpha+1)}{\alpha}\left[\frac{(x^{\rho}-a^{\rho})^{\alpha+1}+(b^{\rho}-x^{\rho})^{\alpha+1}}{b-a}\right];\ x\in[a,b].$$

Theorem 2.6. *Let $f : [a^{\rho}, b^{\rho}] \to \mathbb{R}$, $a, b \geq 0$, $a < b$ be a differentiable function on (a^{ρ}, b^{ρ}) and $f^{'} \in L_1[a, b]$. If $\left|f^{'}\right|^{q}$, is s-Godunova-Levin function of second kind and $\left|f^{'}(x^{\rho})\right| \leq M$, $x \in [a, b]$ then the following inequality for Katugampola fractional integrals holds*

$$\Bigg|\left(\frac{(x^{\rho}-a^{\rho})^{\alpha}+(b^{\rho}-x^{\rho})^{\alpha}}{b-a}\right)f(x^{\rho})-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)}\times$$
$$\left[{}^{\rho}I^{\alpha}_{x^-}f(a^{\rho})+\ {}^{\rho}I^{\alpha}_{x^+}f(b^{\rho})\right]\Bigg|$$
$$\leq M\rho\left[\frac{(x^{\rho}-a^{\rho})^{\alpha+1}+(b^{\rho}-x^{\rho})^{\alpha+1}}{(b-a)(1+p(\alpha\rho+\rho-1))^{\frac{1}{p}}}\right]\left[\frac{1}{1-\rho s}\right]^{\frac{1}{q}};\ x\in[a,b], \tag{4}$$

with $\frac{1}{p}+\frac{1}{q}=1$ where $q > 1$.

Proof. Using Lemma 2.1 and then Holder's inequality, we have

$$\Bigg|\left(\frac{(x^{\rho}-a^{\rho})^{\alpha}+(b^{\rho}-x^{\rho})^{\alpha}}{b-a}\right)f(x^{\rho})-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)}\times$$

$$
\begin{aligned}
&\left[{}^{\rho}I^{\alpha}_{x^-}f(a^{\rho}) + {}^{\rho}I^{\alpha}_{x^+}f(b^{\rho})\right]\Bigg| \\
&\leq \frac{\rho(x^{\rho}-a^{\rho})^{\alpha+1}}{b-a}\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^{\rho}x^{\rho}+(1-t^{\rho})a^{\rho})\right|dt \\
&+\frac{\rho(b^{\rho}-x^{\rho})^{\alpha+1}}{b-a}\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^{\rho}x^{\rho}+(1-t^{\rho})b^{\rho})\right|dt \\
&\leq \frac{\rho(x^{\rho}-a^{\rho})^{\alpha+1}}{b-a}\left(\int_0^1 t^{p(\alpha\rho+\rho-1)}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f^{'}(t^{\rho}x^{\rho}+(1-t^{\rho})a^{\rho})\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{\rho(b^{\rho}-x^{\rho})^{\alpha+1}}{b-a}\left(\int_0^1 t^{p(\alpha\rho+\rho-1)}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f^{'}(t^{\rho}x^{\rho}+(1-t^{\rho})b^{\rho})\right|^q dt\right)^{\frac{1}{q}}. \qquad (5)
\end{aligned}
$$

Since $\left|f^{'}\right|^q$ is s-Godunova-Levin function of second kind and $\left|f^{'}(x^{\rho})\right| \leq M$, we get

$$
\begin{aligned}
&\int_0^1 \left|f^{'}(t^{\rho}x^{\rho}+(1-t^{\rho})a^{\rho})\right|^q dt\times \\
&\leq \int_0^1 \left[\frac{1}{(t^{\rho})^s}\left|f^{'}(x^{\rho})\right|^q + \frac{1}{(1-t^{\rho})^s}\left|f^{'}(a^{\rho})\right|^q\right]dt \\
&\leq M^q\int_0^1 \left[\frac{1}{(t^{\rho})^s}+\frac{1}{(1-t^{\rho})^s}\right]dt = \frac{1}{1-\rho s} \qquad (6)
\end{aligned}
$$

similarly

$$
\int_0^1 \left|f^{'}(t^{\rho}x^{\rho}+(1-t^{\rho})b^{\rho})\right|^q dt \leq \frac{1}{1-\rho s}. \qquad (7)
$$

We also have

$$
\int_0^1 t^{p(\alpha\rho+\rho-1)}dt = \frac{1}{1+p(\alpha\rho+\rho-1)}. \qquad (8)
$$

Using (6), (7) and (8) in (5) we can get (4). □

Remark 2.7. *(i) If we put $\rho = 1$ in (4), then we get* [4, Theorem 3.2].
(ii) If we put $\rho = 1$ and $\alpha = 1$ in (4), then we get [4, Corollary 3.2].

