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COMPUTING TOPOLOGICAL INDICES OF HEX BOARD AND ITS LINE GRAPH

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ABSTRACT. A topological index is a real number related to a molecular graph, which is a graph invariant. Uptill now there are several topological indices are defined. Some of them are distance based while the others are degree based, all have found numerous applications in pharmacy, theoretical chemistry and especially in QSPR/QSAR research. In this paper, we compute some degree based topological indices i.e some versions of Zagreb indices, Randić index, General sum connectivity index and GA index of Hex board and of its line graph.

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Key words and phrases: degree; topological index; Zagreb index.

1. Introduction and Preliminaries

Mathematical chemistry is the branch of theoretical chemistry in which we discuss and predict the behavior of mathematical structure by using mathematical tools. There is lot of research which is done in this area in the last few decades. This theory contributes a major role in the field of chemical sciences.
Let G be the molecular graph in which $V(G)$ represents the set of vertices corresponds the atoms and $E(G)$ the set of edges to the chemical bonds. A line graph $L(G)$ of a simple graph G is obtained by associating a vertex with each edge of the graph and connecting two vertices with an edge if and only if the corresponding edges of G have a vertex in common.
The very first topological index Randić index introduced by Milan Randić in

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1975 (see [1]). and is defined as:

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}.$$

Later, this index was generalized by Bollobás & Erdös(see [2]) to the following form for any real number α, and named the general Randić index:

$$R_\alpha(G) = \sum_{uv \in E(G)} [d_u d_v]^\alpha.$$

The Zagreb indices were first introduced by Gutman in [3], they are important molecular descriptors and have been closely correlated with many chemical properties (see [4]) and defined as:

$$M_1(G) = \sum_{u \in V(G)} d_u^2 \qquad \text{and}$$

$$M_2(G) = \sum_{uv \in E(G)} d_u d_v.$$

The third Zagreb index, introduced by Fath-Tabar in [5]. This index is defined as follows:

$$M_3(G) = \sum_{uv \in E(G)} |d_u - d_v|.$$

The hyper-Zagreb index was first introduced in [6]. This index is defined as follows:

$$HM(G) \quad = \sum_{uv \in E(G)} (d_u + d_v)^2.$$

The Atom-Bond Connectivity index (ABC), introduced by Estrada, *et al.* in [7] and applied up until now to study the stability of alkanes and the strain energy of cycloalkanes. The ABC index of G is defined as:

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}.$$

For more details see the article [8]. In 2010, the general sum-connectivity index $\chi(G)$ has been introduced in [9]. For more detail on sum connectivity we refer the articles [9, 10]. This index is defined as follows:

$$\chi(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u + d_v}}.$$

Vukićević and Furtula introduced the geometric arithmetic (GA) index in [11]. The GA index for G is defined by

$$GA(G) = \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v}.$$

Inspired by the work on the ABC index, Furtula *et al.* proposed the following modified version of the ABC index and called it as augmented Zagreb index (AZI) in [12]. This index is defined as follows:

$$AZI(G) = \sum_{uv \in E(G)} \left(\frac{d_u d_v}{d_u + d_v - 2} \right)^3.$$

The hexagonal and honeycomb networks have also been recognized as crucial evolutionary biology, in particular for the evolution of cooperation, where the overlapping triangles are vital for the propagation of cooperation in social dilemmas. Relevant research that applies this theory and which could benefit further from the insights of the new research in (see [13]).
The following lemma is helpful for computing the degree of a vertex of line graph.

Lemma 1.1. *Let G be a graph with $u, v \in V(G)$ and $e = uv \in E(G)$. Then:*

$$d_e = d_u + d_v - 2.$$

Lemma 1.2. [14] *Let G be a graph of order p and size q, then the line graph $L(G)$ of G is a graph of order p and size $\frac{1}{2}M_1(G) - q$.*

2. Topological indices of Hex board

In this section we will compute the topological indices of Hex board.

