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ANALYTICAL SOLUTION FOR THE FLOW OF A GENERALIZED OLDROYD-B FLUID IN A CIRCULAR CYLINDER

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Abstract. The tangential stress and velocity field corresponding to the flow of a generalized Oldroyd-B fluid in an infinite circular cylinder will be determined by mean of Laplace and finite Hankel transform. The motion is produced by the cylinder, that after $t = 0^+$, begins to rotate about its axis, under the action of oscillating shear stress $\Omega R \sin(\omega t)$ given on boundary. The solutions are based on an important remark regarding the governing equation for the non- trivial shear stress. The solutions that have been obtained satisfy all imposed initial and boundary conditions. The obtained solution will be presented under series form in term of generalized G-function. The similar solutions for the ordinary Oldroyd-B fluid, Maxwell, ordinary Maxwell and Newtonian fluids performing the same motion will be obtained as special cases of our general solutions.

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1. Introduction

The Oldroyd-B fluid models is very important among the fluids of rate type due to its special behavior. Also, this model contains the Newtonian fluid model and Maxwell fluid model as special cases. The Oldroyd-B fluid model [1, 2] considered the memory effects and elastic effects exhibited by a large class of fluids such as the biological and polymeric liquids. The motion of a fluid in the neighborhood of a moving body is of great interest for industry. The flow between cylinders or

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through a rotating cylinder has applications in the food industry and being one of the most important and interesting problem of motion near rotating bodies. Exact solutions for some simple flows of Oldroyd-B fluids were presented by many authors, see for example, Rajagopal and Bhatnagar [3], Hayat et al. [4, 5]. The velocity distriution for different motions of Newtonian fluids through a circular cylinder is given in [6]. Wood [7] has considered the general case of helical flow of an Oldroyd-B fluid, due to the combined action of rotating cylinders(with constant angular velocities) and a constant axial pressure gradient. Accurate solutions regarding motions of Non-Newtonian fluids in cylindrical domains appear to be those of Ting [2], Srivastava [8] and Water and King [9] for second grade, Maxwell and Oldroyd-B fluids respectively.

The most general solution corresponding to the helical flow of a second grade fluid seem to be those of Fetecau and Cornia Fetecau [10], in which the cylinder is rotating around its axis and sliding along the same axis with time-dependent velocities.

There is a vast literature dealing with such fluids, but we shall recall here only a few of the most recent papers [12-16]. Most existing solutions in the literature correspond to problems with boundary conditions on the velocity. Though, all above mentioned papers incorporate motion problem in which velocity is given on the boundary. In [16], Renardy explained how well posed boundary value problems can be formulated using boundary conditions on stress. Water and King [17] were among the first specialists who used the shear stress on boundary to find exact solution for motions of rate type fluids. Our goal is to investigate analytical solution for the flow of a generalized Oldroyd-B fluid in a circular cylinder. We considered the boundary conditions on the shear stress.

The flow of fluid is due to rotation of the cyliner around its axis, under the action of oscillating shear stress $\Omega R \sin(\omega t)$ given on boundary. These solutions are obtained by mean of integral transforms. The obtained solution satisfy the all imposed initial and boundary conditions. Finally, solution of the ordinary Oldroyd-B fluid, Maxwell, ordinary Maxwell and Newtonian fluid flows are obtained as particular cases of our general results.

2. Mathematical formulation of the problem

For an Oldroyd-B fluid constitutive equations is

$$T = -p\mathbf{I} + \mathbf{S} \quad ; \quad S + \lambda\left(\frac{d\mathbf{S}}{dt} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T\right),$$

$$T = \mu\mathbf{A} + \mu\lambda_r\left(\frac{d\mathbf{A}}{dt} - \mathbf{L}\mathbf{A} - \mathbf{A}\mathbf{L}^T\right), \tag{1}$$

where $\mathbf{T}$ (Cauchy stress tensor), -p$\mathbf{I}$ (indeterminate spherical stress), $\mathbf{S}$ (stress tensor), $\mathbf{L}$ (velocity gradient), μ (dynamic viscosity), $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ (first the Rivilin-Erickson tensor,) λ and λ_r ($0 \leqslant \lambda_r < \lambda$) are relaxation and retardation time.