Corollary 2.8. *In Theorem 2.6, if we take $s = 0$, which means that $\left|f^{'}\right|$ is p-function, then (4) becomes the following inequality*

$$
\begin{aligned}
&\left|\left(\frac{(x^{\rho}-a^{\rho})^{\alpha}+(b^{\rho}-x^{\rho})^{\alpha}}{b-a}\right)f(x^{\rho}) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)}\times\right. \\
&\left[{}^{\rho}I^{\alpha}_{x^-}f(a^{\rho}) + {}^{\rho}I^{\alpha}_{x^+}f(b^{\rho})\right]\Bigg| \\
&\leq \frac{M}{(1+p(\alpha\rho+\rho-1))^{\frac{1}{p}}}\left[\frac{(x^{\rho}-a^{\rho})^{\alpha+1}+(b^{\rho}-x^{\rho})^{\alpha+1}}{b-a}\right];\ x\in[a,b].
\end{aligned}
$$

Corollary 2.9. *In Theorem , if we take $s = 1$, which means that $\left|f^{'}\right|$ is Godunova-Levin function, then (4) becomes the following inequality*

$$\Bigg|\left(\frac{(x^\rho - a^\rho)^\alpha + (b^\rho - x^\rho)^\alpha}{b-a}\right) f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)} \times$$
$$\left[{}^\rho I^\alpha_{x^-} f(a^\rho) + {}^\rho I^\alpha_{x^+} f(b^\rho)\right]\Bigg| \le \frac{M\rho}{(1 + p(\alpha\rho + \rho - 1))^{\frac{1}{p}}} \times$$
$$\left[\frac{(x^\rho - a^\rho)^{\alpha+1} + (b^\rho - x^\rho)^{\alpha+1}}{b-a}\right] \left[\frac{1+\alpha}{\alpha\rho}\right]^{\frac{1}{q}}; \; x \in [a, b].$$

Theorem 2.10. *Let $f : [a^\rho, b^\rho] \to \mathbb{R}$, $a, b \ge 0$, $a < b$ be a differentiable function on (a^ρ, b^ρ) and $f^{'} \in L_1[a, b]$. If $\left|f^{'}\right|^q$ is s-Godunova-Levin function of second kind and $\left|f^{'}(x^\rho)\right| \le M$, $x \in [a, b], q \ge 1$, then the following inequality for Katugampola fractional integrals holds*

$$\Bigg|\left(\frac{(x^\rho - a^\rho)^\alpha + (b^\rho - x^\rho)^\alpha}{b-a}\right) f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)} \times$$
$$\left[{}^\rho I^\alpha_{x^-} f(a^\rho) + {}^\rho I^\alpha_{x^+} f(b^\rho)\right]\Bigg|$$
$$\le \frac{M\rho}{(\alpha\rho + \rho)^{1-\frac{1}{q}}} \left[\frac{(x^\rho - a^\rho)^{\alpha+1} + (b^\rho - x^\rho)^{\alpha+1}}{b-a}\right] \times$$
$$\left(\frac{1}{\rho(\alpha - s + 1)} + \frac{{}^\rho\Gamma(\alpha + 1)\, {}^\rho\Gamma(1 - s)}{\rho\, {}^\rho\Gamma(\alpha - s + 2)}\right)^{\frac{1}{q}}; \; x \in [a, b]. \tag{9}$$