Theorem 2.1. *Let G be the Hex board H_n. Then*

(1) $M_1(G) = 36n^2 - 80n + 42$;
(2) $M_2(G) = 108n^2 - 320n + 232$;
(3) $M_3(G) = -16n + 22$;
(4) $HM(G) = 432n^2 - 1248n + 886$;
(5) $R(G) = \frac{1}{2}n^2 + \frac{1}{12}(8n - 20)\sqrt{6} - \frac{5}{3}n + \frac{4}{3}\sqrt{2} + \frac{2}{3}\sqrt{3} + 1$;
(6) $ABC(G) = \frac{1}{6}(3n^2 - 16n + 21)\sqrt{10} + \frac{1}{3}(8n - 20)\sqrt{3} + \frac{1}{4}(4n - 10)\sqrt{6} + \frac{1}{3}\sqrt{14} + \frac{2}{3}\sqrt{15} + 2\sqrt{2}$;
(7) $\chi(G) = \frac{1}{6}(3n^2 - 16n + 21)\sqrt{3} + \frac{1}{10}(8n - 20)\sqrt{10} + \frac{1}{4}(4n - 10)\sqrt{2} + \frac{2}{3}\sqrt{6} + \frac{4}{7}\sqrt{7} + \frac{2}{3}$;
(8) $GA(G) = 3n^2 + \frac{2}{5}(8n - 20)\sqrt{6} - 12n + 4\sqrt{2} + \frac{16}{7}\sqrt{3} + 11$;
(9) $AZI(G) = \frac{17496}{125}n^2 - \frac{1534424}{3375}n + \frac{429997724}{1157625}$.

Proof. The graph G for $n = 4$ is shown in Figure 1. It is easy to see that the order of G is n^2 out of which 2 vertices are of degree 2, 2 vertices are of degree 3, $4(n-2)$ vertices are of degree 4 and $n^2 - 4(n-1)$ vertices are of degree 6 and G has size $3n^2 - 4n + 1$. We partition the size of G into edges of the type $E_{(d_u,d_v)}$ where uv is an edge. In G, we get edges of the type $E_{(2,4)}$, $E_{(3,4)}$, $E_{(3,6)}$, $E_{(4,4)}$, $E_{(4,6)}$ and $E_{(6,6)}$. The number of edges of these types are given in the Table 1. Then we obtain the required results by using Table1 as follows:

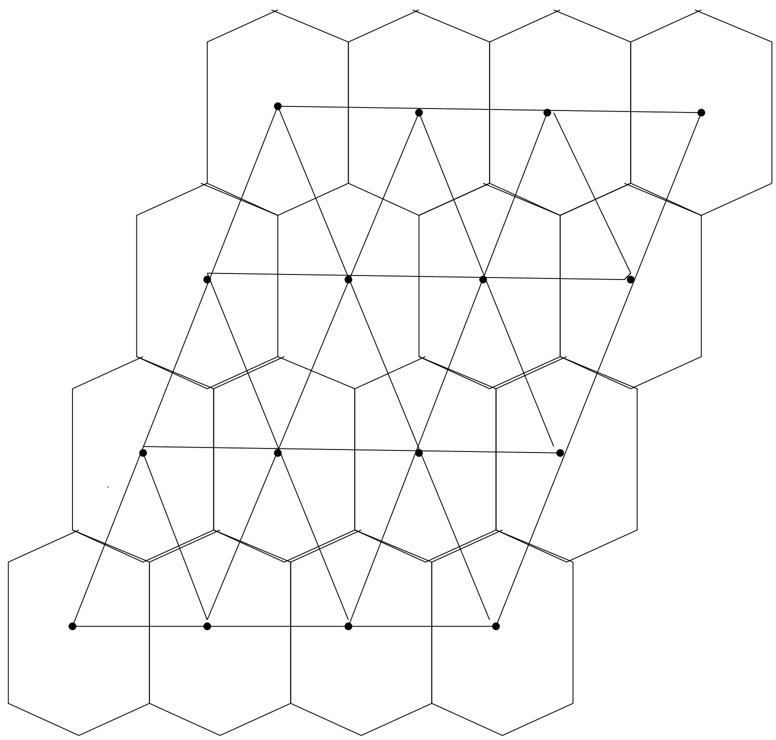

FIGURE 1. Hex board with center dots.