Assume an infinite circular cylinder rotate along z-axis with radius R. Cylinder is filled with an Oldroyd-B fluid which is at rest at time $t = 0$. After $t = 0^+$ the cylinder applies an oscillating rotational shear stress $\Omega R \sin(\omega t)$ to the fluid, where ω is the angular frequency. We assumed that velocity field and the extra shear stress are of the form

$$\mathbf{V} = \mathbf{V}(r,t) = W(r,t)e_\theta \quad ; \quad \mathbf{S} = \mathbf{S}(r,t), \tag{2}$$

where e_θ is unit vector in the θ-direction of the cylindrical coordinate system.We assume that $\mathbf{S}$ and $\mathbf{V}$ is a function of time and radius only. At $t = 0$ there is no motion in fluid i.e; fluid is at rest then

$$\mathbf{V}(r,0) = 0 \quad ; \quad \mathbf{S}(r,0) = 0. \tag{3}$$

Introducing Eqs.(2) in(1) and using (3) we get $S_{rr} = S_{rz} = S_{z\theta} = S_{zz} = 0$ and the meaning partial differential equation.

$$\left(1 + \lambda \frac{\partial}{\partial t}\right)\tau(r,t) = \mu\left(1 + \lambda_r \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\omega(r,t), \tag{4}$$

where $\tau(r,t) = S_{r\theta}(r,t)$ is non-zero component of extra stress tensor. If we neglect the body force, then due to rotation symmetry the balance of linear momentum leads to the relevant equations.

$$\rho \frac{\partial}{\partial t}\omega(r,t) = \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\tau(r,t), \tag{5}$$

where ρ is the constant density of the field. In order to obtained the governing equations for shear stress on boundary we eliminate $\omega(r,t)$ between equation (4) and (5) after elimination $\omega(r,t)$ we get,

$$\left(1 + \lambda \frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\tau(r,t) = \nu\left(1 + \lambda_r \frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2}\right)\tau(r,t), \tag{6}$$

where $\nu = \frac{\mu}{\rho}$ is the Kinematic viscosity of the fluid. The governing model by using fractional derivative is shown as;

$$\left(1 + \lambda D_t^\alpha\right)\tau(r,t) = \mu\left(1 + \lambda_r D_t^\beta\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\omega(r,t), \tag{7}$$

$$\rho D_t^\alpha \omega(r,t) = \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\tau(r,t), \tag{8}$$

$$\left(1 + \lambda D_t^\alpha\right)\frac{\partial}{\partial t}\tau(r,t) = \nu\left(1 + \lambda_r D_t^\beta\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2}\right)\tau(r,t). \tag{9}$$

Where

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(t-\tau)^\alpha}d\tau \quad ; \quad 0 \leqslant \alpha < 1$$

is Caputo fractional derivative operator and $\Gamma(.)$ is the Euler integral of second kind or Gamma function.

At $t = 0$ fluid is at rest because there is no rotation in the cylinder, at $t = 0^+$ cylinder starts its rotation along its axis and boundary of cylinder applies shear stress on fluid and radius of cylinder is R. Appropriate conditions i.e; initial and boundary conditions are,

$$\tau(r,t)|_{t=0} = \frac{\partial \tau(r,t)}{\partial t}|_{t=0} = 0; \qquad r\epsilon[0,R] \tag{10}$$

$$\tau(R,t) = \Omega R \sin(\omega t),: \qquad t \geqslant 0, \tag{11}$$

where Ω is constant.

