FLIP AND HOPF BIFURCATIONS OF DISCRETE-TIME FITZHUGH-NAGUMO MODEL

QAMAR DIN1, SADAF KHALIQ

ABSTRACT. In this paper, dynamics of a two-dimensional Fitzhugh-Nagumo model is discussed. The discrete-time model is obtained with the implementation of forward Euler’s scheme. We present the parametric conditions for local asymptotic stability of steady-states. It is shown that the two-dimensional discrete-time model undergoes period-doubling bifurcation and Neimark-Sacker bifurcation at its positive steady-state. Furthermore, in order to illustrate theoretical discussion some interesting numerical examples are presented.

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Key words and phrases: Fitzhugh-Nagumo model; Flip bifurcation; Hopf bifurcation.

1. Introduction

In 1961 FitzHugh and Nagumo [1] presented the following two-dimensional model:

\[
\begin{pmatrix}
\frac{dx}{dt} \\
\frac{dy}{dt}
\end{pmatrix} = \begin{pmatrix}
c_1 \left( x + y - \frac{x^3}{3} \right) \\
\frac{1}{c_1} \left( x - a_1 + b_1 y \right)
\end{pmatrix},
\]

where \(a_1\), \(b_1\) and \(c_1\) are positive constants. Using the forward Euler method to system (1), we get the discrete-time model as follows:

\[
\begin{pmatrix}
x \\
y
\end{pmatrix} \to \begin{pmatrix}
x + hc_1 \left( x + y - \frac{x^3}{3} \right) \\
y - \frac{1}{c_1} \left( x - a_1 + b_1 y \right)
\end{pmatrix},
\]
where $h > 0$ is step size. For further biological relevance and dynamical analysis of some models that are very close to system (2), we refer to [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15], and references are therein. We investigate the existence of equilibria for (2) and local asymptotic stability of these steady-states by implementing linearized stability analysis techniques. Also, Neimark-Sacker bifurcation and period-doubling bifurcation are discussed.

2. Existence of equilibria and stability

The steady-states of (2) satisfy the following system of algebraic equations:

$$
\begin{align*}
x &= x + hc_1 \left( x + y - \frac{x^3}{3} \right), \\
y &= y - \frac{h}{c_1} \left( x - a_1 + b_1 \ast y \right).
\end{align*}
$$

(3)

From (3), it is quite simple to obtain:

$$
b_1 x s + 3(1 - b_1)x - 3a_1 = 0, \\
y = \frac{a_1 - x}{b_1}.
$$

Next, we define the following quantity:

$$
\Delta := 4b_1(b_1 - 1)^3 - 9a_1^2b_1^2.
$$

(4)

Then, it follows that

- If $\Delta > 0$, then system (2) has three distinct equilibrium points.
- If $\Delta = 0$, then system (2) has a multiple equilibrium points.
- If $\Delta < 0$, then system (2) has a unique positive equilibrium point.

For $a_1 = 4.92$ and $b_1 = 0.16$, we have $\Delta = -5.95649 < 0$ and existence for unique positive equilibrium is depicted in Figure 1. For $a_1 \in [0, 50]$ and $b_1 \in [0, 50]$, the region (blue) where $\Delta < 0$ and region (red) where $\Delta > 0$ are depicted in Figure 2. Mathematically, we have the following conditions for negativity and positivity of $\Delta$:

- $\Delta < 0$ if and only if $0 < b_1 \leq 1$, or $b_1 > 1$ and $a_1 > \frac{2}{3} \sqrt{-1 + 3b_1 - 3b_1^2 + b_1^3}$.
- $\Delta > 0$ if and only if $b_1 > 1$ and $a_1 < \frac{2}{3} \sqrt{-1 + 3b_1 - 3b_1^2 + b_1^3}$.
- $\Delta = 0$ if and only if $b_1 > 1$ and $a_1 = \frac{2}{3} \sqrt{-1 + 3b_1 - 3b_1^2 + b_1^3}$.

Now the Jacobian matrix of (2) evaluated at arbitrary equilibrium $(x, y)$ is given by:

$$
\begin{align*}
J(x, y) := \begin{pmatrix}
1 + \left( h - hx^2 \right) c_1 & h c_1 \\
-\frac{1}{c_1} & 1 - \frac{h}{c_1}
\end{pmatrix}.
\end{align*}
$$

Moreover, the characteristic polynomial of $J(x, y)$ is given by:

$$
P(\lambda) := \lambda^2 - \left( 2 - \frac{b_1 h}{c_1} + c_1 h \left( 1 - x^2 \right) \right) \lambda \\
+ 1 + h \left( c_1 + h - c_1 x^2 \right) - \frac{b_1 h \left( 1 + c_1 h \left( 1 - x^2 \right) \right)}{c_1}.
$$

(5)
Theorem 2.1. [16] Assume that $\Delta < 0$, then unique positive equilibrium $(x, y)$ has the following topological classification:

(i) $(x, y)$ is a sink if and only if

$$\left| 2 - \frac{b_1 h}{c_1} + c_1 h (1 - x^2) \right| < 2 + h (c_1 + h - c_1 x^2) - \frac{b_1 h (1 + c_1 h (1 - x^2))}{c_1} < 2.$$

(ii) $(x, y)$ is a saddle if and only if

$$\left( 2 - \frac{b_1 h}{c_1} + c_1 h (1 - x^2) \right)^2 - 4 \left( 1 + h (c_1 + h - c_1 x^2) - \frac{b_1 h (1 + c_1 h (1 - x^2))}{c_1} \right) > 0,$$

and

$$\left| 2 - \frac{b_1 h}{c_1} + c_1 h (1 - x^2) \right| > \left| 2 + h (c_1 + h - c_1 x^2) - \frac{b_1 h (1 + c_1 h (1 - x^2))}{c_1} \right|.$$