Proof. Using Lemma 2.1 and power mean inequality, we have

$$\Bigg|\left(\frac{(x^\rho - a^\rho)^\alpha + (b^\rho - x^\rho)^\alpha}{b-a}\right) f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)} \times$$
$$\left[{}^\rho I^\alpha_{x^-} f(a^\rho) + {}^\rho I^\alpha_{x^+} f(b^\rho)\right]\Bigg|$$
$$\le \frac{\rho(x^\rho - a^\rho)^{\alpha+1}}{b-a} \int_0^1 t^{\alpha\rho+\rho-1} \left|f^{'}(t^\rho x^\rho + (1 - t^\rho)a^\rho)\right| dt$$
$$+ \frac{\rho(b^\rho - x^\rho)^{\alpha+1}}{b-a} \int_0^1 t^{\alpha\rho+\rho-1} \left|f^{'}(t^\rho x^\rho + (1 - t^\rho)b^\rho)\right| dt$$
$$\le \frac{\rho(x^\rho - a^\rho)^{\alpha+1}}{b-a} \left(\int_0^1 t^{\alpha\rho+\rho-1} dt\right)^{1-\frac{1}{q}} \times$$
$$\left(\int_0^1 t^{\alpha\rho+\rho-1} \left|f^{'}(t^\rho x^\rho + (1 - t^\rho)a^\rho)\right|^q dt\right)^{\frac{1}{q}}$$
$$+ \frac{\rho(b^\rho - x^\rho)^{\alpha+1}}{b-a} \left(\int_0^1 t^{\alpha\rho+\rho-1} dt\right)^{1-\frac{1}{q}} \times$$

$$\left(\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^\rho x^\rho+(1-t^\rho)b^\rho)\right|^q dt\right)^{\frac{1}{q}}. \tag{10}$$

Since $\left|f^{'}\right|^q$ is s-Godunova-Levin function of second kind and $\left|f^{'}(x^\rho)\right| \leq M$, we get

$$\begin{aligned}
&\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^\rho x^\rho+(1-t^\rho)a^\rho)\right|^q dt\\
&\leq \int_0^1 \left[\frac{t^{\alpha\rho+\rho-1}}{(t^\rho)^s}\left|f^{'}(x^\rho)\right|^q + \frac{t^{\alpha\rho+\rho-1}}{(1-t^\rho)^s}\left|f^{'}(a^\rho)\right|^q\right] dt\\
&\leq M^q \int_0^1 \left[\frac{t^{\alpha\rho+\rho-1}}{(t^\rho)^s} + \frac{t^{\alpha\rho+\rho-1}}{(1-t^\rho)^s}\right] dt\times\\
&= M^q \left[\frac{1}{\rho(\alpha-s+1)} + \frac{{}^\rho\Gamma(\alpha+1)\ {}^\rho\Gamma(1-s)}{\rho\ {}^\rho\Gamma(\alpha-s+2)}\right]
\end{aligned} \tag{11}$$

similarly

$$\begin{aligned}
&\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^\rho x^\rho+(1-t^\rho)b^\rho)\right|^q dt\times\\
&\leq M^q \left[\frac{1}{\rho(\alpha-s+1)} + \frac{{}^\rho\Gamma(\alpha+1)\ {}^\rho\Gamma(1-s)}{\rho\ {}^\rho\Gamma(\alpha-s+2)}\right].
\end{aligned} \tag{12}$$

Using (11) and (12) in (10) we can attain (9). □

Remark 2.11. *(i) If we put $\rho = 1$ in (9), then we get* [4, Theorem 3.3].
(ii) If we put $\rho = 1$ and $\alpha = 1$ in (9), then we get [4, Corollary 3.3].

Corollary 2.12. *In Theorem 2.10, if we take $s = 0$, which means that $\left|f^{'}\right|$ is p-function, then (9) becomes the following inequality*

$$\begin{aligned}
&\left|\left(\frac{(x^\rho-a^\rho)^\alpha+(b^\rho-x^\rho)^\alpha}{b-a}\right)f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)}\times\right.\\
&\left.[{}^\rho I^\alpha_{x^-}f(a^\rho) + {}^\rho I^\alpha_{x^+}f(b^\rho)]\right|\\
&\leq \frac{M\rho}{(\alpha\rho+\rho)^{1-\frac{1}{q}}}\left[\frac{(x^\rho-a^\rho)^{\alpha+1}+(b^\rho-x^\rho)^{\alpha+1}}{b-a}\right]\left[\frac{2}{\rho(\alpha+1)}\right]^{\frac{1}{q}};\ x\in[a,b].
\end{aligned}$$

