Open J. Math. Sci., Vol. **2***(2018), No. 1, pp. 351 - 360*
Website: https://pisrt.org/psr-press/journals/oms/
ISSN: 2523-0212 (Online) 2616-4906 (Print)
http://dx.doi.org/10.30538/oms2018.0040

ON TORSION AND FINITE EXTENSION OF FC AND τN_K GROUPS IN CERTAIN CLASSES OF FINITELY GENERATED GROUPS

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Abstract. Let $k > 0$ an integer. F, τ, N, N_k, $N_k^{(2)}$ and A denote the classes of finite, torsion, nilpotent, nilpotent of class at most k, group which every two generator subgroup is N_k and abelian groups respectively. The main results of this paper is, firstly, we prove that, in the class of finitely generated τN-group (respectively FN-group) a $(FC)\tau$-group (respectively $(FC)F$-group) is a τA-group (respectively is FA-group). Secondly, we prove that a finitely generated τN-group (respectively FN-group) in the class $((\tau N_k)\tau,\ \infty)$ (respectively $((FN_k)F,\ \infty)$) is a $\tau N_k^{(2)}$-group (respectively $FN_k^{(2)}$-group). Thirdly we prove that a finitely generated τN-group (respectively FN-group) in the class $((\tau N_k)\tau,\ \infty)^*$ (respectively $((FN_k)F,\ \infty)^*$) is a τN_c-group (respectively FN_c-group) for certain integer c and we extend this results to the class of NF-groups.

Mathematics Subject Classification: 20F22, 20F99.
Key words and phrases: $(FC)\tau$-group; $(FC)F$-group; $((\tau N_k)\tau,\ \infty)$-group; $((FN_k)F,\ \infty)$-group; $((\tau N_k)\tau,\ \infty)^*$-group; $((FN_k)F,\ \infty)^*$-group.

1. Introduction and Preliminaries

Definition 1.1. A group G is said to be with finite conjugacy classes (or shortly FC-group) if and only if every element of G has a finite conjugacy class in G.

It is known that $FIZ \subseteq FA \subseteq FC$, where FIZ denotes the class of center-by-finite groups, and that for finitely generated equalities $FIZ = FA = FC$ hold. These results and other have been studied and developed by Baer, Neumann,

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Received 30 April 2018. Revised 10 October 2018.

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Erdos and Tomkinson and others in [1, 2, 3, 4, 5]. FC-groups have many similar properties with abelian groups and finite groups. It is known that the class of FC-groups is closed under taking subgroup, homomorphic images, quotient and forming restricted direct products, but it is not closed under taking finite extension. We prove in Theorem 2.2, that in the class of finitely generated τN-group (respectively FN-group) a $(FC)\tau$-group (respectively $(FC)F$-group) is τA-group (respectively is FA-group).

Definition 1.2. Let χ is a given property of groups. A group G it is said to be in the class $(\chi,\, \infty)$ (respectively $(\chi,\, \infty)^*$) if and only if every infinite subset X of G contains two distinct elements x, y such that the subgroup $< x,\, y >$ (respectively $< x,\, x^y >$) is a χ-group. Note that if χ is a subgroup closed class, then $\chi \subset (\chi,\, \infty) \ \subset (\chi,\, \infty)^*$.

On the one hand, several authors have studied the class of $(\chi,\, \infty)$-groups, where χ is a given property of groups, with some conditions on these groups. The question that interests mathematicians is the following: If G is a group in the class $(\chi,\, \infty)$ where χ is a given property, then does G have a property in relation to the property χ. For example G has the property $\chi\gamma$ or $\gamma\chi$, etc. where γ is another group property, or in particular G is in the same class χ. For example, in 1976, Neumann in [6], has shown that a group is in the class $(A,\, \infty)$ if and only if it is FIZ-group. In 1981, Lennox and Wiegold in [7] proved that a finitely generated solvable group is in the class $(N,\, \infty)$ (respectively $(P,\, \infty)$, $(C_o,\, \infty)$) if and only if it is FN, (respectively P, C_o), where P and C_o respectively polycyclic and coherent groups.

