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ON THE VISCOSITY RULE FOR COMMON FIXED POINTS OF TWO NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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ABSTRACT. In this paper, we introduce, for the first time, the viscosity rules for common fixed points of two nonexpansive mappings in Hilbert spaces. The strong convergence of this technique is proved under certain assumptions imposed on the sequence of parameters.

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1. Introduction

In this paper, we shall take H as a real Hilbert space, $\langle .,. \rangle$ as inner product, $\|.\|$ as the induced norm, and C as a nonempty closed subset of H.

Definition 1.1. Let $T : H \to H$ be a mapping. T is called *non-expansive* if

$$\|T(x) - T(y)\| \leq \|x - y\|, \quad \forall x, y \in H$$

Definition 1.2. A mapping $f : H \to H$ is called a *contraction* if for all $x, y \in H$ and $\theta \in [0, 1)$

$$\|f(x) - f(y)\| \leq \theta \|x - y\|.$$

Definition 1.3. $P_c : H \to C$ is called a *metric projection* if for every $x \in H$ there exists a unique nearest point in C, denoted by $P_c x$, such that

$$\|x - P_c x\| \leq \|x - y\|, \quad \forall\ y \in C.$$

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In order to verify the weak convergence of an algorithm to a fixed point of a non-expansive mapping we need the demiclosedness principle:

Theorem 1.4. [1] (*The demiclosedness principle*) *Let C be a nonempty closed convex subset of the real Hilbert space H and $T : C \to C$ such that*

$$x_n \rightharpoonup x^* \in C \text{ and } (I - T)x_n \to 0$$

Then $x^* = Tx^*$. Here, $\to$ and $\rightharpoonup$ denotes strong and weak convergence respectively.

Moreover, the following result gives the conditions for the convergence of a non-negative real sequence.

Theorem 1.5. [2] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \le (1 - \gamma_n)a_n + \delta_n, \forall n \ge 0$, where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence with*

(1) $\sum_{n=0}^{\infty} \gamma_n = \infty$,
(2) $\lim_{n\to\infty} \sup \frac{\delta_n}{\gamma_n} \le 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then $a_n \to 0$.

The following strong convergence theorem, which is also called the *viscosity approximation method*, for non-expansive mappings in real Hilbert spaces is given by Moudafi, [3], in (2000).

Theorem 1.6. *Let C be a non-empty closed convex subset of the real Hilbert space H. Let T be a non-expansive mapping of C into itself such that $F(T) = \{x \in C : T(x) = x\}$ is nonempty. Let f be a contraction of C into itself. Consider the sequence*

$$x_{n+1} = \frac{\epsilon_n}{1+\epsilon_n} f(x_n) + \frac{1}{1+\epsilon_n} T(x_n), \quad n \ge 0,$$

where the sequence $\{\epsilon_n\} \in (0, 1)$ satisfies

(1) $\lim_{n\to\infty} \epsilon_n = 0$,
(2) $\sum_{n=0}^{\infty} \epsilon_n = \infty$, *and*
(3) $\lim_{n\to\infty} |\frac{1}{\epsilon_{n+1}} - \frac{1}{\epsilon_n}| = 0$.

Then x_n converges strongly to a fixed point x^* of the non-expansive mapping T, which is also the unique solution of the variational inequality

$$\langle (I - f)x, y - x\rangle \ge 0, \quad \forall \in F(T).$$

In (2015), Xu *et al.* [2] applied viscosity method on the midpoint rule for non-expansive mappings and give the generalized viscosity implicit rule (GVIR):

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), \quad \forall n \ge 0$$

This, using contraction, regularizes the implicit midpoint rule for nonexpansive mappings. They also proved that the sequence generated by GMIR converges strongly to a fixed point of T. Ke *et al.* [4], motivated and inspired by the idea of Xu *et al.* [2], proposed two generalized viscosity implicit rules:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\left(s_n x_n + (1-s_n)x_{n+1}\right),$$

$$x_{n+1} = \alpha_n x_n + \beta f(x_n) + \gamma_n T(s_n x_n + (1-s_n)x_{n+1}).$$

Our contribution in this direction is the following viscosity rule for common fixed points of two nonexpansive mappings in Hilbert spaces:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n S\left(\frac{x_{n+1}+x_n}{2}\right) + \gamma_n T\left(\frac{x_{n+1}+x_n}{2}\right).$$

2. Main Result

Theorem 2.1. *Let C be a nonempty closed convex subset of the real Hilbert space H. Let $S : C \to C$ and $T : C \to C$ be two nonexpansive mappings with $U := F(T) \cap F(S) \neq \phi$ and $f : C \to C$ be a contraction with coefficient $\theta \in [0,1)$. Let $\{x_n\}$ be a sequence in C generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n S\left(\frac{x_{n+1}+x_n}{2}\right) + \gamma_n T\left(\frac{x_{n+1}+x_n}{2}\right), \tag{1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$, satisfying the following conditions:

(1) $\alpha_n + \beta_n + \gamma_n = 1$ *and* $\lim\limits_{n\to\infty} \gamma_n = 1$,

(2) $\sum\limits_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum\limits_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

(3) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$,

(4) $\lim\limits_{n\to\infty} \left\| T\left(\frac{x_{n+1}+x_n}{2}\right) - S\left(\frac{x_{n+1}+x_n}{2}\right) \right\| = 0$ *and* $\lim\limits_{n\to\infty} \alpha_n = \lim\limits_{n\to\infty} \beta_n = 0$.