3. Calculation Of Shear Stress

We shall use the Laplace transform and Finite Hankel transform to determine the exact analytical solution. Taking the Laplace transform of the equations (9) and (11) we have

$$q(1+\lambda q^{\alpha})\overline{\tau}(r,q) = \nu(1+\lambda_r q^{\beta})\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2}\right)\overline{\tau}(r,q), \tag{12}$$

$$\overline{\tau}(R,q) = \Omega R \frac{\omega}{q^2+\omega^2}, \tag{13}$$

where $\overline{\tau}(r,q)$ represent the Laplace transform of the function $\tau(r,t)$. We can write equation (12) as,

$$\overline{\tau}(r,q) = \frac{\nu(1+\lambda_r q^{\beta})}{q+\lambda q^{\alpha+1}}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2}\right)\overline{\tau}(r,q). \tag{14}$$

Finite Hankel transform of the function $\overline{\tau}(r,q)$ defined as,

$$\overline{\tau}_H(r_n,q) = \int_0^R rJ_2(rr_n)\overline{\tau}(r,q)dr, \tag{15}$$

identity which we used here is,

$$\int_0^R rJ_2(rr_n)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2}J_2(rr_n)\right)\overline{\tau}(r,q)dr$$
$$= -R\overline{\tau}(R,q)J_2'(Rr_n)r_n - r_n^2\overline{\tau}_H(r_n,q). \tag{16}$$

Multiplying equation (14) by $rJ_2(rr_n)$ then integrate from 0 to R with respect to r, we get

$$\int_0^R rJ_2(rr_n)\overline{\tau}(r,q)dr = \frac{\nu(1+\lambda_r q^{\beta})}{q+\lambda q^{\alpha+1}}\int_0^R r\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2}\right)J_2(rr_n)\overline{\tau}(r,q)dr.$$

using equations (13), (15) and (16) we get,

$$\overline{\tau}_H(r_n,q)=\frac{\nu(1+\lambda_r q^{\beta})}{q+\lambda q^{\alpha+1}}[-Rr_nJ_2'(Rr_n)\overline{\tau}(R,q)-r_n^2\overline{\tau}_H(r_n,q)].$$

Now simplification for $\overline{\tau}_H(r_n,q)$ we get

$$\overline{\tau}_H(r_n,q)=\frac{-R^2\Omega\omega J_2'(Rr_n)}{(q^2+\omega^2)}\frac{\nu r_n(1+\lambda_r q^{\beta})}{(q+\nu r_n^2+\lambda q^{\alpha+1}+\nu\lambda_r q^{\beta}r_n^2)}. \tag{17}$$

Separating the function in suitable form as,

$$\overline{\tau}_H(r_n,q)=\frac{-R^2\Omega\omega J_2'(Rr_n)}{(q^2+\omega^2)r_n}+\frac{R^2\Omega\omega J_2'(Rr_n)}{(q^2+\omega^2)}\frac{q+\lambda q^{\alpha+1}}{r_n(q+\lambda q^{\alpha+1}+\nu r_n^2+\nu\lambda_r q^{\beta}r_n^2)},$$

$$\overline{\tau}_H(r_n,q)=\overline{\tau}_{1H}(r_n,q)+\overline{\tau}_{2H}(r_n,q), \tag{18}$$

where,

$$\overline{\tau}_{1H}(r_n,q)=\frac{-R^2\Omega\omega J_2'(Rr_n)}{r_n(q^2+\omega^2)}=-R^2\Omega J_1(Rr_n)\frac{\omega}{q^2+\omega^2}, \tag{19}$$

$$\overline{\tau}_{2H}(r_n,q)=\frac{R^2\Omega\omega J_2'(Rr_n)}{r_n(q^2+\omega^2)}\frac{q+\lambda q^{\alpha+1}}{(q+\lambda q^{\alpha+1}+\nu r_n^2+\nu\lambda_r q^{\beta}r_n^2)},$$

$$\overline{\tau}_{2H}(r_n,q)=\frac{R^2\Omega\omega J_1(Rr_n)}{q^2+\omega^2}\frac{q+\lambda q^{\alpha+1}}{(q+\lambda q^{\alpha+1}+\nu r_n^2+\nu\lambda_r q^{\beta}r_n^2)}. \tag{20}$$

