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SOME ESTIMATIONS ČEBYŠEV-GRÜSS TYPE INEQUALITIES INVOLVING FUNCTIONS AND THEIR DERIVATIVES

MEHMET ZEKI SARIKAYA[1], SÜMEYRA KAPLAN

ABSTRACT. In this paper, some inequalities related to Čebyšev's functional are proved.

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1. Introduction

The classical form of Grüss inequality, first published by G. Grüss in 1935, gives an estimate of the difference between the integral of the product and the product of the integrals of two functions. In recent years, several bounds for the Cebysev functional in various cases including convexity assumptions for the functions involved are proved. In the subsequent years, many variants of these inequalities appeared in the literature (see, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]). In 1935, G. Grüss [4] proved the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx\right|\leq\frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma),\quad(1)$$

provided that f and g are two integrable function on $[a,b]$ satisfying the condition

$$\varphi\leq f(x)\leq\Phi \quad\text{and}\quad \gamma\leq g(x)\leq\Gamma \text{ for all } x\in[a,b].$$

The constant $\frac{1}{4}$ is best possible.

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[1] Corresponding Author

In 1882, P. L. Čebyšev [13] gave the following inequality:

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \left\|f'\right\|_\infty \left\|g'\right\|_\infty ,$$

where $f,g:[a,b]\to\mathbb{R}$ are absolutely continuous function, whose first derivatives f' and g' are bounded,

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|p\|_\infty = ess\sup_{t\in[a,b]}|p(t)|$.

In [14], Beesack *et al.* have proved the following Cebysev inequality for absolutely continuous functions whose first derivatives belong to Lp spaces.

$$|T(f,g)| \leq \frac{(b-a)}{4}\left(\frac{2^p-1}{p(p+1)}\right)^{\frac{1}{p}}\left(\frac{2^q-1}{q(q+1)}\right)^{\frac{1}{q}}\|f\|_p\|g\|_q \tag{2}$$

where $\|h\|_p=\left(\int_a^b |h(x)|^p\,dx\right)^{\frac{1}{p}}$, $\forall p>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

In this paper, some inequalities related to Chebyshev's functional are proved. We give our results in the case of differentiable functions whose derivatives and theirself belong to $L_p[a,b]$, $1\leq p\leq\infty$.

2. Main results

Theorem 2.1. *Let $f,g:[a,b]\to R$ be an absolutely continuous function on $[a,b]$ so that $|f'|$ and $|g'|$ are convex on $[a,b]$.*

(1) *If $f,f',g,g'\in L_\infty[a,b]$, then we have*

$$|T(f,g)| \leq \frac{(b-a)}{6}\left[\|g\|_\infty\|f'\|_\infty+\|f\|_\infty\|g'\|_\infty\right], \tag{3}$$

(2) *If $f,f',g,g'\in L_p[a,b]$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then we have*

$$|T(f,g)| \leq \frac{2^{\frac{1}{q}-2}(b-a)^{\frac{2}{q}-1}}{[(q+1)(q+2)]^{\frac{1}{q}}} \left[(b-a)^{\frac{1}{p}}\left(\|gf'\|_p+\|fg'\|_p\right)+\|g\|_p\|f'\|_p+\|f\|_p\|g'\|_p\right], \tag{4}$$

(3) *If $f,f',g,g'\in L_1[a,b]$, then we have*

$$|T(f,g)| \leq \frac{1}{4}\left[\|gf'\|_1+\|fg'\|_1\right]+\frac{1}{4(b-a)}\left[\|g\|_1\|f'\|_1+\|f\|_1\|g'\|_1\right]. \tag{5}$$

Proof. For any $x,t\in[a,b]$, $x\neq t$, we write

$$\frac{f(x)-f(t)}{x-t}=\frac{1}{x-t}\int_t^x f'(u)du=\int_0^1 f'\left[(1-\lambda)x+\lambda t\right]d\lambda$$

and so

$$f(x)=f(t)+(x-t)\int_0^1 f'\left[(1-\lambda)x+\lambda t\right]d\lambda. \tag{6}$$

Let's rewrite (6) as follows

$$g(x)=g(t)+(x-t)\int_0^1 g'\left[(1-\lambda)x+\lambda t\right]d\lambda. \tag{7}$$