Then $\{x_n\}$ converges strongly to a common fixed point x^* of the nonexpansive mappings T and S which is also the unique solution of the variational inequality

$$\langle (I-f)x, y-x \rangle, \quad \forall \quad y \in U$$

In other words, x^* is the unique fixed point of the contraction $P_U f$.

Proof. We will prove this theorem into the following five steps:

Step 1. Firstly, we want to show that the sequence (x_n) is bounded. Indeed, take $p \in U$ arbitrarily, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \left\| -p + \alpha_n f(x_n) + \beta_n S\left(\frac{x_{n+1}+x_n}{2}\right) + \gamma_n T\left(\frac{x_{n+1}+x_n}{2}\right) \right\| \\ &= \left\| -(\alpha_n + \beta_n + \gamma_n)p + \alpha_n f(x_n) + \beta_n S\left(\frac{x_{n+1}+x_n}{2}\right) \right. \\ &\quad \left. + \gamma_n T\left(\frac{x_{n+1}+x_n}{2}\right) \right\| \end{aligned}$$

$$\begin{aligned}
&\le \alpha_n\|f(x_n)-p\| + \beta_n \left\| S\left(\frac{x_{n+1}+x_n}{2}\right) - p \right\| \\
&\quad +\gamma_n \left\| T\left(\frac{x_{n+1}+x_n}{2}\right) - p \right\| \\
&\le \alpha_n\|f(x_n)-f(p)\| + \alpha_n\|f(p)-p\| + \beta_n \left\| \frac{x_{n+1}+x_n}{2} - p \right\| \\
&\quad +\gamma_n \left\| \frac{x_{n+1}+x_n}{2} - p \right\| \\
&\le \theta\alpha_n\|x_n-p\| + \alpha_n\|f(p)-p\| + (\beta_n+\gamma_n) \left\| \frac{x_{n+1}+x_n}{2} - p \right\| \\
&= \theta\alpha_n\|x_n-p\| + \alpha_n\|f(p)-p\| + (1-\alpha_n) \left\| \frac{x_{n+1}+x_n}{2} - p \right\| \\
&\le \theta\alpha_n\|x_n-p\| + \alpha_n\|f(p)-p\| + \frac{1-\alpha_n}{2}\|x_{n+1}-p\| \\
&\quad +\frac{1-\alpha_n}{2}\|x_n-p\|.
\end{aligned}$$

This is equivalent to

$$\left(1-\frac{1-\alpha_n}{2}\right)\|x_{n+1}-p\| \le \left(\frac{1-\alpha_n}{2}+\alpha_n\theta\right)\|x_n-p\| + \alpha_n\|f(p)-p\|$$

$\Rightarrow$

$$(1+\alpha_n)\|x_{n+1}-p\| \le (1-\alpha_n+2\alpha_n\theta)\|x_n-p\| + 2\alpha_n\|f(p)-p\|$$

$\Rightarrow$

$$\begin{aligned}
\|x_{n+1}-p\| &\le \frac{1+\alpha_n-2\alpha_n+2\alpha_n\theta}{1+\alpha_n}\|x_n-p\| + \frac{2\alpha_n}{1+\alpha_n}\|f(p)-p\| \\
&= \left(1-\frac{2\alpha_n(1-\theta)}{1+\alpha_n}\right)\|x_n-p\| \\
&\quad +\frac{2\alpha_n(1-\theta)}{1+\alpha_n}\left(\frac{1}{1-\theta}\|f(p)-p\|\right).
\end{aligned}$$

Thus,

$$\|x_{n+1}-p\| \le \max\left\{\|x_n-p\|, \frac{1}{1-\theta}\|f(p)-p\|\right\}.$$

Similarly,

$$\|x_n-p\| \le \max\left\{\|x_{n-1}-p\|, \left(\frac{1}{1-\theta}\|f(p)-p\|\right)\right\}.$$

From this, we obtain,

$$\|x_{n+1}-p\| \le \max\left\{\|x_n-p\|, \frac{1}{1-\theta}\|f(p)-p\|\right\}$$

$$
\begin{aligned}
&\leq \max\left\{\|x_{n-1}-p\|, \frac{1}{1-\theta}\|f(p)-p\|\right\}\\
&\quad\vdots\\
&\leq \max\left\{\|x_0-p\|, \frac{1}{1-\theta}\|f(p)-p\|\right\}.
\end{aligned}
$$

Hence, we concluded that $\{x_n\}$ is a bounded sequence. Consequently, $\{f(x_n)\}$, $\{S(\frac{x_{n+1}+x_n}{2})\}$ and $\{T(\frac{x_{n+1}+x_n}{2})\}$ are bounded.

Step 2. Now, we prove that $\lim\limits_{n\to\infty}\|x_{n+1}-x_n\|=0$.