(1) $M_1(G) = \sum_{uv \in E(G)} \left[d_u + d_v\right]$
$= 4(2+4) + 4(3+4) + 2(3+6) + (4n-10)(4+4) + (8n-20)(4+6) + (3n^2-16n+21)(6+6)$
$= 36n^2 - 80n + 42.$

(2) $M_2(G) = \sum_{uv \in E(G)} d_u d_v$
$= 4(2 \times 4) + 4(3 \times 4) + 2(3 \times 6) + (4n-10)(4 \times 4) + (8n-20)(4 \times 6) + (3n^2-16n+21)(6 \times 6)$
$= 108n^2 - 320n + 232.$

(3) $M_3(G) = \sum_{uv \in E(G)} |d_u - d_v|$
$= 4(2-4) + 4(3-4) + 2(3-6) + (4n-10)(4-4) + (8n-20)(4-6) + (3n^2-16n+21)(6-6).$
$= -16n + 22.$

(4) $HM(G) = \sum_{uv \in E(G)} (d_u + d_v)^2$
$= 4(2+4)^2 + 4(3+4)^2 + 2(3+6)^2 + (4n-10)(4+4)^2 + (8n-20)(4+6)^2 + (3n^2-16n+21)(6+6)^2.$
$= 432n^2 - 1248n + 886.$

(5) $R(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d_u d_v}}$
$= 4(\frac{1}{\sqrt{2\times 4}}) + 4(\frac{1}{\sqrt{3\times 4}}) + 2(\frac{1}{\sqrt{3\times 6}}) + (4n-10)(\frac{1}{\sqrt{4\times 4}}) + (8n-20)(\frac{1}{\sqrt{4\times 6}}) +$
$(3n^2 - 16n + 21)(\frac{1}{\sqrt{6\times 6}})$.
$\frac{1}{2}n^2 + \frac{1}{12}(8n-20)\sqrt{6} - \frac{5}{3}n + \frac{4}{3}\sqrt{2} + \frac{2}{3}\sqrt{3} + 1$.

(6) $ABC(G) = \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u d_v}}$
$= 4\sqrt{\frac{2+4-2}{2\times 4}} + 4\sqrt{\frac{3+4-2}{3\times 4}} + 2\sqrt{\frac{3+6-2}{3\times 6}} + (4n-10)\sqrt{\frac{4+4-2}{4\times 4}} + (8n-20)\sqrt{\frac{4+6-2}{4\times 6}} +$
$(3n^2 - 16n + 21)\sqrt{\frac{6+6-2}{6\times 6}}$.
$= \frac{1}{6}(3n^2 - 16n + 21)\sqrt{10} + \frac{1}{3}(8n-20)\sqrt{3} + \frac{1}{4}(4n-10)\sqrt{6} + \frac{1}{3}\sqrt{14} +$
$\frac{2}{3}\sqrt{15} + 2\sqrt{2}$.

(7) $\chi(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d_u+d_v}}$
$= 4(\frac{1}{\sqrt{2+4}}) + 4(\frac{1}{\sqrt{3+4}}) + 2(\frac{1}{\sqrt{3+6}}) + (4n-10)(\frac{1}{\sqrt{4+4}}) + (8n-20)(\frac{1}{\sqrt{4+6}}) +$
$(3n^2 - 16n + 21)(\frac{1}{\sqrt{6+6}})$.
$= \frac{1}{6}(3n^2 - 16n + 21)\sqrt{3} + \frac{1}{10}(8n-20)\sqrt{10} + \frac{1}{4}(4n-10)\sqrt{2} + \frac{2}{3}\sqrt{6} + \frac{4}{7}\sqrt{7} + \frac{2}{3}$.

(8) $GA(G) = \sum_{uv\in E(G)} \frac{2\sqrt{d_u d_v}}{d_u+d_v}$
$= 4(\frac{2\sqrt{2\times 4}}{2+4}) + 4(\frac{2\sqrt{3\times 4}}{3+4}) + 2(\frac{2\sqrt{3\times 6}}{3+6}) + (4n-10)(\frac{2\sqrt{4\times 4}}{4+4}) + (8n-20)(\frac{2\sqrt{4\times 6}}{4+6}) +$
$(3n^2 - 16n + 21)(\frac{2\sqrt{6\times 6}}{6+6})$.
$= 3n^2 + \frac{2}{5}(8n-20)\sqrt{6} - 12n + 4\sqrt{2} + \frac{16}{7}\sqrt{3} + 11$.

(9) $AZI(G) = \sum_{uv\in E(G)} \left(\frac{d_u d_v}{d_u+d_v-2}\right)^3$
$= 4(\frac{2\times 4}{2+4-2})^3 + 4(\frac{3\times 4}{3+4-2})^3 + 2(\frac{3\times 6}{3+6-2})^3 + (4n-10)(\frac{4\times 4}{4+4-2})^3 + (8n-$
$20)(\frac{4\times 6}{4+6-2})^3 + (3n^2 - 16n + 21)(\frac{6\times 6}{6+6-2})^3$.
$= \frac{17496}{125}n^2 - \frac{1534424}{3375}n + \frac{429997724}{1157625}$.