Multiplying (6) by $g(x)$ and (7) by $f(x)$, adding the resulting identities, and integrate over $x,t\in[a,b]$, and divide by $(b-a)^2$, we have

$$\begin{aligned}\frac{2}{(b-a)}\int_a^b f(x)g(x)dx &= \frac{2}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx \\ &+\frac{1}{(b-a)^2}\int_a^b\int_a^b (x-t)g(x)\int_0^1 f'\left[(1-\lambda)x+\lambda t\right]d\lambda dt dx \\ &+\frac{1}{(b-a)^2}\int_a^b\int_a^b (x-t)f(x)\int_0^1 g'\left[(1-\lambda)x+\lambda t\right]d\lambda dt dx\end{aligned}$$

and rewriting we get

$$\begin{aligned}T(f,g) &= \frac{1}{2(b-a)^2}\int_a^b\int_a^b (x-t)g(x)\int_0^1 f'\left[(1-\lambda)x+\lambda t\right]d\lambda dt dx \\ &+\frac{1}{2(b-a)^2}\int_a^b\int_a^b (x-t)f(x)\int_0^1 g'\left[(1-\lambda)x+\lambda t\right]d\lambda dt dx.\end{aligned} \tag{8}$$

(1) Thus, using the properties of modulus and the convexity of $|f'|$ and $|g'|$, we have

$$\begin{aligned}|T(f,g)| &\leq \frac{1}{2(b-a)^2}\int_a^b\int_a^b |x-t|\,|g(x)|\int_0^1 \left[(1-\lambda)\,|f'(x)|+\lambda\,|f'(t)|\right]d\lambda dt dx \\ &+\frac{1}{2(b-a)^2}\int_a^b\int_a^b |x-t|\,|f(x)|\int_0^1 \left[(1-\lambda)\,|g'(x)|+\lambda\,|g'(t)|\right]d\lambda dt dx \\ &= \frac{1}{4(b-a)^2}\int_a^b\int_a^b |x-t|\left[|g(x)|\,|f'(x)|+|g(x)|\,|f'(t)|\right]dt dx\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{4(b-a)^2}\int_a^b\int_a^b |x-t|\left[|f(x)|\,|g'(x)|+|f(x)|\,|g'(t)|\right]dtdx \\
\leq\ & \frac{1}{4(b-a)^2} ess \sup_{x\in[a,b]}\left[|g(x)|\,|f'(x)|+|f(x)|\,|g'(x)|\right]\int_a^b\int_a^b |x-t|\,dxdt \\
& +\frac{1}{4(b-a)^2} ess \sup_{x\in[a,b]}|g(x)|\int_a^b\int_a^b |x-t|\,|f'(t)|\,dxdt \\
& +\frac{1}{4(b-a)^2} ess \sup_{x\in[a,b]}|f(x)|\int_a^b\int_a^b |x-t|\,|g'(t)|\,dxdt \\
\leq\ & \frac{1}{4(b-a)^2}\left[\|g\|_\infty\|f'\|_\infty+\|f\|_\infty\|g'\|_\infty\right]\int_a^b\left[\frac{(t-a)^2+(b-t)^2}{2}\right]dt \\
& +\frac{1}{4(b-a)^2}\|g\|_\infty\, ess \sup_{t\in[a,b]}|f'(t)|\int_a^b\left[\frac{(t-a)^2+(b-t)^2}{2}\right]dt \\
& +\frac{1}{4(b-a)^2}\|f\|_\infty\, ess \sup_{t\in[a,b]}|g'(t)|\int_a^b\left[\frac{(t-a)^2+(b-t)^2}{2}\right]dt \\
=\ & \frac{(b-a)}{6}\left[\|g\|_\infty\|f'\|_\infty+\|f\|_\infty\|g'\|_\infty\right]
\end{aligned}
$$

for $x,t\in[a,b]$, and the inequality (3) is proved.
(2) As above, we rewrite

$$
\begin{aligned}
|T(f,g)| \leq\ & \frac{1}{4(b-a)^2}\int_a^b\int_a^b |x-t|\left[|g(x)|\,|f'(x)|+|f(x)|\,|g'(x)|\right]dtdx \qquad (9)\\
& +\frac{1}{4(b-a)^2}\int_a^b\int_a^b |x-t|\left[|g(x)|\,|f'(t)|+|f(x)|\,|g'(t)|\right]dtdx.
\end{aligned}
$$