Using the identity i.e;

$$\frac{1}{q+\nu r_n^2+\lambda q^{\beta+1}+\nu r_n^2\lambda_r q^{\gamma}} =\frac{1}{\lambda}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!}{(k-m)!m!}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\lambda_r^m\frac{q^{\gamma m-k-1}}{(q^{\beta}+\frac{1}{\lambda})^{k+1}}.$$

Now equation (20) becomes,

$$\overline{\tau}_{2H}(r_n,q)=R^2\Omega J_1(Rr_n)\frac{1}{\lambda}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!}{(k-m)!m!}\lambda_r^m\left(\frac{-\nu r_n^2}{\lambda}\right)^k\frac{\omega}{q^2+\omega^2} \left[\frac{q^{\beta m-k}}{(q^{\alpha}+\frac{1}{\lambda})^{k+1}}+\lambda\frac{q^{\alpha+\beta m-k}}{(q^{\alpha}+\frac{1}{\lambda})^{k+1}}\right]. \tag{21}$$

Taking the inverse Laplace transform of equation (19)

$$\tau_{1H}(r_n,t)=-R^2\Omega J_1(Rr_n)\sin(\omega t). \tag{22}$$

Take the inverse Laplace transform and use convolution theorem of equation (21) we get

$$\tau_{2H}(r_n,t)=R^2\Omega J_1(Rr_n)\frac{1}{\lambda}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!}{m!(k-m)!}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\lambda_r^m\Bigg[\int_0^t\sin\omega(t-s)$$

$$G_{\alpha,\beta m-k,k+1}(-\lambda^{-1},s)ds + \lambda \int_0^t \sin\omega(t-s) G_{\alpha,\alpha+\beta m-k,k+1}(-\lambda^{-1},s)ds \Bigg]. \quad (23)$$

Where $G_{a,b,c}(.,t)$ is the generalized G-function with $\pounds^{-1}\{\frac{q^b}{(q^a-d)^c}\} = G_{a,b,c}(d,t)$, $Re(ac-b) > 0$, $Re(q) > 0$, $\mid \frac{d}{q^a} \mid < 1$ and $G_{a,b,c}(d,t) = \sum_{j=0}^{\infty} \frac{t^{(c+j)a-b-1}}{\Gamma((c+j)a-b))} \frac{d^j \Gamma(c+j)}{\Gamma(c)\Gamma(j+1)}$ Taking inverse Laplace transform of equation (18) and using equation (22) and (23)

$$\tau_H(r_n,t) = -R^2\Omega J_1(Rr_n)\sin(\omega t) + \frac{R^2\Omega J_1(Rr_n)}{\lambda} \sum_{k=0}^{\infty}\sum_{m=0}^{k} \frac{k!}{m!(k-m)!}$$
$$\times \lambda_r^m \left(\frac{-\nu r_n^2}{\lambda}\right)^k \Bigg[\int_0^t \sin\omega(t-s) G_{\alpha,\beta m-k,k+1}(-\lambda^{-1},s)ds$$
$$+ \lambda \int_0^t \sin\omega(t-s) G_{\alpha,\alpha+\beta m-k,k+1}(-\lambda^{-1},s)ds \Bigg]. \quad (24)$$

Apply the inverse Hankel transform to equation (24) and using the known formulae

$$H(r^2) = \int_0^R r^3 J_2(rr_n)dr = \frac{-R^3 J_1 Rr_n}{r_n}$$

$$\tau(r,t) = 2\sum_{n=1}^{\infty} \frac{J_2(rr_n)}{[J_2'(r_n)]^2} \tau_H(r_n,t)$$

$$\tau(r,t) = \frac{r^2 r_n \Omega \sin(\omega t)}{R} + 2R^2\Omega \sum_{n=1}^{\infty} \frac{J_2(rr_n)J_1(Rr_n)}{J_1^2(r_n)} \frac{1}{\lambda}$$
$$\times \sum_{k=0}^{\infty}\sum_{m=0}^{k} \frac{k!}{(k-m)!m!} \lambda_r^m \left(\frac{-\nu r_n^2}{\lambda}\right)^k \Bigg[\int_0^t \sin\omega(t-s) G_{\alpha,\beta m-k,k+1}(-\lambda^{-1},s)ds$$
$$+ \lambda \int_0^t \sin\omega(t-s) G_{\alpha,\alpha+\beta m-k,k+1}(-\lambda^{-1},s)ds \Bigg] \quad (25)$$