$$
\begin{aligned}
& \|x_{n+1}-x_n\| \\
= & \left\|\alpha_n f(x_n)+\beta_n S\left(\frac{x_{n+1}+x_n}{2}\right)+\gamma_n T\left(\frac{x_{n+1}+x_n}{2}\right)\right. \\
& \left. -\alpha_{n-1} f(x_{n-1})+\beta_{n-1} S\left(\frac{x_n+x_{n-1}}{2}\right)+\gamma_{n-1} T\left(\frac{x_n+x_{n-1}}{2}\right)\right\| \\
= & \left\|\alpha_n(f(x_n)-f(x_{n-1}))+(\alpha_n-\alpha_{n-1}) f(x_{n-1})+\beta_n\left(S\left(\frac{x_{n+1}+x_n}{2}\right)\right.\right. \\
& \left.-S\left(\frac{x_n+x_{n-1}}{2}\right)\right)+(\beta_n-\beta_{n-1}) S\left(\frac{x_n+x_{n-1}}{2}\right) \\
& +\gamma_n\left(T\left(\frac{x_{n+1}+x_n}{2}\right)-T\left(\frac{x_n+x_{n-1}}{2}\right)\right) \\
& \left.+(\gamma_n-\gamma_{n-1}) T\left(\frac{x_n+x_{n-1}}{2}\right)\right\| \\
= & \left\|\alpha_n(f(x_n)-f(x_{n-1}))+(\alpha_n-\alpha_{n-1}) f(x_{n-1})+\beta_n\left(S\left(\frac{x_{n+1}+x_n}{2}\right)\right.\right. \\
& \left.-S\left(\frac{x_n+x_{n-1}}{2}\right)\right)+(\beta_n-\beta_{n-1}) S\left(\frac{x_n+x_{n-1}}{2}\right) \\
& +\gamma_n\left(T\left(\frac{x_{n+1}+x_n}{2}\right)-T\left(\frac{x_n+x_{n-1}}{2}\right)\right) \\
& \left.+(\alpha_n-\alpha_{n-1}+\beta_n-\beta_{n-1}) T\left(\frac{x_n+x_{n-1}}{2}\right)\right\| \\
= & \left\|\alpha_n(f(x_n)-f(x_{n-1}))+(\alpha_n-\alpha_{n-1})\left(f(x_{n-1})-T\left(\frac{x_n+x_{n-1}}{2}\right)\right)\right. \\
& +\beta_n\left(S\left(\frac{x_{n+1}+x_n}{2}\right)-S\left(\frac{x_n+x_{n-1}}{2}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+(\beta_n-\beta_{n-1})\left(S\left(\frac{x_n+x_{n-1}}{2}\right)-T\left(\frac{x_n+x_{n-1}}{2}\right)\right)\\
&+\gamma_n\left(T\left(\frac{x_{n+1}+x_n}{2}\right)-T\left(\frac{x_n+x_{n-1}}{2}\right)\right)\Bigg\|\\
\leq\ &\alpha_n\left\|f(x_n)-f(x_{n-1})\right\|+|\alpha_n-\alpha_{n-1}|\left\|f(x_{n-1})-T\left(\frac{x_n+x_{n-1}}{2}\right)\right\|\\
&+\beta_n\left\|S\left(\frac{x_{n+1}+x_n}{2}\right)-S\left(\frac{x_n+x_{n-1}}{2}\right)\right\|\\
&+|\beta_n-\beta_{n-1}|\left\|S\left(\frac{x_n+x_{n-1}}{2}\right)-T\left(\frac{x_n+x_{n-1}}{2}\right)\right\|\\
&+\gamma_n\left\|T\left(\frac{x_{n+1}+x_n}{2}\right)-T\left(\frac{x_n+x_{n-1}}{2}\right)\right\|.
\end{aligned}
$$

Let M_2 be a number such that $M_2 \geq \max\left\{\sup\limits_{n\geq 0}\left\|S\left(\frac{x_{n+1}+x_n}{2}\right)-T\left(\frac{x_{n+1}+x_n}{2}\right)\right\|,\right.$ $\left.\sup\limits_{n\geq 0}\left\|f(x_n)-T\left(\frac{x_{n+1}+x_n}{2}\right)\right\|\right\}$. Thus, the above is equivalent to

$$
\begin{aligned}
&\|x_{n+1}-x_n\|\\
\leq\ &\alpha_n\theta\left\|x_n-x_{n-1}\right\|+\beta_n\left\|\frac{x_{n+1}+x_n}{2}-\frac{x_n+x_{n-1}}{2}\right\|\\
&+\gamma_n\left\|\frac{x_{n+1}+x_n}{2}-\frac{x_n+x_{n-1}}{2}\right\|+\left(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|\right)M_2\\
\leq\ &\alpha_n\theta\left\|x_n-x_{n-1}\right\|\\
&+\frac{\beta_n}{2}\left\|x_{n+1}-x_n\right\|+\frac{\beta_n}{2}\left\|x_n-x_{n-1}\right\|+\frac{\gamma_n}{2}\left\|x_{n+1}-x_n\right\|\\
&+\frac{\gamma_n}{2}\left\|x_n-x_{n-1}\right\|+\left(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|\right)M_2\\
=\ &\left(\alpha_n\theta+\frac{\beta_n}{2}+\frac{\gamma_n}{2}\right)\left\|x_n-x_{n-1}\right\|+\left(\frac{\beta_n}{2}+\frac{\gamma_n}{2}\right)\left\|x_{n+1}-x_n\right\|\\
&+(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|)M_2\\
=\ &\left(\alpha_n\theta+\frac{1-\alpha_n}{2}\right)\left\|x_n-x_{n-1}\right\|+\frac{1-\alpha_n}{2}\|x_{n+1}-x_n\|\\
&+(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|)M_2.
\end{aligned}
$$

Combining the common terms from left and right hand sides, we get,

$$
\begin{aligned}
\left(1-\frac{1-\alpha_n}{2}\right)\left\|x_{n+1}-x_n\right\| \quad\leq\quad &\left(\alpha_n\theta+\frac{1-\alpha_n}{2}\right)\left\|x_n-x_{n-1}\right\|\\
&+(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|)M_2.
\end{aligned}
$$