In 2000, 2002 and 2005, Abdollahi and Trabelsi, proved in [8, 9, 10] that a finitely generated solvable group is in the class $(FN_k,\, \infty)$ (respectively $(FN,\, \infty)$, $(NF,\, \infty)$, $(\tau N,\, \infty)$) if and only if it is $FN_k^{(2)}$, (respectively $FN, NF, \tau N$). Other results of this type have been obtained, for example in [8, 11, 12, 13, 14, 15, 16]. In this note, we prove that a finitely generated τN-group G which is in the class $((\tau N_k)\tau,\, \infty)$ is in the class $\tau N_k^{(2)}$ and deduce that a finitely generated FN-group (respectively NF-group) G in the class of $((FN_k)F,\, \infty)$-groups, is in the class of $FN_k^{(2)}$-groups (respectively in the class of $N_k^{(2)}F$-groups) and particularly a group G is in the class $((FC)F,\, \infty)$ if and only if, it is FA-group.

On the other hand, in 2005, Trabelsi in [10] (respectively in 2007, Rouabehi and Trabelsi in [17]) proved that a finitely generated soluble group in the class $(CN,\, \infty)^*$ where C is the class of cernikov group (respectively in the class in the class $(\tau N,\, \infty)^*$) is FN-group (respectively τN-group) and in 2007 too, Guerbi and Rouabhi in [14] proved that a finitely generated Hyper(abelian-by-finite) group in the class $(\Omega,\, \infty)^*$ where Ω is the class of finite depth group, is FN-group. In this paper, we prove that a finitely generated τN-group in the class $((FN_k)\tau,\, \infty)^*$ is in the class τN_c for certain integer c and deduce that a finitely generated FN-group (respectively NF-group) G in the class $((FN_k)F,\, \infty)^*$) is in the class FN_c (respectively N_cF). In particular, if G is a finitely generated FN-group in the class $((FC)F, \infty)^*$ (respectively $((FN_2)F, \infty)^*$) then G is in the class of FN_2-groups (respectively in the class of $FN_3^{(2)}$-groups).

2. Main Results

2.1. Torsion and finite extension of property FC. It is known that the property FC is not closed under the formation of extension. The following example shows that even, a finite extension (respectively torsion extension) of an FC-group is not always an FC-group(respectively τA-group).
Note that if the center of an infinite finitely generated group is trivial or finite then this group is not FC.

Example 2.1. Let $G = D_\infty =< a, b/ \ a^2 = 1$ and $aba = b^{-1} >$ the infinite dihedral group, which is a finitely generated soluble group, generated by the involutions a, b. We have $K = C_\infty =< b >$ which is a infinite cyclic group isomorphic to $\mathbb{Z}$ therefore it is a FC-group and the quotient group G/K is isomorphic to $C_2 =< a >$ which is finite of order 2, thus G is a finite extension of a FC-group, but as the center of the infinite dihedral group is trivial then it is not a FC-group.

This example shows also that D_∞ is a torsion extension of a FC-group but it is not a τA-group, so we consider the class of finitely generated τN-groups (respectively FN-group) and we prove that, in this class, a $(FC)\tau$-group (respectively $(FC)F$-group) is a τA-group (respectively FC-group).

Theorem 2.2. *Let G a finitely generated torsion-by-nilpotent group. If G is FC-by-torsion group then G is τA-group.*

Lemma 2.3. *If G is a nilpotent group of nipotency class d and g an element of G. The subgroup $< G', g >$ generated by the derived group G' and G is a nilpotent group of class $\leq d$.*

Lemma 2.4. *If G is nilpotent and torsion-free group, m, n two non-zero integers and x,$y \in G$, then,*

1. *If $x^n = y^n$ then $x = y$.*
2. *If $[x^m y^n] = 1$ in G, then $[x \ y] = 1$ in G.*
3. *If $[x^m y x^n y] = 1$ in G, then $[x \ y] = 1$ in G.*

