Abstract. In this paper, we use the Delta Riemann-Liouville fractional integrals to establish some new integral inequalities for the Chebyshev functional in the case of two synchronous functions on time scales. Our results improve the inequalities for the discrete and continuous cases.

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1. Introduction

Integral inequalities and their extensions have received considerable attention in the theory of differential and difference equations. Recently, there has been much interest in the study of integral inequalities on time scales. Researchers have studied various aspects of inequalities on time scales [1, 2, 3]. New researches on dynamic inequalities using time scales was done by Agarwal in a monograph [4].

The fractional calculus is an extensions of derivatives and integrals to noninteger orders. This subject came to the attention of many researchers and fractional calculus on time scales has been studied [5]. In [6], some fractional integral inequalities have been studied in the real case. For more investigations, we refer [7, 8] to the readers.
In this paper, we obtain some generalizations and refinements for some existing inequalities on time scales. We consider the functional \[ T(f, g) := \frac{1}{t-a} \int_a^t f(x) g(x) \Delta x - \frac{1}{t-a} \left( \int_a^t f(x) \Delta x \right) \left( \frac{1}{t-a} \int_a^t g(x) \Delta x \right) \]

where \( f \) and \( g \) are two integrable functions which are synchronous on \([a, t]_\mathbb{T}\). (i.e. \((f(x) - f(y))(g(x) - g(y)) \geq 0\) for any \( x, y \in [a, t]_\mathbb{T}\).) The intervals with \( \mathbb{T} \) subscript are used to denote the intersection of the usual interval with \( \mathbb{T} \); i.e., \([a, t]_\mathbb{T} := [a, t] \cap \mathbb{T}\) for convenience.

The main purpose of this paper is to establish some new fractional inequalities for synchronous functions using delta Reimann-Liouville fractional integrals on time scales. Our results unify and extend some continuous inequalities and their corresponding discrete analogues. In the following, we present some basic concepts about time scale calculus and refer the reader to resource [7] for more detailed information on subject.

2. Preliminaries

A time scale \( \mathbb{T} \) is an arbitrary nonempty closed subset of the real numbers \( \mathbb{R} \). For \( t \in \mathbb{T} \) we define the forward jump operator \( \sigma : \mathbb{T} \to \mathbb{T} \) by

\[ \sigma (t) := \inf \{ s \in \mathbb{T} : s > t \} \]

while the backward jump operator \( \rho : \mathbb{T} \to \mathbb{T} \) is defined by

\[ \rho (t) := \sup \{ s \in \mathbb{T} : s < t \} . \]

If \( \sigma (t) > t \), we say that \( t \) is right-scattered, while if \( \rho (t) < t \) we say that \( t \) is left-scattered. Also, if \( \sigma (t) = t \), then \( t \) is called right-dense, and if \( \rho (t) = t \), then \( t \) is called left-dense. The graininess function \( \mu : \mathbb{T} \to [0, \infty) \) is defined by

\[ \mu (t) := \sigma (t) - t. \]

We introduce the set \( \mathbb{T}^\kappa \) which is derived from the time scale \( \mathbb{T} \) as follows. If \( \mathbb{T} \) has left-scattered maximum \( m \), then \( \mathbb{T}^\kappa = \mathbb{T} - \{ m \} \), otherwise \( \mathbb{T}^\kappa = \mathbb{T} \).

**Definition 2.1.** Let \( f \) and \( g \) be two integrable functions defined on \([a, t]_\mathbb{T}\). If for any \( x, y \in [a, t]_\mathbb{T}\)

\[ |f(x) - f(y)| |g(x) - g(y)| \geq 0, \]

then \( f \) and \( g \) are called synchronous functions on \([a, t]_\mathbb{T}\).

**Definition 2.2.** The rd continuous functions \( h_\alpha : \mathbb{T} \times \mathbb{T} \to \mathbb{R} \) is called as generalized polynomials on time scales such that, for all \( s, t \in \mathbb{T} \) and \( \alpha \geq 0 \);

\[ h_0 (t, s) = 1 \]

\[ h_{\alpha+1} (t, s) = \int_s^t h_\alpha (\tau, s) \Delta \tau \]
Definition 2.3. The function \( f : \mathbb{T} \to \mathbb{R} \) is called \( \text{rd-continuous} \) provided it is continuous at right-dense points in \( \mathbb{T} \) and its left-sided limits exist (finite) at left-dense points in \( \mathbb{T} \).