Using the Hölder's inequality for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, we have

$$
\begin{aligned}
& |T(f,g)| \\
& \leq\frac{1}{4(b-a)^2}\left(\int_a^b\int_a^b |x-t|^q\,dtdx\right)^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&\times\left\{\left(\int_a^b\int_a^b\left[|g(x)|\,|f'(x)|+|f(x)|\,|g'(x)|\right]^p dtdx\right)^{\frac{1}{p}}\right.\\
&\left.+\left(\int_a^b\int_a^b |g(x)|^p\,|f'(t)|^p\,dtdx\right)^{\frac{1}{p}}+\left(\int_a^b\int_a^b |f(x)|^p\,|g'(t)|^p\,dtdx\right)^{\frac{1}{p}}\right\}\\
=\ & \frac{1}{4\,(b-a)^2}\left[(b-a)^{\frac{1}{p}}\,\||gf'|+|fg'|\|_p+\|g\|_p\,\|f'\|_p+\|f\|_p\,\|g'\|_p\right]\\
\times\ & \left(\int_a^b\left[\frac{(t-a)^{q+1}+(b-t)^{q+1}}{q+1}\right]dt\right)^{\frac{1}{q}}\\
\leq\ & \frac{2^{\frac{1}{q}-2}\,(b-a)^{\frac{2}{q}-1}}{\left[(q+1)(q+2)\right]^{\frac{1}{q}}}\left[(b-a)^{\frac{1}{p}}\left(\|gf'\|_p+\|fg'\|_p\right)+\|g\|_p\,\|f'\|_p+\|f\|_p\,\|g'\|_p\right]
\end{aligned}
$$

and the inequality (4) is proved.
(3) We consider the inequality (9) that

$$
\begin{aligned}
& |T\,(f,g)|\\
\leq\ & \frac{1}{4\,(b-a)}\int_a^b\left[|g(x)|\,|f'(x)|+|f(x)|\,|g'(x)|\right]\sup_{t\in[a,b]}|x-t|\,dx\\
&+\frac{1}{4\,(b-a)^2}\int_a^b\sup_{t\in[a,b]}|x-t|\int_a^b\left[|g(x)|\,|f'(t)|+|f(x)|\,|g'(t)|\right]dtdx\\
=\ & \frac{1}{4\,(b-a)}\int_a^b\left[|g(x)|\,|f'(x)|+|f(x)|\,|g'(x)|\right]\max\{x-a,b-x\}\,dx\\
&+\frac{1}{4\,(b-a)^2}\int_a^b\max\{x-a,b-x\}\left[|g(x)|\,\|f'\|_1+|f(x)|\,\|g'\|_1\right]dx\\
=\ & \frac{1}{4\,(b-a)}\int_a^b\left[|g(x)|\,|f'(x)|+|f(x)|\,|g'(x)|\right]\left(\frac{(b-a)+|2x-b-a|}{2}\right)dx\\
&+\frac{1}{4\,(b-a)^2}\int_a^b\left(\frac{(b-a)+|2x-b-a|}{2}\right)\left[|g(x)|\,\|f'\|_1+|f(x)|\,\|g'\|_1\right]dx\\
\leq\ & \frac{1}{8}\left[\|gf'\|_1+\|fg'\|_1\right]+\frac{1}{8\,(b-a)}\left[\|g\|_1\,\|f'\|_1+\|f\|_1\,\|g'\|_1\right]
\end{aligned}
$$

$$+\frac{1}{8(b-a)}\sup_{x\in[a,b]}|2x-b-a|\left\{\int_a^b [|g(x)||f'(x)|+|f(x)||g'(x)|]\,dx\right.$$

$$\left.+\frac{1}{(b-a)}\int_a^b [|g(x)|\,\|f'\|_1+|f(x)|\,\|g'\|_1]\,dx\right\}$$

$$=\frac{1}{4}[\|gf'\|_1+\|fg'\|_1]+\frac{1}{4(b-a)}[\|g\|_1\|f'\|_1+\|f\|_1\|g'\|_1]$$

and the inequality (5) is proved. □

Theorem 2.2. *Let $f,g:[a,b]\to R$ be an absolutely continuous function on $[a,b]$ so that $|f'|^p$ and $|g'|^p$ with $p>1$are convex on $[a,b]$.*

(1) *If $f,f',g,g'\in L_\infty[a,b]$, then we have*

$$|T(f,g)|\le\frac{(b-a)}{6}[\|g\|_\infty\|f'\|_\infty+\|f\|_\infty\|g'\|_\infty],\tag{10}$$