Proof. **(1)** We proceed by induction on the nipotency class d of the group G. If $d = 1$ so G is abelian: $x^n = y^n \iff x^n y^{-n} = 1 \iff (xy^{-1})^n = 1$ and as G is torsion-free then $xy^{-1} = 1 \iff x = y$. we suppose now that G is torsion-free nilpotent and non abelian of nilpotecy class d. We consider the subgroup $H =< G' \ x >$ generated by the derived group G' and the element x, by the Lemma 2.3 above the nilpotency class of H is less than d. Then by the inductive hypothesis the Lemma is verified for H. we have $x \in H$ and $x^y = y^{-1}xy = x[x \ y] \in H$ and $x^n = y^n$. So as $(y^{-1}xy)^n = y^{-1}x^n y = y^{-1}y^n y = y^n = x^n$. The (1) in lemma applied to H give us that $y^{-1}xy = x$ which means that x and y

commute. So we have in G: $x^n = y^n \iff x^n y^{-n} = 1 \iff (xy^{-1})^n = 1 \iff x = y$.
(2) We have: $x^m y^n = y^n x^m \iff y^{-n} x^m y^n = x^m \iff (y^{-n} x y^n)^m = x^m \iff y^{-n} x y^n = x$ (according to 1)$\iff xy^n = y^n x \iff xy^n x^{-1} = y^n \iff (xyx^{-1})^n = y^n$ also by (1) we obtain $xyx^{-1} = y$ and so $xy = yx$.
(3) $[x^m y x^n y] = 1 \iff x^m y x^n y = x^n y x^m y \iff x^{m-n} y x^n = y x^m \iff x^{m-n} y = y x^{m-n} \iff [x^{m-n} y] = 1$ by the (2) we obtain $xy = yx$. □

Proof of Theorem 2.2

Proof. Since G is finitely generated torsion-by-nilpotent group, there exists a normal and torsion subgroup F of G such that the quotient group G/F is nilpotent group. As the property FC-by-torsion is closed under quotient, it is enough to show that G/F is a FC-group. For this it is sufficient to show that every FC-by-torsion group G in the class of finitely generated nilpotent groups in a τA-group. Assume that G is (FC)-by-torsion, so there exists a normal FC-subgroup N such that the quotient G/N is torsion. Since G is finitely generated and nilpotent, it checks the maximal condition on subgroups. So N is finitely generated FC-subgroup. According to ([1], Theorem 6.2) N is center-by-finite which means that $Z(N)$ is of finite index in N. Or G/N is torsion group. It follows that the quotient $G/Z(N)$ is torsion group. So for all x and y in G, there exist non-zero integers m and n such that x^m and y^n belong to $Z(N)$, it follows that $[x^m,\, y^n] = 1$ in G. Let $T = Tor(G)$, the torsion subgroup of G. Since G is nilpotent, the quotient group G/T is a finitely generated nilpotent torsion-free group, and therefore $[x^m T,\, y^n T] = T$ in G/T. By the result (2) in Lemma 2.4 above, we deduce that $[xT,\, yT] = T$ in G/T, which shows that the group G/T is an abelian group. More, since G is nilpotent and checks the maximal condition, by ([2], Theorem 5.1) the subgroup T as finitely generated nilpotent torsion group, is finite. Thus G is finite-by-Abelian so G is τA. Since G/F is FC-by-torsion group in the class of finitely generated nilpotent-group, then G/F is τA and as F is torsion group it follows that G is $\tau(\tau A) = \tau A$. This completes the proof. □

Remark 2.1. The example below shows that Theorem 2.2. is falls when the condition "finitely generated" is omitted.

Example 2.5. Let $A = F_2[X]$ algebra of polynomials on the field F_2 and the isomorphism $\varphi : A \times A \to A \times A$, $(P, Q) \mapsto (P + Q, Q)$. We put $H = A \times A$ and $K =< \varphi >$ such that $\varphi^2 = Id_{A\times A}$ the identity application on $A \times A$. Since H is an abelian group, it is a FC-group. K is a finite group of order 2 and so it is FC too. We consider $G = H \rtimes K$, the semi-direct product of H by K. G is a non-finitely generated nilpotent group, which is a finite extension of the FC-group H. But G is not a FC-group.