This implies that

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
\leq\ & \frac{1 + \alpha_n - 2\alpha_n + 2\alpha_n\theta}{1 + \alpha_n}\|x_n - x_{n-1}\| \\
&+ \left(\frac{2(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|)}{1 + \alpha_n}\right) M_2 \\
=\ & \left(1 - \frac{2\alpha_n(1-\theta)}{1 + \alpha_n}\right)\|x_n - x_{n-1}\| \\
&+ \left(\frac{2(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|)}{1 + \alpha_n}\right) M_2.
\end{aligned}$$

Note that $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, $\sum\limits_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum\limits_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$. Using Theorem 1.5, we have $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

Step 3. Now, we will show that $\lim\limits_{n\to\infty} \|x_n - Sx_n\| = 0$ and $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$. Consider

$$\begin{aligned}
&\|x_n - S(x_n)\| \\
=\ & \left\|x_n - x_{n+1} + x_{n+1} - T\left(\frac{x_{n+1} + x_n}{2}\right) + T\left(\frac{x_{n+1} + x_n}{2}\right)\right. \\
&\left. - S\left(\frac{x_{n+1} + x_n}{2}\right) + S\left(\frac{x_{n+1} + x_n}{2}\right) - S(x_n)\right\| \\
\leq\ & \|x_n - x_{n+1}\| + \left\|x_{n+1} - T\left(\frac{x_{n+1} + x_n}{2}\right)\right\| + \left\|T\left(\frac{x_{n+1} + x_n}{2}\right)\right. \\
&\left. -S\left(\frac{x_{n+1} + x_n}{2}\right)\right\| + \left\|S\left(\frac{x_{n+1} + x_n}{2}\right) - S(x_n)\right\| \\
\leq\ & \|x_n - x_{n+1}\| + \left\|\alpha_n f(x_n) + \beta_n S\left(\frac{x_{n+1} + x_n}{2}\right) + \gamma_n T\left(\frac{x_{n+1} + x_n}{2}\right)\right. \\
&\left. -T\left(\frac{x_{n+1} + x_n}{2}\right)\right\| + \left\|\frac{x_{n+1} + x_n}{2} - x_n\right\| \\
&+ \left\|T\left(\frac{x_{n+1} + x_n}{2}\right) - S\left(\frac{x_{n+1} + x_n}{2}\right)\right\| \\
\leq\ & \|x_n - x_{n+1}\| + \alpha_n\left\|f(x_n) - T\left(\frac{x_{n+1} + x_n}{2}\right)\right\| \\
&+\beta_n\left\|S\left(\frac{x_{n+1} + x_n}{2}\right) - T\left(\frac{x_{n+1} + x_n}{2}\right)\right\| + \frac{1}{2}\|x_{n+1} - x_n\| \\
&+ \left\|S\left(\frac{x_{n+1} + x_n}{2}\right) - T\left(\frac{x_{n+1} + x_n}{2}\right)\right\|
\end{aligned}$$

$$\leq \frac{3}{2}\|x_n - x_{n+1}\| + \alpha_n \left\| f(x_n) - T\left(\frac{x_{n+1}+x_n}{2}\right)\right\| + (1+\beta_n)\left\| S\left(\frac{x_{n+1}+x_n}{2}\right) - T\left(\frac{x_{n+1}+x_n}{2}\right)\right\|.$$

Since, $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \left\| T\left(\frac{x_{n+1}+x_n}{2}\right) - S\left(\frac{x_{n+1}+x_n}{2}\right)\right\| = 0$ and $\lim_{n\to\infty} \|x_{n+1} - x_n\| \to 0$, we get $\|x_n - S(x_n)\| \to 0$ as $n \to \infty$.

Moreover, we have

$$\begin{aligned} \left\| S\left(\frac{x_{n+1}+x_n}{2}\right) - x_n \right\| &= \left\| S\left(\frac{x_{n+1}+x_n}{2}\right) - S(X_n) + S(x_n) - x_n \right\| \\ &\leq \left\| S\left(\frac{x_{n+1}+x_n}{2}\right) - S(x_n)\right\| + \|S(x_n) - x_n\| \\ &\leq \left\| \frac{x_{n+1}+x_n}{2} - x_n \right\| + \|S(x_n) - x_n\| \\ &= \frac{1}{2}\|x_{n+1} - x_n\| + \|S(x_n) - x_n\| \\ &\to 0, \qquad \text{as } (n \to \infty). \end{aligned}$$

Now, consider

$$\begin{aligned} &\|x_n - T(x_n)\| \\ = &\left\| x_n - x_{n+1} + x_{n+1} - S\left(\frac{x_{n+1}+x_n}{2}\right) + S\left(\frac{x_{n+1}+x_n}{2}\right) \right. \\ &\left. - T\left(\frac{x_{n+1}+x_n}{2}\right) + T\left(\frac{x_{n+1}+x_n}{2}\right) - T(x_n) \right\| \\ \leq &\|x_n - x_{n+1}\| + \left\| x_{n+1} - S\left(\frac{x_{n+1}+x_n}{2}\right)\right\| + \left\| T\left(\frac{x_{n+1}+x_n}{2}\right) - T(x_n)\right\| \\ &+ \left\| S\left(\frac{x_{n+1}+x_n}{2}\right) - T\left(\frac{x_{n+1}+x_n}{2}\right)\right\| \\ \leq &\|x_n - x_{n+1}\| + \left\| T\left(\frac{x_{n+1}+x_n}{2}\right) - S\left(\frac{x_{n+1}+x_n}{2}\right)\right\| + \\ &\left\| \alpha_n f(x_n) + \beta_n S\left(\frac{x_{n+1}+x_n}{2}\right) + \gamma_n T\left(\frac{x_{n+1}+x_n}{2}\right) - S\left(\frac{x_{n+1}+x_n}{2}\right)\right\| \\ &+ \left\| \frac{x_{n+1}+x_n}{2} - x_n \right\| \\ \leq &\|x_n - x_{n+1}\| + \alpha_n \left\| f(x_n) - T\left(\frac{x_{n+1}+x_n}{2}\right)\right\| + \frac{1}{2}\|x_{n+1} - x_n\| \\ &+\gamma_n \left\| S\left(\frac{x_{n+1}+x_n}{2}\right) - T\left(\frac{x_{n+1}+x_n}{2}\right)\right\| \end{aligned}$$