Definition 2.4. The Delta-Riemann-Liouville fractional integral operator of order \( \alpha \geq 1 \) on time scales, for a function \( f \in C_{rd} \) is defined as

\[
D_\alpha^a f(t) = \int_a^t h_{\alpha-1}(t, \sigma(\tau)) f(\tau) \Delta \tau,
\]

\( D_0^a f = f \).

3. Main Results

In this section we present our main results.

Theorem 3.1. Let \( f \) and \( g \) be two synchronous functions on \( [0, \infty)_{\mathbb{T}} \). Then for all \( t > a, \alpha \geq 1, a \geq 0 \) we have

\[
D_\alpha^a (fg)(t) \geq (h_\alpha(t, a))^{-1} D_\alpha^a f(t) D_\alpha^a g(t). \tag{1}
\]

Proof. Since \( f \) and \( g \) are synchronous functions on \( [0, \infty)_{\mathbb{T}} \), then for all \( \tau, \phi \geq 0 \),

\[
(f(\tau) - f(\phi))(g(\tau) - g(\phi)) \geq 0 \tag{2}
\]

and

\[
f(\tau)g(\tau) - f(\tau)g(\phi) - f(\phi)g(\tau) + f(\phi)g(\phi) \geq 0. \tag{3}
\]

Therefore,

\[
f(\tau)g(\tau) + f(\phi)g(\phi) \geq f(\tau)g(\phi) + f(\phi)g(\tau). \tag{4}
\]

For \( \tau \in (a, t) \), multiplying both sides of (4) by \( h_{\alpha-1}(t, \sigma(\tau)) \), we have

\[
h_{\alpha-1}(t, \sigma(\tau)) f(\tau) g(\tau) + h_{\alpha-1}(t, \sigma(\tau)) f(\phi) g(\phi) \geq h_{\alpha-1}(t, \sigma(\phi)) f(\tau) g(\phi) + h_{\alpha-1}(t, \sigma(\tau)) f(\phi) g(\tau). \tag{5}
\]

Integrating (5) over \((a, t)\), we get

\[
\int_a^t h_{\alpha-1}(t, \sigma(\tau)) f(\tau) g(\tau) \Delta \tau + \int_a^t h_{\alpha-1}(t, \sigma(\tau)) f(\phi) g(\phi) \Delta \tau \\
\geq \int_a^t h_{\alpha-1}(t, \sigma(\tau)) f(\tau) g(\phi) \Delta \tau + \int_a^t h_{\alpha-1}(t, \sigma(\tau)) f(\phi) g(\tau) \Delta \tau.
\]

Since \( f(\phi), g(\phi) \) and \( f(\phi)g(\phi) \) are independent from \( \tau \), we take them out of the integral and we obtain

\[
\int_a^t h_{\alpha-1}(t, \sigma(\tau)) f(\tau) g(\tau) \Delta \tau + f(\phi)g(\phi) \int_a^t h_{\alpha-1}(t, \sigma(\tau)) \Delta \tau
\]
Integrating (7) over \(a\) and finally we get
\[
\geq g(\phi) \int_a^t h_{a-1}(t, \sigma(\tau)) f(\tau) \Delta \tau + f(\phi) \int_a^t h_{a-1}(t, \sigma(\tau)) g(\tau) \Delta \tau.
\]