If we replace the property τ by the property F we obtain a necessary and sufficient condition for the property FC to be closed under finite extension in the class of finitely generated FN-group.

Corollary 2.6. *Let G a finitely generated finite-by-nilpotent group. G is FC-by-finite group if and only if G is FA-group.*

Proof. It clear that if G is FA-group then G is FC and so FC-by-finite. If G is finitely generated finite-by-nilpotent group, as the same case in theorem above there exists a normal and finite subgroup F of G such that the quotient group G/F is nilpotent group. As the property FC-by-finite is closed under quotient, it is enough to show that G/F is a FC-group. For this it is sufficient to show that every FC-by-finite group G in the class of finitely generated nilpotent groups is a FA-group. Assume that G is (FC)-by-finite, so there exists a normal FC-subgroup N such that the quotient G/N is finite. As the same way in the above theorem, we found that $Z(N)$ is of finite index in N and the quotient $G/Z(N)$ is finite group. So for all x and y in G, there exist non-zero integers m and n such that x^m and y^n belong to $Z(N)$, it follows that $[x^m, y^n] = 1$ in G. If $T = Tor(G)$, we have $[x^mT, y^nT] = T$ in G/T. By the result (2) in Lemma 2.3, we deduce that $[xT, yT] = T$ in G/T, which shows that the group G/T is an Abelian group.
Moreover, as in the above theorem we found that T is finite. Thus G is finite-by-Abelian. Since G/F is FC-by-finite group in the class of finitely generated nilpotent-group, it is FA and so G is $F(FA) = FA$. This completes the proof. □

2.2. $\tau\mathbf{N}_k$ and FN_k-groups and conditions on infinite subsets. Our first elementary propositions below follows from a results in [8, 12] and [10].

Proposition 2.7. *If G is a finitely generated finite-by-soluble group in the class (FN_k, ∞), then G is in the class of $FN_k^{(2)}$-groups.*

Proof. Suppose that G is finite-by-soluble, there exists finite normal subgroup N such that G/N is soluble. As the class of (FN_k, ∞)-group, is closed under taking quotient, then the quotient group G/N is a finitely generated soluble group in the class of (FN_k, ∞)-group. By ([8] Corollary 1.8), G/N is in the class of $FN_k^{(2)}$-groups. Therefore G is finite-by-$FN_k^{(2)}$-group, and this gives that G is $FN_k^{(2)}$-group. □

Proposition 2.8. *If G is a finitely generated torsion-by-soluble group in the class $(\tau N_k, \infty)$, then G is in the class of $\tau N_k^{(2)}$-groups.*

Proof. Suppose that G is finite-by-soluble, there exists a torsion and normal subgroup N such that G/N is soluble. As the class of $(\tau N_k, \infty)$-group, is closed under taking quotient, then the quotient group G/N is a finitely generated soluble group in the class of $(\tau N_k, \infty) \subset (\tau N, \infty)$. By ([10], Theorem 1) G/N is in the class of τN-groups.

So G/N admits a torsion group $\tau(G/N) = T/N$ such that T is torsion and the quotient G/T is torsion-free in the class $(\tau N_k,\ \infty)$. So G/T is a finitely generated soluble group in the class $(N_k,\ \infty)$. It results by ([12]) that $G/T \in FN_k^{(2)}$, therefore G is torsion-by-$FN_k^{(2)}$, and this gives that G is $\tau N_k^{(2)}$-group. $\square$

Theorem 2.9. *Let G a finitely generated τN-group. If G is in the class $((\tau N_k)\tau,\ \infty)$, then*

1. *G is $\tau N_k^{(2)}$-group.*
2. *There exist integers d such that G is in the class $\tau N_{k^{d-1}}$.*