$$
\begin{aligned}
&+\left\|S\left(\frac{x_{n+1}+x_n}{2}\right)-T\left(\frac{x_{n+1}+x_n}{2}\right)\right\| \\
\leq\ & \frac{3}{2}\|x_n-x_{n+1}\|+\alpha_n\left\|f(x_n)-T\left(\frac{x_{n+1}+x_n}{2}\right)\right\| \\
&+(1+\gamma_n)\left\|S\left(\frac{x_{n+1}+x_n}{2}\right)-T\left(\frac{x_{n+1}+x_n}{2}\right)\right\|.
\end{aligned}
$$

Since, $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\left\|T\left(\frac{x_{n+1}+x_n}{2}\right)-S\left(\frac{x_{n+1}+x_n}{2}\right)\right\| = 0$ and $\lim_{n\to\infty}\|x_{n+1}-x_n\|\to 0$, we get $\|x_n-Tx_n\|\to 0$ as $n\to\infty$.
Also,

$$
\begin{aligned}
\left\|T\left(\frac{x_{n+1}+x_n}{2}\right)-x_n\right\| &= \left\|T\left(\frac{x_{n+1}+x_n}{2}\right)-T(X_n)+T(x_n)-x_n\right\| \\
&\leq \left\|T\left(\frac{x_{n+1}+x_n}{2}\right)-T(x_n)\right\|+\|T(x_n)-x_n\| \\
&\leq \left\|\frac{x_{n+1}+x_n}{2}-x_n\right\|+\|T(x_n)-x_n\| \\
&= \frac{1}{2}\|x_{n+1}-x_n\|+\|T(x_n)-x_n\| \\
&\to 0 \qquad (\text{ as } n\to\infty)
\end{aligned}
$$

Step 4. In this step, we will show that $\limsup_{n\to\infty}\langle x^*-f(x^*),x^*-x_n\rangle\leq 0$, where, $x^*=P_U f(x^*)$.

Indeed, we take a subsequence, $\{x_{n_i}\}$ of $\{x_n\}$, which converges weakly to a fixed point $p\in U=F(T)\cap F(S)$. Without loss of generality, we may assume that $\{x_{n_i}\}\rightharpoonup p$. From $\lim_{n\to\infty}\|x_n-S(x_n)\|=0$, $\lim_{n\to\infty}\|x_n-T(x_n)\|=0$ and Theorem 1.4, we have $p=S(p)$ and $p=T(p)$. This together with the property of the metric projection implies that

$$\limsup_{n\to\infty}\langle x^*-f(x^*),x^*-x_n\rangle=\langle x^*-f(x^*),x^*-p\rangle\leq 0.$$

Step 5. Finally, we show that $x_n\to x^*$ as $n\to\infty$. Again, take $x^*\in U$ to be the unique fixed point of the contraction $P_U f$. Consider