4. Calculation for Velocity Field

Rewrite equation (8) and used equation (25) we get the non-integer order differential equation for velocity.

$$\rho D_t^{\alpha} \omega(r,t) = \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\tau(r,t)$$

where

$$\tau(r,t) = \tau_1(r,t) + \tau_2(r,t)$$

So above equation becomes;

$$\rho D_t^{\alpha}\omega(r,t) = \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\tau_1(r,t) + \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\tau_2(r,t) \tag{26}$$

where

$$\tau_1(r,t) = \frac{r^2 r_n \Omega \sin(\omega t)}{R}$$

$$\tau_2(r,t) = 2R^2\Omega \sum_{n=1}^{\infty} \frac{J_2(rr_n)J_1(Rr_n)}{J_1^2(r_n)} \frac{1}{\lambda} \sum_{k=0}^{\infty}\sum_{m=0}^{k} \frac{k!}{(k-m)!m!}\lambda_r^m \left(\frac{-\nu r_n^2}{\lambda}\right)^k$$

$$\times\left[\int_0^t \sin\omega(t-s)G_{\alpha,\beta m-k,k+1}(-\lambda^{-1},s)ds\right.$$

$$\left.+\lambda\int_0^t \sin\omega(t-s)G_{\alpha,\alpha+\beta m-k,k+1}(-\lambda^{-1},s)ds\right].$$

So,

$$\left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\tau_1(r,t) = \frac{4rr_n\Omega\sin(\omega t)}{R}, \tag{27}$$

and

$$\left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\tau_2(r,t) = \frac{2R^2\Omega}{\lambda} \sum_{n=1}^{\infty} \frac{J_1(rr_n)J_1(Rr_n)r_n}{J_1^2(r_n)} \sum_{k=0}^{\infty}\sum_{m=0}^{k} \frac{k!}{(k-m)!m!}$$

$$\times\left(\frac{-\nu r_n^2}{\lambda}\right)^k \lambda_r^m \left[\int_0^t \sin\omega(t-s)G_{\alpha,\beta m-k,k+1}(-\lambda^{-1},s)ds\right.$$

$$\left.+\lambda\int_0^t \sin\omega(t-s)G_{\alpha,\alpha+\beta m-k,k+1}(-\lambda^{-1},s)ds\right]. \tag{28}$$

Use equation (27) and (28) in equation (26) we get,

$$\rho D_t^{\alpha}\omega(r,t) = \frac{4rr_n\Omega\sin(\omega t)}{R} + \frac{2R^2\Omega}{\lambda} \sum_{n=1}^{\infty} \frac{J_1(rr_n)J_1(Rr_n)r_n}{J_1^2(r_n)}$$

$$\times \sum_{k=0}^{\infty}\sum_{m=0}^{k} \frac{k!}{(k-m)!m!}\lambda_r^m \left(\frac{-\nu r_n^2}{\lambda}\right)^k \left[\int_0^t \sin\omega(t-s)G_{\alpha,\beta m-k,k+1}(-\lambda^{-1},s)ds\right.$$

$$+\lambda\int_0^t \sin\omega(t-s)G_{\alpha,\alpha+\beta m-k,k+1}(-\lambda^{-1},s)ds\Bigg]. \quad (29)$$

The Laplace transform of equation (29) is

$$\varpi(r,q)=\frac{4rr_n\Omega}{\rho R}\frac{1}{q^\alpha}\frac{\omega}{q^2+\omega^2}+\frac{2R^2\Omega}{\rho\lambda}\sum_{n=1}^{\infty}\frac{J_1(rr_n)J_1(Rr_n)r_n}{J_1^2(r_n)}$$
$$\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!}{(k-m)!m!}\left(\frac{-\nu r_n^2}{\lambda}\right)^k$$
$$\lambda_r^m\left[\frac{\omega}{q^2+\omega^2}\frac{q^{\beta m-\alpha-k}}{(q^\alpha+\frac{1}{\lambda})^{k+1}}+\lambda\frac{\omega}{q^2+\omega^2}\frac{q^{\beta m-k}}{(q^\alpha+\frac{1}{\lambda})^{k+1}}\right]. \quad (30)$$