Proof. **(1)** Assume that G is finitely generated τN-group in the class $((\tau N_k)\tau, \infty)$. There exist a normal and torsion subgroup H of G such that G/H is nilpotent quotient group. Since G/H is finitely generated nilpotent group, it has a torsion subgroup T/H of finite order and as H is torsion group then T is torsion group too. So G/T is torsion-free nilpotent group in the class $((\tau N_k)\tau, \infty)$, which gives that G/T is in the class $(N_k\tau, \infty)$. We deduce by ([18], Lemma 6.33) that G/T is in the class (N_k, ∞) and so G/T is a finitely generated soluble group in the class (N_k, ∞). It follows by [12] that G/T belongs in the class of $FN_k^{(2)}$-groups and torsion-free so G/T is in the class $N_k^{(2)}$, it gives that G is in the class of $\tau N_k^{(2)}$-groups.

(2) In (1) we have G/T is a torsion-free nilpotent group in the class $N_k^{(2)}$ which is included in ε_k, so G/T is k-Engel torsion-free nilpotent (so soluble) group. If the integer d is the derived4length of G/T as a soluble group, then by a result of Gruenberg [18], Theorem 7.36, G/T is in the class $N_{k^{d-1}}$. So as T is torsion. It gives that G is $\tau N_{k^{d-1}}$. This completes the proof. $\square$

If we replace the property τN by the property FN, we obtain the results in the Lemma bellow.

Lemma 2.10. *Let G a finitely generated FN-group in the class $((FN_k)F,\ \infty)$, then,*

1. *G is in the class of $FN_k^{(2)}$-groups.*
2. *There exist integers d =d(k) and c=c(k, d) such that G is in the class $FN_{k^{d-1}}$ and $G/Z_c(G)$ is finite.*

Proof. **(1)** Assume that G is finitely generated FN-group in the class $((FN_k)F, \infty) \subset ((\tau N_k)\tau, \infty)$. As G is FN-group, there exist a normal and finite subgroup H of G such that G/H is nilpotent. We found that the torsion subgroup T/H of G/H is finite and so T is finite too. As the property $((\tau N_k)\tau, \infty)$ is closed under quotient then the quotient group G/T verifies the conditions of the above theorem. It follows that G/T belongs in the class of $\tau N_k^{(2)}$-groups which gives that G/T is in the class of $N_k^{(2)}$-groups and so G is $FN_k^{(2)}$.

(2) In one hand as the same way in (2) of the above theorem we found that G/T is in the class $N_{k^{d-1}}$ and T is finite. So G is in the class $FN_{k^{d-1}}$. In the other hand and by Hall [15], there exist an integer $c = c(k, d)$ depending on k, d such that $G/Z_c(G)$. ☐

The example 2.5 above shows that nilpotency is necessary for the results of the above theorem to remain true. Recall that FN-groups are NF-groups (see [15]).

Theorem 2.11. *Let G a finitely generated NF-group. If G in the class $((FN_k)F, \infty)$, then*

1. *G is in the class of $N_k^{(2)}F$-groups. In particular, if G in the class $((FA)F, \infty)$, then G is in the class of AF-groups.*
2. *There exist integers $d = d(k)$ such that G is in the class $N_{k^{d-1}}F$.*

Proof. **(1)** Assume that G is an infinite finitely generated NF- group in the class $((FN_k)\tau, \infty)$. As the group G is NF- group, and then it contains a normal nilpotent subgroup N such that G/N is finite. As the subgroup N is finitely generated and nilpotent of finite index then N is polycyclic so by ([19], Theorem 5.4.15) there exist a subgroup M normal in N and poly-infinite cyclic, hence torsion-free and of finite index in N. Let $K = M_G$ the core of the subgroup M, so K is nilpotent torsion-free of finite index in G. Since the class $((FN_k)\tau, \infty)$ is closed under taking subgroups, then K is nilpotent subgroup in the class $((FN_k)\tau, \infty)$ and according to (1) of lemma 2.10, we deduce that K is torsion-free subgroup in the class of $\tau N_k^{(2)}$-groups which gives that K is $N_k^{(2)}$-group and so G is $N_k^{(2)}F$-group. In particular, for $k = 1$ we have: $FN_1)\tau = (FC)\tau = (FA)\tau$ and $N_1^{(2)}\tau = A\tau$.