$$
\begin{aligned}
&\|x_{n+1}-x_n\|^2 \\
=\ & \left\|\alpha_n f(x_n)+\beta_n S\left(\frac{x_{n+1}+x_n}{2}\right)+\gamma_n T\left(\frac{x_{n+1}+x_n}{2}\right)-x^*\right\|^2 \\
=\ & \left\|\alpha_n f(x_n)+\beta_n S\left(\frac{x_{n+1}+x_n}{2}\right)+\gamma_n T\left(\frac{x_{n+1}+x_n}{2}\right)-(\alpha_n+\beta_n+\gamma_n)x^*\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \left\|\alpha_n(f(x_n)-x^*)+\beta_n\left(S\left(\frac{x_{n+1}+x_n}{2}\right)-x^*\right)+\gamma_n\left(T\left(\frac{x_{n+1}+x_n}{2}\right)-x^*\right)\right\|^2\\
&= \alpha_n^2\left\|f(x_n)-x^*\right\|^2+\beta_n^2\left\|S\left(\frac{x_{n+1}+x_n}{2}\right)-x^*\right\|^2\\
&\quad+\gamma_n^2\left\|T\left(\frac{x_{n+1}+x_n}{2}\right)-x^*\right\|^2\\
&\quad+2\alpha_n\beta_n\left\langle f(x_n)-x^*,S\left(\frac{x_{n+1}+x_n}{2}\right)-x^*\right\rangle\\
&\quad+2\alpha_n\gamma_n\left\langle f(x_n)-x^*,T\left(\frac{x_{n+1}+x_n}{2}\right)-x^*\right\rangle\\
&\quad+2\beta_n\gamma_n\left\langle S\left(\frac{x_{n+1}+x_n}{2}\right)-x^*,T\left(\frac{x_{n+1}+x_n}{2}\right)-x^*\right\rangle\\
&\le \alpha_n^2\|f(x_n)-x^*\|^2+\beta_n^2\left\|\frac{x_{n+1}+x_n}{2}-x^*\right\|^2+\gamma_n^2\left\|\frac{x_{n+1}+x_n}{2}-x^*\right\|^2\\
&\quad+2\alpha_n\beta_n\left\langle f(x_n)-f(x^*),S\left(\frac{x_{n+1}+x_n}{2}\right)-x^*\right\rangle\\
&\quad+2\alpha_n\beta_n\left\langle f(x^*)-x^*,S\left(\frac{x_{n+1}+x_n}{2}\right)-x^*\right\rangle\\
&\quad+2\alpha_n\gamma_n\left\langle f(x_n)-f(x^*),T\left(\frac{x_{n+1}+x_n}{2}\right)-x^*\right\rangle\\
&\quad+2\alpha_n\gamma_n\left\langle f(x^*)-x^*,T\left(\frac{x_{n+1}+x_n}{2}\right)-x^*\right\rangle\\
&\quad+2\beta_n\gamma_n\left\langle S\left(\frac{x_{n+1}+x_n}{2}\right)-x^*,T\left(\frac{x_{n+1}+x_n}{2}\right)-x^*\right\rangle\\
&\le (\beta_n^2+\gamma_n^2)\left\|\frac{x_{n+1}+x_n}{2}-x^*\right\|^2+2\alpha_n\beta_n\|f(x_n)-f(x^*)\|.\\
&\quad\left\|S\left(\frac{x_{n+1}+x_n}{2}\right)-x^*\right\|\\
&\quad+2\alpha_n\gamma_n\|f(x_n)-f(x^*)\|.\left\|T\left(\frac{x_{n+1}+x_n}{2}\right)-x^*\right\|\\
&\quad+2\beta_n\gamma_n\left\|S\left(\frac{x_{n+1}+x_n}{2}\right)-x^*\right\|.\left\|T\left(\frac{x_{n+1}+x_n}{2}\right)-x^*\right\|+K_n,
\end{aligned}
$$

where,

$$
\begin{aligned}
K_n &= \alpha_n^2\|f(x_n)-x^*\|^2+2\alpha_n\beta_n\left\langle f(x^*)-x^*,S\left(\frac{x_{n+1}+x_n}{2}\right)-x^*\right\rangle\\
&\quad+2\alpha_n\gamma_n\left\langle f(x^*)-x^*,T\left(\frac{x_{n+1}+x_n}{2}\right)-x^*\right\rangle
\end{aligned}
$$

This implies that

$$\begin{aligned}
& \|x_{n+1}-x_n\|^2 \\
\leq\ & (\beta_n^2+\gamma_n^2)\left\|\frac{x_{n+1}+x_n}{2}-x^*\right\|^2+2\alpha_n\beta_n\theta\,\|x_n-x^*\|\,.\left\|\frac{x_{n+1}+x_n}{2}-x^*\right\| \\
& +2\alpha_n\gamma_n\theta\,\|x_n-x^*\|\,.\left\|\frac{x_{n+1}+x_n}{2}-x^*\right\| \\
& +2\beta_n\gamma_n\left\|\frac{x_{n+1}+x_n}{2}-x^*\right\|.\left\|\frac{x_{n+1}+x_n}{2}-x^*\right\|+K_n \\
=\ & (\beta_n^2+\gamma_n^2+2\beta_n\gamma_n)\left\|\frac{x_{n+1}+x_n}{2}-x^*\right\|^2 \\
& +2\alpha_n\theta(\beta_n+\gamma_n)\,\|x_n-x^*\|\,.\left\|\frac{x_{n+1}+x_n}{2}-x^*\right\|+K_n \\
=\ & (\beta_n+\gamma_n)^2\left\|\frac{x_{n+1}+x_n}{2}-x^*\right\|^2 \\
& +2\alpha_n\theta(\beta_n+\gamma_n)\,\|x_n-x^*\|\,.\left\|\frac{x_{n+1}+x_n}{2}-x^*\right\|+K_n \\
=\ & (1-\alpha_n)^2\left\|\frac{x_{n+1}+x_n}{2}-x^*\right\|^2 \\
& +2\alpha_n\theta(1-\alpha_n)\left\|x_n-x^*\|.\|\frac{x_{n+1}+x_n}{2}-x^*\right\|+K_n.
\end{aligned}$$

The above calculation shows that

$$\begin{aligned}
0 \ \leq\ & 2\alpha_n\theta(1-\alpha_n)\,\|x_n-x^*\|\,.\left\|\frac{x_{n+1}+x_n}{2}-x^*\right\| \\
& +(1-\alpha_n)^2\left\|\frac{x_{n+1}+x_n}{2}-x^*\right\|^2-\|x_{n+1}-x^*\|^2+K_n,
\end{aligned}$$

which is a quadratic inequality in $\left\|\frac{x_{n+1}+x_n}{2}-x^*\right\|$. Solving the above inequality for $\left\|\frac{x_{n+1}+x_n}{2}-x^*\right\|$, we have,

$$\begin{aligned}
\left\|\frac{x_{n+1}+x_n}{2}-x^*\right\| \ \geq\ & \frac{-2\theta\alpha_n(1-\alpha_n)\|x_n-x^*\|}{2(1-\alpha_n)^2} \\
& +\frac{\sqrt{4\theta^2\alpha_n^2(1-\alpha_n)^2\|x_n-x^*\|^2-4(1-\alpha_n)^2(K_n-\|x_{n+1}-x^*\|^2)}}{2(1-\alpha_n)^2} \\
=\ & \frac{-\theta\alpha_n\|x_n-x^*\|+\sqrt{\theta^2\alpha_n^2\|x_n-x^*\|^2-K_n+\|x_{n+1}-x^*\|^2}}{1-\alpha_n}.
\end{aligned}$$