Corollary 2.13. *In Theorem 2.10, if we take $s = 1$, which means that $\left|f^{'}\right|$ is Godunova-Levin function, then (9) becomes the following inequality*

$$\begin{aligned}
&\left|\left(\frac{(x^\rho-a^\rho)^\alpha+(b^\rho-x^\rho)^\alpha}{b-a}\right)f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)}\times\right.\\
&\left.[{}^\rho I^\alpha_{x^-}f(a^\rho) + {}^\rho I^\alpha_{x^+}f(b^\rho)]\right|\\
&\leq \frac{M\rho}{(\alpha\rho+\rho)^{1-\frac{1}{q}}}\left[\frac{(x^\rho-a^\rho)^{\alpha+1}+(b^\rho-x^\rho)^{\alpha+1}}{b-a}\right]\left[\frac{1+\alpha}{\alpha\rho}\right]^{\frac{1}{q}};\ x\in[a,b].
\end{aligned}$$

We use the following lemma to establish some new results. Its proof is similar to Lemma 2.1.

Lemma 2.14. *Let $f:[a^\rho,b^\rho]\to\mathbb{R}$ be a differentiable mapping on (a^ρ,b^ρ) with $a^\rho<b^\rho$ such that $f^{'}\in L_1[a^\rho,b^\rho]$, where $\rho>0$. Then we have the following equality*

$$\begin{aligned}&f(x^\rho)-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-}f(a^\rho)}{2(x^\rho-a^\rho)^\alpha}+\frac{{}^\rho I^\alpha_{x^+}f(b^\rho)}{2(b^\rho-x^\rho)^\alpha}\right]\\&=\frac{\rho(x^\rho-a^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}f^{'}(t^\rho x^\rho+(1-t^\rho)a^\rho)dt\\&-\frac{(b^\rho-x^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}f^{'}(t^\rho x^\rho+(1-t^\rho)b^\rho)dt;\ x\in[a,b].\end{aligned}\tag{13}$$

Theorem 2.15. *Let $f:[a^\rho,b^\rho]\to\mathbb{R}$, $a,b\geq 0$, $a<b$ be a differentiable function on (a^ρ,b^ρ) and $f^{'}\in L_1[a,b]$. If $\left|f^{'}\right|$ is s-Godunova-Levin function of second kind and $\left|f^{'}(x^\rho)\right|\leq M$, $x\in[a,b]$, then the following inequality holds*

$$\begin{aligned}&\left|f(x^\rho)-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-}f(a^\rho)}{2(x^\rho-a^\rho)^\alpha}+\frac{{}^\rho I^\alpha_{x^+}f(b^\rho)}{2(b^\rho-x^\rho)^\alpha}\right]\right|\\&\leq\frac{M(b^\rho-a^\rho)}{2}\left[\frac{1}{\alpha-s+1}+\frac{{}^\rho\Gamma(\alpha+1)\ {}^\rho\Gamma(1-s)}{{}^\rho\Gamma(\alpha-s+2)}\right];\ x\in[a,b].\end{aligned}\tag{14}$$

Proof. Using Corollary 2.8 and s-Godunova-Levin function of second kind of $\left|f^{'}\right|$ we proceed as follows

$$\begin{aligned}&\left|f(x^\rho)-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-}f(a^\rho)}{2(x^\rho-a^\rho)^\alpha}+\frac{{}^\rho I^\alpha_{x^+}f(b^\rho)}{2(b^\rho-x^\rho)^\alpha}\right]\right|\\&\leq\frac{\rho(x^\rho-a^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^\rho x^\rho+(1-t^\rho)a^\rho)\right|dt\\&+\frac{\rho(b^\rho-x^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^\rho x^\rho+(1-t^\rho)b^\rho)\right|dt\\&\leq\frac{\rho(x^\rho-a^\rho)}{2}\int_0^1\left[\frac{t^{\alpha\rho+\rho-1}}{(t^\rho)^s}\left|f^{'}(x^\rho)\right|+\frac{t^{\alpha\rho+\rho-1}}{(1-t^\rho)^s}\left|f^{'}(a^\rho)\right|\right]dt\\&+\frac{\rho(b^\rho-x^\rho)}{2}\int_0^1\left[\frac{t^{\alpha\rho+\rho-1}}{(t^\rho)^s}\left|f^{'}(x^\rho)\right|+\frac{t^{\alpha\rho+\rho-1}}{(1-t^\rho)^s}\left|f^{'}(b^\rho)\right|\right]dt\\&\leq\frac{M\rho(x^\rho-a^\rho)}{2}\int_0^1\left[\frac{t^{\alpha\rho+\rho-1}}{(t^\rho)^s}+\frac{t^{\alpha\rho+\rho-1}}{(1-t^\rho)^s}\right]dt\\&+\frac{M\rho(b^\rho-x^\rho)}{2}\int_0^1\left[\frac{t^{\alpha\rho+\rho-1}}{(t^\rho)^s}+\frac{t^{\alpha\rho+\rho-1}}{(1-t^\rho)^s}\right]dt\\&=M\rho\left[\frac{(x^\rho-a^\rho)+(b^\rho-x^\rho)}{2}\right]\int_0^1\left[t^{\alpha\rho-\rho s+\rho-1}+t^{\alpha\rho+\rho-1}(1-t^\rho)^{-s}\right]dt.\end{aligned}$$