(2) As K is a torsion-free nilpotent subgroup in the class $N_k^{(2)}$ then it is in the class ε_k of k-Engel groups. So by Gruenberg [[18], Theorem 7.36(1)], there exist integer $d = d(k)$ such that K is in the class $N_{k^{d-1}}$ and as K is of finite index then G is in the class of $N_{k^{d-1}}F$. This completes the proof. ☐

In 2007 T.Rouabehi and N.Trabelsi in [17] proved that a finitely generated soluble group in the class $(\tau N_k, \infty)^*$ is in the class τN_c for certain integer c depending only on k.

If we replace the properties $((\tau N_k)\tau, \infty)$ and $((FN_k)F, \infty)$ in the above results by the properties $((\tau N_k)\tau, \infty)^*$ and $((FN_k)F, \infty)^*$, we obtain the next results.

Theorem 2.12. *Let G a finitely generated τN-group. G is in the class $((\tau N_k)\tau, \infty)^*$, then there exist an integer $c = c(k)$ such that G is in the class of τN_c-group.*

Proof. Assume that G is finitely generated τN- group in the class $((\tau N_k)\tau, \infty)^*$. There exist a normal and torsion subgroup F of G such that G/F is nilpotent quotient group. Since G/F is finitely generated nilpotent group, it has a finite and so torsion subgroup T/F such that T is a normal and torsion subgroup containing F. So G/T is torsion-free nilpotent group in the class $((\tau N_k)\tau, \infty)^*$ and hence G/T is in the class $(N_k\tau, \infty)^*$. We deduce by ([18], Lemma 6.33) that G/T is in the class $(N_k, \infty)^*$. It is known that the class $(N_k, \infty)^*$ is included in the class $\varepsilon_{k+1}(\infty)$, where $\varepsilon_{k+1}(\infty)$ is the class of groups whose every

infinite subset X contain two distinct elements x, y such that $[x,_{k+1} y] = 1$. We deduce that G/T belongs in $\varepsilon_{k+1}(\infty)$. Since G/T is nilpotent so soluble then by ([20], Theorem 3) there exist an integer $c = c(k)$ depending only on k such that $(G/T)/Z_c(G/T)$ is finite. By a result in ([15], Theorem 1) $\gamma_{c+1}(G/T) = \gamma_{c+1}(G)T/T$ is finite and so is torsion, and since T is torsion group , we deduce that $\gamma_{c+1}(G)$ is torsion group. Therefore G is in the class of τN_c-group.
This completes the proof. □

Lemma 2.13. *Let G a finitely generated FN-group. Then*

1. *if G is in the class $((FN_k)F, \infty)^*$, then there exist an integer $c = c(k)$ depending only on k such that G is in the class of FN_c-group.*
2. *if G is in the class $((FC)F, \infty)^*$, then, $G/Z_2(G)$ is finite and G is in the class of FN_2-groups.*
3. *if G is in the class $((FN_2)F, \infty)^*$, then, G is in the class of $FN_3^{(2)}$-groups and there exist an integer d such that G is $FN_{3^{d-1}}$.*