This will give

$$\begin{aligned} & \frac{1}{2}\left(\|x_{n+1}-x^*\|+\|x_n-x^*\|\right) \\ \geq \quad & \frac{-\theta\alpha_n\|x_n-x^*\|+\sqrt{\theta^2\alpha_n^2\|x_n-x^*\|^2-K_n+\|x_{n+1}-x^*\|^2}}{1-\alpha_n} \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} & \frac{1}{2}\left((1-\alpha_n)\|x_{n+1}-x^*\|+(1+(2\theta-1)\alpha_n)\|x_n-x^*\|\right) \\ \geq \quad & \sqrt{\theta^2\alpha_n^2\|x_n-x^*\|^2-K_n+\|x_{n+1}-x^*\|^2} \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} & \frac{1}{4}\left((1-\alpha_n)\|x_{n+1}-x^*\|+(1+(2\theta-1)\alpha_n)\|x_n-x^*\|\right)^2 \\ \geq \quad & \theta^2\alpha_n^2\|x_n-x^*\|^2-K_n+\|x_{n+1}-x^*\|^2, \end{aligned}$$

which is reduced to

$$\begin{aligned} & \frac{1}{4}(1-\alpha_n)^2\|x_{n+1}-x^*\|^2+\frac{1}{4}(1+(2\theta-1)\alpha_n)^2\|x_n-x^*\|^2 \\ + \quad & \frac{1}{2}(1-\alpha_n)(1+(2\theta-1)\alpha_n)\|x_{n+1}-x^*\|\|x_n-x^*\| \\ \geq \quad & \theta^2\alpha_n^2\|x_n-x^*\|^2-K_n+\|x_{n+1}-x^*\|^2. \end{aligned}$$

This inequality is further reduced by using the elementary inequality

$$2\|x_{n+1}-x^*\|\|x_n-x^*\| \quad \leq \quad \|x_{n+1}-x^*\|^2+\|x_n-x^*\|^2,$$

to the following inequality

$$\begin{aligned} & \frac{1}{4}(1-\alpha_n)^2\|x_{n+1}-x^*\|^2+\frac{1}{4}(1+(2\theta-1)\alpha_n)^2\|x_n-x^*\|^2 \\ + \quad & \frac{1}{4}(1-\alpha_n)(1+(2\theta-1)\alpha_n)(\|x_{n+1}-x^*\|^2+\|x_n-x^*\|^2) \\ \geq \quad & \theta^2\alpha_n^2\|x_n-x^*\|^2-K_n+\|x_{n+1}-x^*\|^2. \end{aligned}$$

This implies that

$$\begin{aligned} & \left(1-\frac{1}{4}(1-\alpha_n)^2-\frac{1}{4}(1-\alpha_n)(1+(2\theta-1)\alpha_n)\right)\|x_{n+1}-x^*\|^2 \\ \leq \quad & \left(\frac{1}{4}(1+(2\theta-1)\alpha_n)^2+\frac{1}{4}(1-\alpha_n)(1+(2\theta-1)\alpha_n)-\theta^2\alpha_n^2\right)\|x_n-x^*\|^2 \\ & +K_n, \end{aligned}$$

or

$$\begin{aligned} \|x_{n+1}-x^*\|^2 \quad \leq \quad & \frac{\frac{1}{4}(1-\alpha_n)(1+(2\theta-1)\alpha_n)-\theta^2\alpha_n^2}{1-\frac{1}{4}(1-\alpha_n)^2-\frac{1}{4}(1-\alpha_n)(1+(2\theta-1)\alpha_n)}\|x_n-x^*\|^2 \\ & +\frac{\frac{1}{4}(1+(2\theta-1)\alpha_n)^2}{1-\frac{1}{4}(1-\alpha_n)^2-\frac{1}{4}(1-\alpha_n)(1+(2\theta-1)\alpha_n)}+K_n', \quad (2) \end{aligned}$$

where,

$$K_n' = \frac{K_n}{1 - \frac{1}{4}(1-\alpha_n)^2 - \frac{1}{4}(1-\alpha_n)(1+(2\theta-1)\alpha_n)}.$$

Note that,

$$\begin{aligned}
& 1 - \frac{1}{4}(1-\alpha_n)^2 - \frac{1}{4}(1-\alpha_n)(1+(2\theta-1)\alpha_n) \\
= \; & 1 - \frac{1}{4}(1-\alpha_n)(1-\alpha_n+1+(2\theta-1)\alpha_n) \\
= \; & 1 - \frac{1}{4}(1-\alpha_n)(1-\alpha_n+1+2\theta\alpha_n-\alpha_n) \\
= \; & 1 - \frac{1}{4}(1-\alpha_n)(2-2\alpha_n+2\theta\alpha_n) \\
= \; & 1 - \frac{1}{2}(1-\alpha_n)(1-\alpha_n+\theta\alpha_n) \\
= \; & 1 - \frac{1}{2}(1-\alpha_n)(1-\alpha_n(1-\theta)),
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{4}(1+(2\theta-1)\alpha_n)^2 + \frac{1}{4}(1-\alpha_n)(1+(2\theta-1)\alpha_n) - \theta^2\alpha_n^2 \\
= \; & \frac{1}{4}(1+(2\theta-1)\alpha_n)(1+(2\theta-1)\alpha_n+1-\alpha_n) - \theta^2\alpha_n^2 \\
= \; & \frac{1}{4}(1+(2\theta-1)\alpha_n)(2+2\theta\alpha_n-2\alpha_n) - \theta^2\alpha_n^2 \\
= \; & \frac{1}{2}(1+(2\theta-1)\alpha_n)(1+\theta\alpha_n-\alpha_n) - \theta^2\alpha_n^2 \\
= \; & \frac{1}{2}(1+(2\theta-1)\alpha_n)(1-(1-\theta)\alpha_n) - \theta^2\alpha_n^2.
\end{aligned}$$