TABLE 1. The size partition of G

(d_u, d_v) where $uv \in E(G)$	$(2,4)$	$(3,4)$	$(3,6)$
Number of edges	4	4	2
(d_u, d_v) where $uv \in E(G)$	$(4,4)$	$(4,6)$	$(6,6)$
Number of edges	$4n-10$	$8n-20$	$3n^2 - 16n + 21$

□

3. Topological indices of line graph of Hex board

In this section we will compute the topological indices of the line graph of Hex board.

Theorem 3.1. *Let G be the line graph of the Hex board H_n. Then*

(1) $M_1(G) = 300n^2 - 944n + 722$;
(2) $M_2(G) = 1500n^2 - 5616n + 5194$;

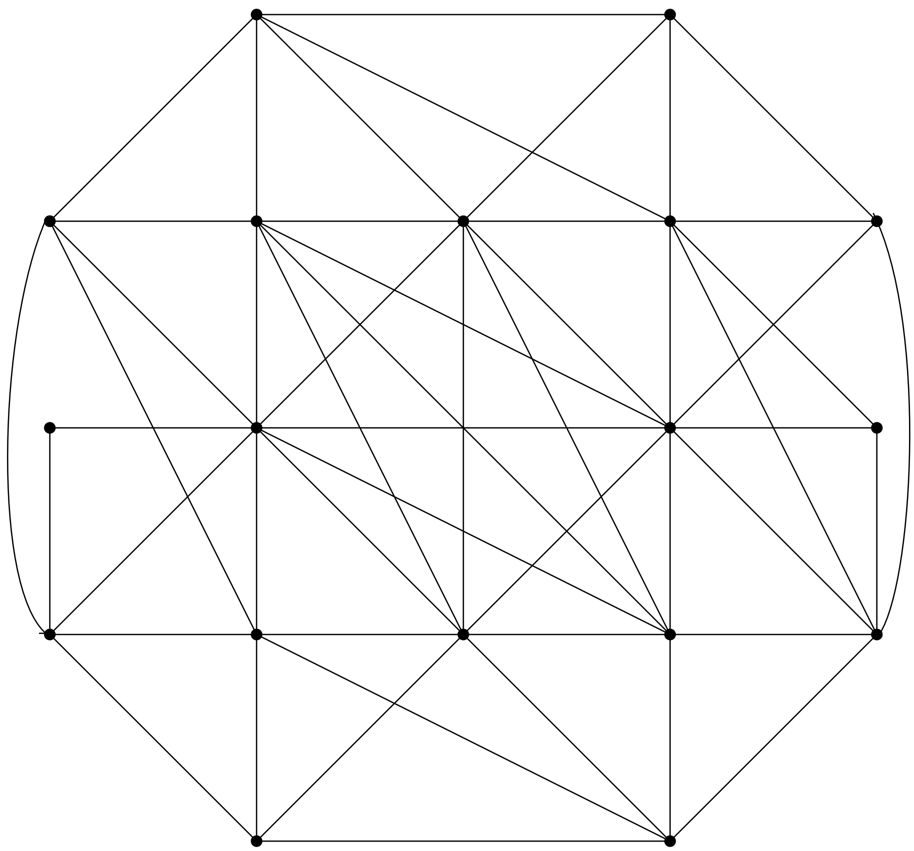

FIGURE 2. Line graph of hex board with center dots.