Proof. **(1)** Assume that G is finitely generated FN-group in the class $((FN_k)F, \infty)^*$. As G is FN-group, there exist a normal and finite subgroup F of G such that G/F is nilpotent quotient group. Since G/F finitely generated nilpotent group it has a torsion subgroup T/F of finite order. So the subgroup T of torsion elements of G is normal and finite in G and as the same way in the above theorem, we deduce by ([18], Lemma 6.33) that G/T is nilpotent torsion -free in the class $(N_k, \infty)^* \subset \varepsilon_{k+1}(\infty)$ and according to ([20], Theorem 3) we found that there exist an integer $c = c(k)$ depending only on k such that $(G/T)/Z_c(G/T)$ is finite. Also by ([15], Theorem 1) we find that $\gamma_{c+1}(G)T/T$ is finite and as T is finite then $\gamma_{c+1}(G)$ is finite too. Therefore G is in the class of FN_c-group.
(2) If G is in the class $((FC)F, \infty)^* = ((FA)F, \infty)^* = ((FN_1)F, \infty)^*$, then as in (i) G/T is in the class $(N_1, \infty)^* \subset \varepsilon_2(\infty)$, by a result of Abdollahi [11] $(G/T)/Z_2(G/T)$ is finite and by ([15], Theorem 1) we find that $\gamma_3(G)T/T$ is finite and as T is finite then $\gamma_3(G)$ is finite too and as G is finitely generated then by [15],G/Z_2 is finite. Therefore G is in the class of FN_2-group.
(3) For $k = 2$, as the same way in (1) we found that G/T is in the class $(N_2, \infty)^*$which is included in the class $\varepsilon 3(\infty)$, where $\varepsilon_3(\infty)$ is the class of groups whose every infinite subset X contain two distinct elements x, y such that $[x,_3 y] = 1$. We deduce by ([20], Theorem 1) that G/T is torsion-free in the class $FN_3^{(2)}$ so G/T is in $N_3^{(2)}$ and as the torsion subgroup T is finite, then G is $FN_3^{(2)}$-group. As G/T is torsion-free soluble group in $N_3^{(2)} \subset \varepsilon_3$ (the 3-Engel group) then by Gruenberg [[18], Theorem 7.36 (1)], there exist integer d such that G/T is in the class $N_{3^{d-1}}$ which gives that G is $FN_{3^{d-1}}$This completes the proof. □

Theorem 2.14. *Let G a finitely generated NF-group. Then*

1. *if G is in the class $((FN_k)F, \infty)^*$, then there exist an integer $c = c(k)$ depending only on such that G is in the class of N_cF-groups.*

(2) *if G is in the class of $((FC)F,\ \infty)^*$-groups, then, G is in the class of N_2F-group.*

(3) *if G is in the class $((FN_2)F,\ \infty)^*$, then, G is in the class of $N_3^{(2)}F$-groups and there exist an integer d such that G is $FN_{3^{d-1}}$.*

Proof. **(1)** As the group G is NF- group, and then it contains a normal nilpotent subgroup N such that G/N is finite. As the subgroup N is finitely generated and nilpotent of finite index then N is polycyclic so by ([19], Theorem 5.4.15) there exist a normal subgroup M in N and poly-infinite cyclic hence torsion-free and of finite index in N. Let $K = M_G$ the core of the subgroup M, so K is nilpotent torsion-free of finite index in G. Since the class $((FN_k)F,\ \infty)^*$is closed under taking subgroups, then K is in this class too, so by (1) of lemma 2.13, we obtains that there exist an integer $c = c(k)$ depending only on k such that K is FN_c-group and as K is torsion-free, it is N_c-group and so G is N_cF-group.
(2) Particularly for $k = 1$, we have $((FC)F,\ \infty)^* = ((FN_1)F, \infty)^*$, in this case the subgroup K is a finitely generated torsion-free nilpotent group in the class $((FN_1)F,\ \infty)^*$ and according to (2) of lemma 2.13, we deduce that K is in the class FN_2-groups and as K is torsion-free, it is N_2-group of finite index in G, this gives that G is N_2F-group.
(3) In particular for $k = 2$, we have the subgroup K in (1) is a finitely generated torsion-free nilpotent group in the class $((FN_2)F,\ \infty)^*$ and according to (3) of lemma 2.13, we deduce that K is in the class $FN_3^{(2)}$-groups and as K is torsion-free it is the class $N_3^{(2)}$-group and as G/K if finite this gives that G is in the class of $N_3^{(2)}F$-groups. As K is nilpotent torsion- free in the class $N_3^{(2)}$then it is in the class ε_3 of 3-Enjel group, then by Gruenberg [[18], Theorem 7.36 (1)], there exist integer d such that K is in the class $N_{3^{d-1}}$ and so G is in the class $N_{3^{d-1}}F$-group. □

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments and suggestions in advance.

Competing Interests

The author(s) do not have any competing interests in the manuscript.

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