Now Apply inverse Laplace transform to equation (30) and using the Convolution theorem

$$\omega(r,t)=\frac{4rr_n\Omega}{\rho R}\int_0^t \sin\omega(t-\tau)\frac{t^{\alpha-1}}{\Gamma(\alpha)}d\tau+\frac{2R^2\Omega}{\rho\lambda}\sum_{n=1}^{\infty}\frac{J_1(rr_n)J_1(Rr_n)r_n}{J_1^2(r_n)}$$
$$\times\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!}{(k-m)!m!}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\lambda_r^m\Bigg[\int_0^t \sin\omega(t-s)G_{\alpha,\beta m-\alpha-k,k+1}(-\lambda^{-1},s)ds$$
$$+\lambda\int_0^t \sin\omega(t-s)G_{\alpha,\beta m-k,k+1}(-\lambda^{-1},s)ds\Bigg]. \quad (31)$$

5. Limiting case

5.1. Ordinary Oldroyd-B fluid. Letting $\alpha\to 1$, $\beta\to 1$ into equations (25) and (31)we get the result of shear stress and velocity field respectively for ordinary Oldroyd-B fluid.

$$\tau_{OB}(r,t)=\frac{r^2r_n\Omega\sin(\omega t)}{R}+2R^2\Omega\sum_{n=1}^{\infty}\frac{J_2(rr_n)J_1(Rr_n)}{J_1^2(r_n)}\frac{1}{\lambda}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!}{(k-m)!m!}$$
$$\times\left(\frac{-\nu r_n^2}{\lambda}\right)^k\lambda_r^m\Bigg[\int_0^t \sin\omega(t-s)G_{1,m-k,k+1}(-\lambda^{-1},s)ds$$
$$+\lambda\int_0^t \sin\omega(t-s)G_{1,1+m-k,k+1}(-\lambda^{-1},s)ds\Bigg], \quad (32)$$

$$\omega_{OB}(r,t) = \frac{4rr_n\Omega}{\rho R}\int_0^t \sin\omega(t-\tau)d\tau + \frac{2R^2\Omega}{\rho\lambda}\sum_{n=1}^{\infty}\frac{J_1(rr_n)J_1(Rr_n)r_n}{J_1^2(r_n)}$$
$$\times\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!}{(k-m)!m!}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\lambda_r^m\left[\int_0^t \sin\omega(t-s)G_{1,m-k-1,k+1}(-\lambda^{-1},s)ds\right.$$
$$\left. + \lambda\int_0^t \sin\omega(t-s)G_{1,m-k,k+1}(-\lambda^{-1},s)ds\right]. \quad (33)$$

5.2. Generalized Maxwell Fluid. By placing $\lambda_r \to 0$, $\beta \to 0$ into equations (25) and (31)we get the results of shear stress and velocity field for generalized Maxwell fluid respectively.

$$\tau_{GM}(r,t) = \frac{r^2r_n\Omega\sin(\omega t)}{R} + 2R^2\Omega\sum_{n=1}^{\infty}\frac{J_2(rr_n)J_1(Rr_n)}{J_1^2(r_n)}\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k$$
$$\left[\int_0^t \sin\omega(t-s)G_{\alpha,-k,k+1}(-\lambda^{-1},s)ds + \lambda\int_0^t \sin\omega(t-s)G_{\alpha,\alpha-k,k+1}(-\lambda^{-1},s)ds\right], \quad (34)$$

$$\omega_{GM}(r,t) = \frac{4rr_n\Omega}{\rho R}\int_0^t \sin\omega(t-\tau)\frac{t^{\alpha-1}}{\Gamma(\alpha)}d\tau + \frac{2R^2\Omega}{\rho\lambda}\sum_{n=1}^{\infty}\frac{J_1(rr_n)J_1(Rr_n)r_n}{J_1^2(r_n)}$$
$$\times\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\left[\int_0^t \sin\omega(t-s)G_{\alpha,-\alpha-k,k+1}(-\lambda^{-1},s)ds\right.$$
$$\left. + \lambda\int_0^t \sin\omega(t-s)G_{\alpha,-k,k+1}(-\lambda^{-1},s)ds\right]. \quad (35)$$