(3) $M_3(G) = -96n + 206$;
(4) $HM(G) = 6000n^2 - 22272n + 20430$;
(5) $R(G) = \frac{3}{2}n^2 - \frac{119}{15}n + \frac{1}{20}(32n-100)\sqrt{5} + \frac{3}{35}\sqrt{100} + \frac{1}{7}\sqrt{14} + \frac{1}{12}(16n-48)\sqrt{3} + \frac{2}{5}\sqrt{10} + \frac{4}{35}\sqrt{35} + \frac{2}{15}\sqrt{30} + \frac{1}{2}\sqrt{2} + \frac{2}{3}\sqrt{6} + \frac{247}{20}$;
(6) $ABC(G) = \frac{3}{10}(15n^2-96n+152)\sqrt{2} + \frac{1}{5}(32n-100)\sqrt{5} + \frac{1}{6}(4n-12)\sqrt{10} + 8n + \frac{1}{8}(8n-14)\sqrt{14} + \frac{3}{7}\sqrt{42} + \frac{1}{7}\sqrt{82} + \frac{2}{5}\sqrt{110} + \frac{4}{7}\sqrt{14} + \frac{2}{5}\sqrt{30} + \frac{4}{5}\sqrt{2} + \sqrt{5} + \frac{8}{3}\sqrt{3} + \frac{1}{2}\sqrt{6} - 24$;
(7) $\chi(G) = \frac{1}{10}(15n^2-96n+152)\sqrt{5} + \frac{1}{6}(32n-100)\sqrt{2} + \frac{1}{14}(16n-48)\sqrt{14} + \frac{1}{6}(4n-12)\sqrt{3} + 2n - \frac{7}{2} + \frac{6}{17}\sqrt{17} + \frac{4}{15}\sqrt{15} + \frac{8}{13}\sqrt{13} + \frac{4}{11}\sqrt{11} + \frac{4}{3}\sqrt{3} + \sqrt{10} + \frac{1}{2}\sqrt{2}$;
(8) $GA(G) = 15n^2 + \frac{4}{9}(32n-100)\sqrt{5} + \frac{4}{7}(16n-48)\sqrt{3} - 84n + \frac{12}{17}\sqrt{70} + \frac{16}{15}\sqrt{14} + \frac{32}{13}\sqrt{10} + \frac{2}{3}\sqrt{35} + \frac{8}{11}\sqrt{30} + \frac{8}{3}\sqrt{2} + \frac{16}{5}\sqrt{6} + 130$;
(9) $AZI(G) = \frac{625000}{243}n^2 - \frac{109249838528}{10418625}n + \frac{1427097955303080667}{133703165756160}$.

Proof. The graph G for $n = 4$ is shown in Figure 2. By using Lemma 1.1, It is easy to see that the order of G is $3n^2 - 4n + 1$ out of which 4 vertices are of degree 4, 4 vertices are of degree 5, 2 vertices are of degree 7, $4n - 10$ vertices are of degree 6, $8n - 20$ vertices are of degree 8 and $3n^2 - 16n + 21$ vertices are of degree 10. Therefore by using Lemma 1.2, G has size $15n^2 - 36n + 20$. We partition the size of G into edges of the type $E_{(d_u,d_v)}$ where uv is an edge. In G, we get edges of the type $E_{(4,4)}$, $E_{(4,6)}$, $E_{(4,8)}$, $E_{(5,5)}$, $E_{(5,6)}$, $E_{(5,7)}$, $E_{(5,8)}$, $E_{(6,6)}$, $E_{(6,8)}$, $E_{(7,8)}$, $E_{(7,10)}$, $E_{(8,8)}$, $E_{(8,10)}$ and $E_{(10,10)}$. The number of edges of these types are given in the Table 2. Then we obtain the required results by using Table2 as follows:

(1) $M_1(G) = \sum_{uv\in E(G)} \left[d_u + d_v\right]$
$= 2(4+4) + 8(4+6) + 4(4+8) + 2(5+5) + 4(5+6) + 4(5+7) + 8(5+8) + (4n-12)(6+6) + (16n-48)(6+8) + 4(7+8) + 6(7+10) + (8n-14)(8+8) + (32n-100)(8+10) + (15n^2-96n+152)(10+10).$
$= 300n^2 - 944n + 722.$

(2) $M_2(G) = \sum_{uv\in E(G)} d_u d_v$
$= 2(4\times4) + 8(4\times6) + 4(4\times8) + 2(5\times5) + 4(5\times6) + 4(5\times7) + 8(5\times8) + (4n-12)(6\times6) + (16n-48)(6\times8) + 4(7\times8) + 6(7\times10) + (8n-14)(8\times8) + (32n-100)(8\times10) + (15n^2-96n+152)(10\times10).$
$= 1500n^2 - 5616n + 5194.$

(3) $M_3(G) = \sum_{uv\in E(G)} |d_u - d_v$
$= 2(4-4) + 8(4-6) + 4(4-8) + 2(5-5) + 4(5-6) + 4(5-7) + 8(5-8) + (4n-12)(6-6) + (16n-48)(6-8) + 4(7-8) + 6(7-10) + (8n-14)(8-8) + (32n-100)(8-10) + (15n^2-96n+152)(10-10).$
$= -96n + 206.$