(2) *If $f,f',g,g'\in L_p[a,b]$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then we have*

$$|T(f,g)|\le\frac{2^{\frac{1}{q}-\frac{1}{p}-1}(b-a)^{\frac{2}{q}-1}}{[(q+1)(q+2)]^{\frac{1}{q}}}\tag{11}$$

$$\times\left\{\left((b-a)\|gf'\|_p^p+\|g\|_p^p\|f'\|_p^p\right)^{\frac{1}{p}}+\left((b-a)\|fg'\|_p^p+\|f\|_p^p\|g'\|_p^p\right)^{\frac{1}{p}}\right\},$$

(3) *If $f,f',g,g'\in L_p[a,b]$, then we have*

$$|T(f,g)|\le\frac{(b-a)^{\frac{1}{q}}}{2}\left\{\left[\|f'\|_p^p+\|gf'\|_p^p\right]^{\frac{1}{p}}+\left[\|g'\|_p^p+\|fg'\|_p^p\right]^{\frac{1}{p}}\right\}.\tag{12}$$

Proof. By Hölder's inequality and using the convexity of $|f'|^p$, we get

$$\begin{aligned}\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda &\le \left(\int_0^1 1^q\right)^{\frac{1}{q}}\left(\int_0^1|f'[(1-\lambda)x+\lambda t]|^p\,d\lambda\right)^{\frac{1}{p}}\\ &=\left(\int_0^1|f'[(1-\lambda)x+\lambda t]|^p\,d\lambda\right)^{\frac{1}{p}}\\ &\le\left(\int_0^1\left[(1-\lambda)|f'(x)|^p+\lambda|f'(t)|^p\right]d\lambda\right)^{\frac{1}{p}}\\ &=\left(\frac{|f'(x)|^p+|f'(t)|^p}{2}\right)^{\frac{1}{p}}.\end{aligned}$$

From (8) and using the properties of modulus and the convexity of $|f'|^p$ and $|g'|^p$, we have

$$
\begin{aligned}
|T(f,g)| &\leq \frac{1}{2(b-a)^2}\int_a^b\int_a^b |x-t|\,|g(x)|\left(\frac{|f'(x)|^p+|f'(t)|^p}{2}\right)^{\frac{1}{p}} dtdx \quad (13)\\
&\quad+\frac{1}{2(b-a)^2}\int_a^b\int_a^b |x-t|\,|f(x)|\left(\frac{|g'(x)|^p+|g'(t)|^p}{2}\right)^{\frac{1}{p}} dtdx
\end{aligned}
$$

for $x,t\in[a,b]$.

(1) If we take $f,f',g,g'\in L_\infty[a,b]$, then from (13), we have

$$
\begin{aligned}
&|T(f,g)|\\
\leq\ & \frac{1}{2(b-a)^2} ess\sup_{x\in[a,b]}|g(x)|\, ess\sup_{x,t\in[a,b]}\left(\frac{|f'(x)|^p+|f'(t)|^p}{2}\right)^{\frac{1}{p}}\int_a^b\int_a^b |x-t|\,dtdx\\
+\ & \frac{1}{2(b-a)^2} ess\sup_{x\in[a,b]}|f(x)|\, ess\sup_{x,t\in[a,b]}\left(\frac{|g'(x)|^p+|g'(t)|^p}{2}\right)^{\frac{1}{p}}\int_a^b\int_a^b |x-t|\,dtdx\\
\leq\ & \frac{(b-a)}{6}\left[\|g\|_\infty\|f'\|_\infty+\|f\|_\infty\|g'\|_\infty\right]
\end{aligned}
$$

for any $x,t\in[a,b]$, the inequality is proved.

(2) If $f,f',g,g'\in L_p[a,b]$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then from (13) and by Hölder's inequality we have

$$
\begin{aligned}
|T(f,g)| &\leq \frac{1}{2(b-a)^2}\left(\int_a^b\int_a^b |x-t|^q\,dtdx\right)^{\frac{1}{q}}\\
&\times\left\{\left(\int_a^b\int_a^b |g(x)|^p\left(\frac{|f'(x)|^p+|f'(t)|^p}{2}\right)dtdx\right)^{\frac{1}{p}}\right.\\
&+\left.\left(\int_a^b\int_a^b |f(x)|^p\left(\frac{|g'(x)|^p+|g'(t)|^p}{2}\right)dtdx\right)^{\frac{1}{p}}\right\}\\
&= \frac{1}{2^{1+\frac{1}{p}}(b-a)^2}\left(\int_a^b\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}dx\right)^{\frac{1}{q}}\\
&\quad\times\left((b-a)\|gf'\|_p^p+\|g\|_p^p\|f'\|_p^p\right)^{\frac{1}{p}}+\left((b-a)\|fg'\|_p^p+\|f\|_p^p\|g'\|_p^p\right)^{\frac{1}{p}}
\end{aligned}
$$