5.3. Ordinary Maxwell Fluid. By putting $\alpha \to 1$ in equation (34) and (35) we get expressing for the shear stress and velocity field for ordinary Maxwell fluid respectively.

$$\tau_{OM}(r,t) = \frac{r^2r_n\Omega\sin(\omega t)}{R} + 2R^2\Omega\sum_{n=1}^{\infty}\frac{J_2(rr_n)J_1(Rr_n)}{J_1^2(r_n)}\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k$$
$$\left[\int_0^t \sin\omega(t-s)G_{1,-k,k+1}(-\lambda^{-1},s)ds + \lambda\int_0^t \sin\omega(t-s)G_{1,1-k,k+1}(-\lambda^{-1},s)ds\right], \quad (36)$$

$$\omega_{OM}(r,t) = \frac{4rr_n\Omega}{\rho R}\int_0^t \sin\omega(t-\tau)d\tau + \frac{2R^2\Omega}{\rho\lambda}\sum_{n=1}^{\infty}\frac{J_1(rr_n)J_1(Rr_n)r_n}{J_1^2(r_n)}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k$$
$$\left[\int_0^t \sin\omega(t-s)G_{1,-1-k,k+1}(-\lambda^{-1},s)ds + \lambda\int_0^t \sin\omega(t-s)G_{1,-k,k+1}(-\lambda^{-1},s)ds\right]. \quad (37)$$

5.4. Newtonian Fluid. Let $\lambda_r \to 0$ and using the limit, i.e;

$$\lim_{\lambda\to 0}\frac{1}{\lambda^k}G_{1,b,k}\left(\frac{-1}{\lambda},t\right) = \frac{t^{-b-1}}{\Gamma(-b)},$$

$b < 0$ in (25) and (31) we can get results for Newtonian fluid as,

$$\tau_{NF}(r,t) = \frac{r^2 r_n\Omega\sin(\omega t)}{R} + \frac{2R^2\Omega}{\omega^2+\nu^2 r_n^4}\sum_{n=1}^{\infty}\frac{J_2(rr_n)J_1(Rr_n)}{J_1^2(r_n)}\{\omega^2\sin(\omega t) + \nu r_n^2\omega\cos(\omega t) - \omega\nu r_n^2 e^{-\nu r_n^2 t}\}, \quad (38)$$

$$\omega_{NF}(r,t) = \frac{4rr_n\Omega}{\rho R}\int_0^t \sin\omega(t-\tau)\frac{t^{\alpha-1}}{\Gamma(\alpha)}d\tau + \frac{2R^2\Omega}{\rho(\omega^2+\nu^2 r_n^4)}\sum_{n=1}^{\infty}\frac{J_1(rr_n)J_1(Rr_n)r_n}{J_1^2(r_n)}$$
$$\left[-\omega\nu r_n^2\int_0^t e^{-\nu r_n^2(t-s)}\frac{\tau^{\alpha-1}}{\Gamma(\alpha)}ds + \omega^2\int_0^t \sin\omega(t-s)\frac{\tau^{\alpha-1}}{\Gamma(\alpha)}ds + \nu r_n^2\omega\int_0^t \cos\omega(t-s)\frac{\tau^{\alpha-1}}{\Gamma(\alpha)}ds\right]. \quad (39)$$

6. Conclusion

The idea presented in this paper is to find a formula which is useful to find out exact solutions for shear stress and velocity field of any Oldroyd-B fluid which is present in rotationally oscillating cylinders. We used two transformation i.e; Hankel transform and Laplace transform. At time $t = 0^+$, cylinder starts its rotation about its axis. To obtain the solutions we used the finite Hankel and Laplace transforms. We express our results in the form of generalized G-function, which satisfy the governing equations and fulfilled all imposed initial and boundary conditions. Furthermore, the results for Ordinary Oldroyd-B fluid, fractional Maxwell fluid, ordinary Maxwell fluid and classical Newtonian fluid are obtained as limiting cases.