(4) $HM(G) = \sum_{uv\in E(G)}(d_u + d_v)^2$
$= 2(4+4)^2 + 8(4+6)^2 + 4(4+8)^2 + 2(5+5)^2 + 4(5+6)^2 + 4(5+7)^2 + 8(5+8)^2 + (4n-12)(6+6)^2 + (16n-48)(6+8)^2 + 4(7+8)^2 + 6(7+10)^2 + (8n-14)(8+8)^2 + (32n-100)(8+10)^2 + (15n^2-96n+152)(10+10)^2.$
$= 6000n^2 - 22272n + 20430.$

(5) $R(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d_u d_v}}$
$= 2(\frac{1}{\sqrt{4\times4}}) + 8(\frac{1}{\sqrt{4\times6}}) + 4(\frac{1}{\sqrt{4\times8}}) + 2(\frac{1}{\sqrt{5\times5}}) + 4(\frac{1}{\sqrt{5\times6}}) + 4(\frac{1}{\sqrt{5\times7}}) + 8(\frac{1}{\sqrt{5\times8}}) + (4n-12)(\frac{1}{\sqrt{6\times6}}) + (16n-48)(\frac{1}{\sqrt{6\times8}}) + 4(\frac{1}{\sqrt{7\times8}}) + 6(\frac{1}{\sqrt{7\times10}}) + (8n-14)(\frac{1}{\sqrt{8\times8}}) + (32n-100)(\frac{1}{\sqrt{8\times10}}) + (15n^2-96n+152)(\frac{1}{\sqrt{10\times10}}).$
$= \frac{3}{2}n^2 - \frac{119}{15}n + \frac{1}{20}(32n-100)\sqrt{5} + \frac{3}{35}\sqrt{100} + \frac{1}{7}\sqrt{14} + \frac{1}{12}(16n-48)\sqrt{3} + \frac{2}{5}\sqrt{10} + \frac{4}{35}\sqrt{35} + \frac{2}{15}\sqrt{30} + \frac{1}{2}\sqrt{2} + \frac{2}{3}\sqrt{6} + \frac{247}{20}.$

(6) $ABC(G) = \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u d_v}}$
$= 2(\sqrt{\frac{4+4-2}{4\times4}}) + 8(\sqrt{\frac{4+6-2}{4\times6}}) + 4(\sqrt{\frac{4+8-2}{4\times8}}) + 2(\sqrt{\frac{5+5-2}{5\times5}}) + 4(\sqrt{\frac{5+6-2}{5\times6}}) + 4(\sqrt{\frac{5+7-2}{5\times7}}) + 8(\sqrt{\frac{5+8-2}{5\times8}}) + (4n-12)(\sqrt{\frac{6+6-2}{6\times6}}) + (16n-48)(\sqrt{\frac{6+8-2}{6\times8}}) +$

$4(\sqrt{\frac{7+8-2}{7\times8}})+6(\sqrt{\frac{7+10-2}{7\times10}})+(8n-14)(\sqrt{\frac{8+18-2}{8\times8}})+(32n-100)(\sqrt{\frac{8+10-2}{8\times10}})+(15n^2-96n+152)(\sqrt{\frac{10+10-2}{10\times10}})$.
$=\frac{3}{10}(15n^2-96n+152)\sqrt{2}+\frac{1}{5}(32n-100)\sqrt{5}+\frac{1}{6}(4n-12)\sqrt{10}+8n+\frac{1}{8}(8n-14)\sqrt{14}+\frac{3}{7}\sqrt{42}+\frac{1}{7}\sqrt{82}+\frac{2}{5}\sqrt{110}+\frac{4}{7}\sqrt{14}+\frac{2}{5}\sqrt{30}+\frac{4}{5}\sqrt{2}+\sqrt{5}+\frac{8}{3}\sqrt{3}+\frac{1}{2}\sqrt{6}-24;$