$$= M\rho\left[\frac{(x^\rho - a^\rho) + (b^\rho - x^\rho)}{2}\right]\left[\frac{1}{\rho(\alpha - s + 1)} + \frac{{}^\rho\Gamma(\alpha+1)\ {}^\rho\Gamma(1-s)}{\rho\ {}^\rho\Gamma(\alpha - s + 2)}\right]$$
$$= \frac{M(b^\rho - a^\rho)}{2}\left[\frac{1}{\alpha - s + 1} + \frac{{}^\rho\Gamma(\alpha+1)\ {}^\rho\Gamma(1-s)}{{}^\rho\Gamma(\alpha - s + 2)}\right].$$

Here we use (1). The proof is completed. □

Corollary 2.16. *In Theorem 2.15, if we take $s = 0$, which means that $|f'|$ is p-function, then (14) becomes the following inequality*

$$\left|f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x^+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha}\right]\right|$$
$$\leq \frac{M(b^\rho - a^\rho)}{\alpha + 1};\ x \in [a, b].$$

Corollary 2.17. *In Theorem 2.15, if we take $s = 1$, which means that $|f'|$ is Godunova-Levin function, then (14) becomes the following inequality*

$$\left|f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x^+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha}\right]\right|$$
$$\leq \frac{M(\alpha + 1)(b^\rho - a^\rho)}{2\alpha};\ x \in [a, b].$$

Theorem 2.18. *Let $f : [a^\rho, b^\rho] \to \mathbb{R}$, $a, b \geq 0$, $a < b$ be a differentiable function on (a^ρ, b^ρ) and $f' \in L_1[a, b]$. If $|f'|^q$, is s-Godunova-Levin function of second kind and $|f'(x^\rho)| \leq M$, $x \in [a, b]$ then the following inequality for Katugampola fractional integrals holds*

$$\left|f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x^+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha}\right]\right|$$
$$\leq \frac{M\rho(b^\rho - a^\rho)}{2(1 + p(\alpha\rho + \rho - 1))^{\frac{1}{p}}}\left[\frac{1}{1 - \rho s}\right]^{\frac{1}{q}};\ x \in [a, b], \tag{15}$$

with $\frac{1}{p} + \frac{1}{q} = 1$ where $q > 1$.

Proof. Using Corollary 2.8 and then Holder's inequality, we have

$$\left|f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x^+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha}\right]\right|$$
$$\leq \frac{\rho(x^\rho - a^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho + (1 - t^\rho)a^\rho)\right|dt$$
$$+ \frac{\rho(b^\rho - x^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho + (1 - t^\rho)b^\rho)\right|dt$$
$$\leq \frac{\rho(x^\rho - a^\rho)}{2}\left(\int_0^1 t^{p(\alpha\rho+\rho-1)}dt\right)^{\frac{1}{p}} \times$$

$$\left(\int_0^1 \left|f^{'}(t^\rho x^\rho+(1-t^\rho)a^\rho)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{\rho(b^\rho-x^\rho)}{2}\left(\int_0^1 t^{p(\alpha\rho+\rho-1)}dt\right)^{\frac{1}{p}}\times$$
$$\left(\int_0^1 \left|f^{'}(t^\rho x^\rho+(1-t^\rho)b^\rho)\right|^q dt\right)^{\frac{1}{q}}. \tag{16}$$