Now from (2),

$$\begin{aligned}
& \|x_{n+1}-x^*\|^2 \\
\leq \; & \frac{\frac{1}{2}(1+(2\theta-1)\alpha_n)(1-(1-\theta)\alpha_n) - \theta^2\alpha_n^2}{1-\frac{1}{2}(1-\alpha_n)(1-\alpha_n(1-\theta))}\|x_n-x^*\|^2 + K_n'. \qquad (3)
\end{aligned}$$

Consider the following function, for $t > 0$.

$$g(t) := \frac{1}{t}\left\{1 - \frac{\frac{1}{2}(1+(2\theta-1)t)(1-(1-\theta)t) - \theta^2t^2}{1-\frac{1}{2}(1-t)(1-t(1-\theta))}\right\}$$

$$\begin{aligned}
g(t) &= \frac{1}{t}\left\{\frac{1-\frac{1}{2}(1-t)(1-t(1-\theta)) - \frac{1}{2}(1+(2\theta-1)t)(1-(1-\theta)t) + \theta^2t^2}{1-\frac{1}{2}(1-t)(1-t(1-\theta))}\right\} \\
&= \frac{1}{t}\left\{\frac{1-\frac{1}{2}(1-t(1-\theta))(1-t+1+2\theta t-t) + \theta^2t^2}{1-\frac{1}{2}(1-t)(1-t(1-\theta))}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{t}\left\{\frac{1-\frac{1}{2}(1-t(1-\theta))(2-2t+2\theta t)+\theta^2t^2}{1-\frac{1}{2}(1-t)(1-t(1-\theta))}\right\}\\
&= \frac{1}{t}\left\{\frac{1-(1-t+\theta t))(1-t+\theta t))+\theta^2t^2}{1-\frac{1}{2}(1-t)(1-t(1-\theta))}\right\}\\
&= \frac{1}{t}\left\{\frac{1-(1+t^2+\theta^2t^2-2t-2\theta t^2+2\theta t)+\theta^2t^2}{1-\frac{1}{2}(1-t)(1-t(1-\theta))}\right\}\\
&= \frac{1}{t}\left\{\frac{1-1-t^2-\theta^2t^2+2t+2\theta t^2-2\theta t+\theta^2t^2}{1-\frac{1}{2}(1-t)(1-t(1-\theta))}\right\}\\
&= \frac{-t+2+2\theta t-2\theta}{1-\frac{1}{2}(1-t)(1-t(1-\theta))}.
\end{aligned}
$$

By applying limit $t \to 0$, we have

$$\lim_{t\to 0} g(t) = 4(1-\theta) > 0.$$

Let $\delta > 0$ be such that for all $0 < t < \delta$, $g(t) > \epsilon := 4(1-\theta) > 0$. This is equivalent to

$$\frac{1}{t}\left\{1-\frac{\frac{1}{2}(1+(2\theta-1)t)(1-(1-\theta)t)-\theta^2t^2}{1-\frac{1}{2}(1-t)(1-t(1-\theta))}\right\} \quad > \quad \epsilon$$

This implies,

$$1-t\epsilon > \frac{\frac{1}{2}(1+(2\theta-1)t)(1-(1-\theta)t)-\theta^2t^2}{1-\frac{1}{2}(1-t)(1-t(1-\theta))}.$$

Since $\alpha_n \to 0$ as $n \to \infty$, there exist some integer N, such that $\alpha_n < \delta$, $\forall\, n \geq N$. From (3), we have

$$\|x_{n+1}-x^*\|^2 \leq (1-\alpha_n\epsilon)\|x_n-x^*\|^2 + K'_n$$

On the other hand, we have

$$
\begin{aligned}
\limsup_{n\to\infty}\frac{K_n}{\alpha_n} \quad = \quad & \limsup_{n\to\infty}\left\{2\beta_n\left\langle f(x^*)-x^*, S\left(\frac{x_{n+1}+x_n}{2}\right)-x^*\right\rangle\right.\\
& +2\gamma_n\left\langle f(x^*)-x^*, T\left(\frac{x_{n+1}+x_n}{2}\right)-x^*\right\rangle\\
& \left.+\alpha_n\|f(x_n)-x^*\|^2\right\} \leq 0.
\end{aligned}
$$

The above inequality implies that

$$\limsup_{n\to\infty}\frac{K'_n}{\alpha_n} \quad \leq \quad 0.$$

From the above two inequalities and Theorem 1.4 we have

$$\lim_{n\to\infty}\|x_{n+1}-x^*\|^2 = 0,$$

which implies that $x_n \to x^*$ as $n \to \infty$. This completes the proof.

□

Competing Interests

None of the authors have any competing interests in the manuscript.

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