(7) $\chi(G)=\sum_{uv\in E(G)}\frac{1}{\sqrt{d_u+d_v}}$
$=2(\frac{1}{\sqrt{4+4}})+8(\frac{1}{\sqrt{4+6}})+4(\frac{1}{\sqrt{4+8}})+2(\frac{1}{\sqrt{5+5}})+4(\frac{1}{\sqrt{5+6}})+4(\frac{1}{\sqrt{5+7}})+8(\frac{1}{\sqrt{5+8}})+(4n-12)(\frac{1}{\sqrt{6+6}})+(16n-48)(\frac{1}{\sqrt{6+8}})+4(\frac{1}{\sqrt{7+8}})+6(\frac{1}{\sqrt{7+10}})+(8n-14)(\frac{1}{\sqrt{8+8}})+(32n-100)(\frac{1}{\sqrt{8+10}})+(15n^2-96n+152)(\frac{1}{\sqrt{10+10}})$.
$=\frac{1}{10}(15n^2-96n+152)\sqrt{5}+\frac{1}{6}(32n-100)\sqrt{2}+\frac{1}{14}(16n-48)\sqrt{14}+\frac{1}{6}(4n-12)\sqrt{3}+2n-\frac{7}{2}+\frac{6}{17}\sqrt{17}+\frac{4}{15}\sqrt{15}+\frac{8}{13}\sqrt{13}+\frac{4}{11}\sqrt{11}+\frac{4}{3}\sqrt{3}+\sqrt{10}+\frac{1}{2}\sqrt{2}.$

(8) $GA(G)=\sum_{uv\in E(G)}\frac{2\sqrt{d_ud_v}}{d_u+d_v}$
$=2\frac{2\sqrt{4\times4}}{4+4}+8\frac{2\sqrt{4\times6}}{4+6}+4\frac{2\sqrt{4\times8}}{4+8}+2\frac{2\sqrt{5\times5}}{5+5}+4\frac{2\sqrt{5\times6}}{5+6}+4\frac{2\sqrt{5\times7}}{5+7}+8\frac{2\sqrt{5\times8}}{5+8}+(4n-12)\frac{2\sqrt{6\times6}}{6+6}+(16n-48)\frac{2\sqrt{6\times8}}{6+8}+4\frac{2\sqrt{7\times8}}{7+8}+6\frac{2\sqrt{7\times10}}{7+10}+(8n-14)\frac{2\sqrt{8\times8}}{8+8}+(32n-100)\frac{2\sqrt{8\times10}}{8+10}+(15n^2-96n+152)\frac{2\sqrt{10\times4}}{10+10}$.
$=15n^2+\frac{4}{9}(32n-100)\sqrt{5}+\frac{4}{7}(16n-48)\sqrt{3}-84n+\frac{12}{17}\sqrt{70}+\frac{16}{15}\sqrt{14}+\frac{32}{13}\sqrt{10}+\frac{2}{3}\sqrt{35}+\frac{8}{11}\sqrt{30}+\frac{8}{3}\sqrt{2}+\frac{16}{5}\sqrt{6}+130.$

(9) $AZI(G)=\sum_{uv\in E(G)}(\frac{d_ud_v}{d_u+d_v-2})^3$
$=2(\frac{4\times4}{4+4-2})^3+8(\frac{4\times6}{4+6-2})^3+4(\frac{4\times8}{4+8-2})^3+2(\frac{5\times5}{5+5-2})^3+4(\frac{5\times6}{5+6-2})^3+4(\frac{5\times7}{5+7-2})^3+8(\frac{5\times8}{5+8-2})^3+(4n-12)(\frac{6\times6}{6+6-2})^3+(16n-48)(\frac{6\times8}{6+8-2})^3+4(\frac{7\times8}{7+8-2})^3+6(\frac{7\times10}{7+10-2})^3+(8n-14)(\frac{8\times8}{8+8-2})^3+(32n-100)(\frac{8\times10}{8+10-2})^3+(15n^2-96n+152)(\frac{10\times10}{10+10-2})^3$
$=\frac{625000}{243}n^2-\frac{109249838528}{10418625}n+\frac{1427097955303080667}{133703165756160}.$

TABLE 2. The size partition of G

(d_u,d_v) where $uv\in E(G)$	$(4,4)$	$(4,6)$	$(4,8)$	$(5,5)$	$(5,6)$
Number of edges	2	8	4	2	4
(d_u,d_v) where $uv\in E(G)$	$(5,7)$	$(5,8)$	$(6,6)$	$(6,8)$	$(7,8)$
Number of edges	4	8	$4n-12$	$16n-48$	4
(d_u,d_v) where $uv\in E(G)$	$(7,10)$	$(8,8)$	$(8,10)$	$(10,10)$	
Number of edges	6	$8n-14$	$32n-100$	$15n^2-96n+152$	

□

Competing Interests

The author(s) do not have any competing interests in the manuscript.

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