Since $\left|f^{'}\right|^q$ is s-Godunova-Levin function of second kind and $\left|f^{'}(x^\rho)\right|\le M$, we get

$$\int_0^1 \left|f^{'}(t^\rho x^\rho+(1-t^\rho)a^\rho)\right|^q dt$$
$$\le \int_0^1\left[\frac{1}{(t^\rho)^s}\left|f^{'}(x^\rho)\right|^q+\frac{1}{(1-t^\rho)^s}\left|f^{'}(a^\rho)\right|^q\right]dt$$
$$\le M^q\int_0^1\left[\frac{1}{(t^\rho)^s}+\frac{1}{(1-t^\rho)^s}\right]dt=\frac{M^q}{1-\rho s} \tag{17}$$

similarly

$$\int_0^1 \left|f^{'}(t^\rho x^\rho+(1-t^\rho)b^\rho)\right|^q dt\le\frac{M^q}{1-\rho s}. \tag{18}$$

We also have

$$\int_0^1 t^{p(\alpha\rho+\rho-1)}dt=\frac{1}{1+p(\alpha\rho+\rho-1)}. \tag{19}$$

Using (17), (18) and (19) in (16) we can get (15). □

Corollary 2.19. *In Theorem 2.18, if we take $s=0$, which means that $\left|f^{'}\right|$ is p-function, then (15) becomes the following inequality*

$$\left|f(x^\rho)-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-}f(a^\rho)}{2(x^\rho-a^\rho)^\alpha}+\frac{{}^\rho I^\alpha_{x^+}f(b^\rho)}{2(b^\rho-x^\rho)^\alpha}\right]\right|$$
$$\le\frac{M\rho(b^\rho-a^\rho)}{2(p(\alpha\rho+\rho-1)+1)^{\frac{1}{p}}};\ x\in[a,b].$$

Corollary 2.20. *In Theorem 2.18, if we take $s=1$, which means that $\left|f^{'}\right|$ is Godunova-Levin function, then (15) becomes the following inequality*

$$\left|f(x^\rho)-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-}f(a^\rho)}{2(x^\rho-a^\rho)^\alpha}+\frac{{}^\rho I^\alpha_{x^+}f(b^\rho)}{2(b^\rho-x^\rho)^\alpha}\right]\right|$$
$$\le\frac{M\rho(b^\rho-a^\rho)}{2(p(\alpha\rho+\rho-1)+1)^{\frac{1}{p}}}\left[\frac{1}{1-\rho}\right]^{\frac{1}{q}};\ x\in[a,b].$$

Theorem 2.21. *Let $f:[a^\rho,b^\rho]\to\mathbb{R}$, $a,b\ge 0$, $a<b$ be a differentiable function on (a^ρ,b^ρ) and $f^{'}\in L_1[a,b]$. If $\left|f^{'}\right|^q$ is s-Godunova-Levin function of second kind*

and $\left|f^{'}(x^{\rho})\right| \leq M$, $x \in [a,b], q \geq 1$, *then the following inequality for Katugampola fractional integrals holds*

$$\left|f(x^{\rho}) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^{\rho}I^{\alpha}_{x^-}f(a^{\rho})}{2(x^{\rho}-a^{\rho})^{\alpha}} + \frac{{}^{\rho}I^{\alpha}_{x^+}f(b^{\rho})}{2(b^{\rho}-x^{\rho})^{\alpha}}\right]\right|$$
$$\leq \frac{M\rho(b^{\rho}-a^{\rho})}{2(\alpha\rho+\rho)^{1-\frac{1}{q}}}\left(\frac{1}{\rho(\alpha-s+1)} + \frac{{}^{\rho}\Gamma(\alpha+1)\ {}^{\rho}\Gamma(1-s)}{\rho\ {}^{\rho}\Gamma(\alpha-s+2)}\right)^{\frac{1}{q}};\ x \in [a,b]. \tag{20}$$