(iii) $(x, y)$ is a source if and only if

$$\left| 1 + h (c_1 + h - c_1 x^2) - \frac{b_1 h (1 + c_1 h (1 - x^2))}{c_1} \right| > 1,$$

and

$$\left| 2 - \frac{b_1 h}{c_1} + c_1 h (1 - x^2) \right| < \left| 2 + h (c_1 + h - c_1 x^2) - \frac{b_1 h (1 + c_1 h (1 - x^2))}{c_1} \right|.$$
(iv) \((x, y)\) is non-hyperbolic if and only if
\[
\left| 2 - \frac{b_1 h}{c_1} + c_1 h (1 - x^2) \right| = \left| 2 + h (c_1 + h - c_1 x^2) - \frac{b_1 h (1 + c_1 h (1 - x^2))}{c_1} \right|
\]
or
\[
(c_1 + h - c_1 x^2) - \frac{b_1 (1 + c_1 h (1 - x^2))}{c_1} = 0
\]
and
\[
\left| 2 - \frac{b_1 h}{c_1} + c_1 h (1 - x^2) \right| \leq 2.
\]
If we choose \(b_1 = 0.45\), \(c_1 = 0.15\), \(h \in [0, 1]\) and \(x \in [0, 10]\), then topological classification for unique positive point of system (2) is shown in Figure 3.
3. Bifurcation analysis

Studying bifurcation analysis for discrete-time models is a topic of great interest. Recently, there are many articles have published for the investigation for period-doubling and Neimark-Sacker bifurcations in discrete-time models [17, 18, 19, 20, 21, 22, 23]. In this section, we explore the parametric conditions under which system (2) undergoes period-doubling and Neimark-Sacker bifurcations at its unique positive equilibrium point. For this, first we discuss the emergence of period-doubling bifurcation at positive equilibrium of system (2). Assume that
$P(-1) = 0$, where $P(\lambda)$ is defined in (5), then system (2) undergoes period-doubling bifurcation as $h$ varies in a small neighborhood of $h_0$ defined by

$$h_0 := \frac{b_1 + (-1 + x^2) c_1^2 - \sqrt{-4c_1^4 + (b_1 - (-1 + x^2) c_1^2)^2}}{(1 + (-1 + x^2) b_1) c_1},$$

or

$$h_0 := \frac{b_1 + (-1 + x^2) c_1^2 + \sqrt{-4c_1^4 + (b_1 - (-1 + x^2) c_1^2)^2}}{(1 + (-1 + x^2) b_1) c_1}.$$

Secondly, we assume that

$$\left( b_1 + (-1 + x^2) c_1^2 \right)^2 \left( -4c_1^4 + (b_1 - (-1 + x^2) c_1^2)^2 \right) < 0.$$ 

Then system (2) undergoes Neimark-Sacker bifurcation as parameter $h$ varies in a small neighborhood of $h_1$ defined by:

$$h_1 := \frac{b_1 + (-1 + x^2) c_1^2}{(1 + (-1 + x^2) b_1) c_1}.$$

In order to verify aforementioned mathematical investigation for existence of period-doubling and Neimark-Sacker bifurcations, we choose particular parametric values for system (2) as follows:

**Period-doubling bifurcation:** Let $a_1 = 2.6$, $b_1 = 1.2$, $c_1 = 1.9$ and $h \in [0.3, 0.5]$. In this case, system (2) undergoes period-doubling bifurcation as $h$ varies in a small neighborhood of $h_0 = 0.39$. Moreover, the bifurcation diagrams for period-doubling bifurcation are shown in Figure 4 and Figure 5. Moreover, maximum Lyapunov exponents (MLE) are shown in Figure 6 and a chaotic attractor is depicted in Figure 7.

**Neimark-Sacker bifurcation:** Taking $a_1 = 2.7$, $b_1 = 2.5$, $c_1 = 0.95$ and $h \in [0.65, 0.72]$. Then system (2) undergoes Neimark-Sacker bifurcation as $h$ varies in a small neighborhood of $h_1 = 0.69$. The diagrams for Neimark-Sacker bifurcation are given in Figure 8 and Figure 9. Furthermore, MLE are shown in Figure 10 and phase portrait at $h = 0.69$ is depicted in Figure 11.

### 4. Conclusion

The qualitative behavior for a two-dimensional discrete-time Fitzhugh-Nagumo model is investigated. Euler’s forward scheme is implemented to obtain the discrete counterpart of the continuous Fitzhugh-Nagumo model. It is investigated that discrete-time model has rich dynamical behavior as compare to its continuous counterpart. The topological classification for steady-state solutions is discussed. Furthermore, parametric conditions for the existence of period-doubling bifurcation and Neimark-Sacker bifurcation are analyzed by taking $h$ as bifurcation parameter. At the end numerical simulations are provided to illustrate the theoretical discussion.
Figure 4. Bifurcation diagram for $x_n$

Figure 5. Bifurcation diagram for $y_n$
Figure 6. Maximum Lyapunov exponents

Figure 7. A chaotic attractor at $h = 0.5$
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Figure 8. Bifurcation diagram for $x_n$

Figure 9. Bifurcation diagram for $y_n$
Figure 10. Maximum Lyapunov exponents

Figure 11. Phase portrait at $h = 0.69$
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Competing Interests

The author(s) do not have any competing interests in the manuscript.

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Qamar Din
Department of Mathematics, The University of Poonch Rawalakot, Pakistan.
e-mail: qamar.sms@gmail.com

Sadaf Khaliq
Department of Mathematics, The University of Poonch Rawalakot, Pakistan.
e-mail: sadafkhaliq00@gmail.com