$$= \frac{2^{\frac{1}{q}-\frac{1}{p}-1}(b-a)^{\frac{2}{q}-1}}{[(q+1)(q+2)]^{\frac{1}{q}}}\left[\left((b-a)\|gf'\|_p^p+\|g\|_p^p\|f'\|_p^p\right)^{\frac{1}{p}}\right.$$
$$\left.+\left((b-a)\|fg'\|_p^p+\|f\|_p^p\|g'\|_p^p\right)^{\frac{1}{p}}\right]$$

which is proved the inequality (11).
(3) If $f, f', g, g' \in L_p[a,b]$, then from (13) and by Hölder's inequality we also have

$$|T(f,g)|$$
$$\leq \frac{1}{2(b-a)^2}\int_a^b \sup_{x\in[a,b]}|x-t|\int_a^b |g(x)|\left(\frac{|f'(x)|^p+|f'(t)|^p}{2}\right)^{\frac{1}{p}}dxdt$$
$$+ \frac{1}{2(b-a)^2}\int_a^b \sup_{x\in[a,b]}|x-t|\int_a^b |f(x)|\left(\frac{|g'(x)|^p+|g'(t)|^p}{2}\right)^{\frac{1}{p}}dxdt$$
$$= \frac{1}{2^{1+\frac{1}{p}}(b-a)^2}\int_a^b \max\{t-a,b-t\}\int_a^b |g(x)|\left(|f'(x)|^p+|f'(t)|^p\right)^{\frac{1}{p}}dxdt$$
$$+ \frac{1}{2^{1+\frac{1}{p}}(b-a)^2}\int_a^b \max\{t-a,b-t\}\int_a^b |f(x)|\left(|g'(x)|^p+|g'(t)|^p\right)^{\frac{1}{p}}dxdt$$
$$\leq \frac{1}{2(b-a)^2}\int_a^b\left(\frac{(b-a)+|2t-b-a|}{2}\right)$$
$$\times \left[\left(\int_a^b |g(x)|^p\left(|f'(x)|^p+|f'(t)|^p\right)dx\right)^{\frac{1}{p}}\left(\int_a^b 1^q dx\right)^{\frac{1}{q}}\right]dt$$
$$+ \frac{1}{2(b-a)^2}\int_a^b\left(\frac{(b-a)+|2t-b-a|}{2}\right)$$
$$\times \left[\left(\int_a^b |f(x)|^p\left(|g'(x)|^p+|g'(t)|^p\right)dx\right)^{\frac{1}{p}}\left(\int_a^b 1^q dx\right)^{\frac{1}{q}}\right]dt$$

$$\leq \frac{(b-a)^{\frac{1}{q}-2}}{2}\sup_{t\in[a,b]}\left(\frac{(b-a)+|2t-b-a|}{2}\right)\int_a^b\left[(b-a)|f'(t)|^p+\|gf'\|_p^p\right]^{\frac{1}{p}}dt$$

$$+\frac{(b-a)^{\frac{1}{q}-2}}{2}\sup_{t\in[a,b]}\left(\frac{(b-a)+|2t-b-a|}{2}\right)\int_a^b\left[(b-a)\left|g'(t)\right|^p+\|fg'\|_p^p\right]^{\frac{1}{p}}dt$$

$$\leq \frac{(b-a)^{\frac{1}{q}-1}}{2}\left(\int_a^b\left[(b-a)\left|f'(t)\right|^p+\|gf'\|_p^p\right]dt\right)^{\frac{1}{p}}\left(\int_a^b 1^q dt\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)^{\frac{1}{q}-1}}{2}\left(\int_a^b\left[(b-a)\left|g'(t)\right|^p+\|fg'\|_p^p\right]dt\right)^{\frac{1}{p}}\left(\int_a^b 1^q dt\right)^{\frac{1}{q}}$$

$$= \frac{(b-a)^{\frac{1}{q}}}{2}\left\{\left[\|f'\|_p^p+\|gf'\|_p^p\right]^{\frac{1}{p}}+\left[\|g'\|_p^p+\|fg'\|_p^p\right]^{\frac{1}{p}}\right\}$$

which is proved the inequality (12). □

Competing interests

The authors declare that they have no competing interests.

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Mehmet Zeki Sarikaya
Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-Turkey.
e-mail: sarikayamz@gmail.com

Sümeyra Kaplan
Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-Turkey.
e-mail: sumeyrakaplan81@hotmail.com