Proof. Using Corollary 2.8 and power mean inequality, we have

$$\left|f(x^{\rho}) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^{\rho}I^{\alpha}_{x^-}f(a^{\rho})}{2(x^{\rho}-a^{\rho})^{\alpha}} + \frac{{}^{\rho}I^{\alpha}_{x^+}f(b^{\rho})}{2(b^{\rho}-x^{\rho})^{\alpha}}\right]\right|$$
$$\leq \frac{\rho(x^{\rho}-a^{\rho})}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^{\rho}x^{\rho}+(1-t^{\rho})a^{\rho})\right|dt$$
$$+ \frac{\rho(b^{\rho}-x^{\rho})}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^{\rho}x^{\rho}+(1-t^{\rho})b^{\rho})\right|dt$$
$$\leq \frac{\rho(x^{\rho}-a^{\rho})}{2}\left(\int_0^1 t^{\alpha\rho+\rho-1}dt\right)^{1-\frac{1}{q}} \times$$
$$\left(\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^{\rho}x^{\rho}+(1-t^{\rho})a^{\rho})\right|^q dt\right)^{\frac{1}{q}}$$
$$+ \frac{\rho(b^{\rho}-x^{\rho})}{2}\left(\int_0^1 t^{\alpha\rho+\rho-1}dt\right)^{1-\frac{1}{q}} \times$$
$$\left(\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^{\rho}x^{\rho}+(1-t^{\rho})b^{\rho})\right|^q dt\right)^{\frac{1}{q}}. \tag{21}$$

Since $\left|f^{'}\right|^q$ is s-Godunova-Levin function of second kind on $[a^{\rho}, b^{\rho}]$ and $\left|f^{'}(x^{\rho})\right| \leq M$, we get

$$\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^{\rho}x^{\rho}+(1-t^{\rho})a^{\rho})\right|^q dt$$
$$\leq \int_0^1 \left[\frac{t^{\alpha\rho+\rho-1}}{(t^{\rho})^s}\left|f^{'}(x^{\rho})\right|^q + \frac{t^{\alpha\rho+\rho-1}}{(1-t^{\rho})^s}\left|f^{'}(a^{\rho})\right|^q\right]dt$$
$$\leq M^q\int_0^1\left[\frac{t^{\alpha\rho+\rho-1}}{(t^{\rho})^s} + \frac{t^{\alpha\rho+\rho-1}}{(1-t^{\rho})^s}\right]dt$$
$$= M^q\left[\frac{1}{\rho(\alpha-s+1)} + \frac{{}^{\rho}\Gamma(\alpha+1)\ {}^{\rho}\Gamma(1-s)}{\rho\ {}^{\rho}\Gamma(\alpha-s+2)}\right] \tag{22}$$

similarly

$$\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^{\rho}x^{\rho}+(1-t^{\rho})b^{\rho})\right|^q dt$$

$$\leq M^q \left[\frac{1}{\rho(\alpha-s+1)} + \frac{{}^{\rho}\Gamma(\alpha+1)\ {}^{\rho}\Gamma(1-s)}{\rho\ {}^{\rho}\Gamma(\alpha-s+2)}\right]. \tag{23}$$

Using (22) and (23) in (21) we can attain (20). □

Corollary 2.22. *In Theorem 2.21, if we take $s=0$, which means that $\left|f'\right|$ is p-function, then (20) becomes the following inequality*

$$\left|f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^{\rho}I^{\alpha}_{x^-}f(a^\rho)}{2(x^\rho-a^\rho)^\alpha} + \frac{{}^{\rho}I^{\alpha}_{x^+}f(b^\rho)}{2(b^\rho-x^\rho)^\alpha}\right]\right|$$
$$\leq \frac{M\rho(b^\rho-a^\rho)}{2(\rho(\alpha+1))^{1-\frac{1}{q}}}\left[\frac{2}{\rho(\alpha+1)}\right]^{\frac{1}{q}};\ x\in[a,b].$$

Corollary 2.23. *In Theorem 2.21, if we take $s=1$, which means that $\left|f'\right|$ is Godunova-Levin function, then (20) becomes the following inequality*

$$\left|f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^{\rho}I^{\alpha}_{x^-}f(a^\rho)}{2(x^\rho-a^\rho)^\alpha} + \frac{{}^{\rho}I^{\alpha}_{x^+}f(b^\rho)}{2(b^\rho-x^\rho)^\alpha}\right]\right|$$
$$\leq \frac{M\rho(b^\rho-a^\rho)}{2(\rho(\alpha+1))^{1-\frac{1}{q}}}\left[\frac{1+\alpha}{\rho\alpha}\right]^{\frac{1}{q}};\ x\in[a,b].$$

3. Conclusion

All results proved in this research paper can also be deduced for Hadamard fractional integrals just by taking limits when parameter $\rho \to 0^+$.

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Competing Interests

The author(s) do not have any competing interests in the manuscript.

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