Competing Interests

The author(s) do not have any competing interests in the manuscript.

References

1. Calamai, G. (1970). BATCHELOR, GK-An *Introduction to Fluid Dynamics.*
2. Ting, T. W. (1963). Certain non-steady flows of second-order fluids. *Archive for Rational Mechanics and Analysis, 14*(1), 1-26.
3. Rajagopal, K. R., & Bhatnagar, R. K. (1995). Exact solutions for some simple flows of an Oldroyd-B fluid. *Acta Mechanica, 113*(1-4), 233-239.
4. Hayat, T., Siddiqui, A. M., & Asghar, S. (2001). Some simple flows of an Oldroyd-B fluid. *International Journal of Engineering Science, 39*(2), 135-147.
5. Hayat, T., Khan, M., & Ayub, M. (2004). Exact solutions of flow problems of an Oldroyd-B fluid. *Applied Mathematics and Computation, 151*(1), 105-119.
6. Jamil, M., Fetecau, C., & Imran, M. (2011). Unsteady helical flows of Oldroyd-B fluids. *Communications in Nonlinear Science and Numerical Simulation, 16*(3), 1378-1386.
7. Wood, W. P. (2001). Transient viscoelastic helical flows in pipes of circular and annular cross-section. *Journal of non-newtonian fluid mechanics, 100*(1-3), 115-126.
8. Srivastava, P. N. (1966). Non-steady helical flow of a visco-elastic liquid(Nonsteady helical flow of viscoelastic liquid contained in circular cylinder, noting occurrence of oscillations in fluid decaying exponentially with time). *Archiwum Mechaniki Stosowanej, 18*(2), 145-150.
9. Waters, N. D., & King, M. J. (1971). The unsteady flow of an elastico-viscous liquid in a straight pipe of circular cross section. *Journal of Physics D: Applied Physics, 4*(2), 204–211.
10. Fetecau, C., & Fetecau, C. (1985). On the uniqueness of some helical flows of a second grade fluid. *Acta mechanica, 57*(3-4), 247-252.
11. Hayat, T., Nadeem, S., & Asghar, S. (2004). Periodic unidirectional flows of a viscoelastic fluid with the fractional Maxwell model. *Applied Mathematics and Computation, 151*(1), 153-161.
12. Khan, M., Nadeem, S., Hayat, T., & Siddiqui, A. M. (2005). Unsteady motions of a generalized second-grade fluid. *Mathematical and Computer Modelling, 41*(6-7), 629-637.
13. Shaowei, W., & Mingyu, X. (2006). Exact solution on unsteady Couette flow of generalized Maxwell fluid with fractional derivative. *Acta mechanica, 187*(1-4), 103-112.
14. Mahmood, A., Parveen, S., Ara, A., & Khan, N. A. (2009). Exact analytic solutions for the unsteady flow of a non-Newtonian fluid between two cylinders with fractional derivative model. *Communications in Nonlinear Science and Numerical Simulation, 14*(8), 3309-3319.
15. Vieru, D., Akhtar, W., Fetecau, C., & Fetecau, C. (2007). Starting solutions for the oscillating motion of a Maxwell fluid in cylindrical domains. *Meccanica, 42*(6), 573-583.
16. Renardy, M. (1990). An alternative approach to inflow boundary conditions for Maxwell fluids in three space dimensions. *Journal of Non-Newtonian Fluid Mechanics, 36*, 419-425.
17. Waters, N. D., & King, M. J. (1970). Unsteady flow of an elastico-viscous liquid. *Rheologica Acta, 9*(3), 345-355.

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