Using Definition 2.2 and Definition 2.4, we find
\[
D^\alpha_a(fg)(t) + f(\phi) g(\phi) (h_\alpha(t,a)) \geq g(\phi) D^\alpha_a f(t) + f(\phi) D^\alpha_a g(t).
\]
\[
(6)
\]
For \(\phi \in (a,t)\), multiplying both sides of (6) by \(h_{a-1}(t,\sigma(\phi))\), we obtain
\[
\begin{align*}
& \geq h_{a-1}(t,\sigma(\phi)) D^\alpha_a(fg)(t) + h_{a-1}(t,\sigma(\phi)) f(\phi) g(\phi)(h_\alpha(t,a)) \\
& \geq h_{a-1}(t,\sigma(\phi)) g(\phi) D^\alpha_a f(t) + h_{a-1}(t,\sigma(\phi)) f(\phi) D^\alpha_a g(t).
\end{align*}
\]
Integrating (7) over \((a,t)\), we get
\[
\begin{align*}
& \int_a^t h_{a-1}(t,\sigma(\phi)) D^\alpha_a(fg)(t) \Delta \phi + \int_a^t h_{a-1}(t,\sigma(\phi)) f(\phi) g(\phi)(h_\alpha(t,a)) \Delta \phi \\
& \geq h_{a-1}(t,\sigma(\phi)) g(\phi) D^\alpha_a f(t) \Delta \phi + \int_a^t h_{a-1}(t,\sigma(\phi)) f(\phi) D^\alpha_a g(t) \Delta \phi.
\end{align*}
\]
Since \(D^\alpha_a(fg)(t)\), \(D^\alpha_a f(t)\), \(D^\alpha_a g(t)\) and \(h_\alpha(t,a)\) are independent from \(\phi\), we obtain
\[
\begin{align*}
& D^\alpha_a(fg)(t) \int_a^t h_{a-1}(t,\sigma(\phi)) \Delta \phi + (h_\alpha(t,a)) \int_a^t h_{a-1}(t,\sigma(\phi)) f(\phi) g(\phi) \Delta \phi \\
& \geq D^\alpha_a f(t) \int_a^t h_{a-1}(t,\sigma(\phi)) g(\phi) \Delta \phi + D^\alpha_a g(t) \int_a^t h_{a-1}(t,\sigma(\phi)) f(\phi) \Delta \phi.
\end{align*}
\]
Therefore from Definition 2.2 and Definition 2.4, we can write
\[
\begin{align*}
& D^\alpha_a(fg)(t) (h_\alpha(t,a)) + (h_\alpha(t,a)) D^\alpha_a(fg)(t) \\
& \geq D^\alpha_a f(t) D^\alpha_a g(t) + D^\alpha_a g(t) D^\alpha_a f(t).
\end{align*}
\]
So we have
\[
2D^\alpha_a(fg)(t) (h_\alpha(t,a)) \geq 2D^\alpha_a f(t) D^\alpha_a g(t),
\]
\[
D^\alpha_a(fg)(t) (h_\alpha(t,a)) \geq D^\alpha_a f(t) D^\alpha_a g(t),
\]
and finally we get
\[
D^\alpha_a(fg)(t) \geq \frac{1}{(h_\alpha(t,a))} D^\alpha_a f(t) D^\alpha_a g(t).
\]
\(\square\)

**Theorem 3.2.** Let \(f\) and \(g\) be two synchronous functions on \([0,\infty)\). Then for all \(t > a\), \(\alpha, \beta \geq 1\), \(\alpha \geq 0\) we have:
\[
h_\alpha(t,a) D^\beta_a(fg)(t) + h_\beta(t,a) D^\alpha_a(fg)(t) \geq D^\alpha_a f(t) D^\alpha_a g(t) + D^\alpha_a g(t) D^\alpha_a f(t).
\]
Proof. Since $f$ and $g$ are synchronous functions on $[0, \infty]$, then for all $\tau, \phi \geq 0$, we have 
\[(f(\tau) - f(\phi))(g(\tau) - g(\phi)) \geq 0.\]
Therefore,
\[f(\tau)g(\tau) + f(\phi)g(\phi) \geq f(\tau)g(\phi) + f(\phi)g(\tau). \tag{8}\]
For $\tau \in (a, t)$, multiplying both sides of (8) by $h_{\alpha-1}(t, \sigma(\tau))$, we obtain
\[h_{\alpha-1}(t, \sigma(\tau))f(\tau)g(\tau) + h_{\alpha-1}(t, \sigma(\tau))f(\phi)g(\phi) \geq h_{\alpha-1}(t, \sigma(\tau))f(\tau)g(\phi) + h_{\alpha-1}(t, \sigma(\tau))f(\phi)g(\tau). \tag{9}\]
Integrating (9) over $(a, t)$, we get
\[
\int_a^t h_{\alpha-1}(t, \sigma(\tau)) f(\tau) g(\tau) \Delta \tau + \int_a^t h_{\alpha-1}(t, \sigma(\tau)) f(\phi) g(\phi) \Delta \tau \\
\geq \int_a^t h_{\alpha-1}(t, \sigma(\tau)) f(\tau) g(\phi) \Delta \tau + \int_a^t h_{\alpha-1}(t, \sigma(\tau)) f(\phi) g(\tau) \Delta \tau.
\]
Consequently
\[D_a^\alpha (fg)(t) + (fg)(\phi)(h_\alpha(t, a)) \geq g(\phi) D_a^\alpha f(t) + f(\phi) D_a^\alpha g(t). \tag{10}\]
For $\phi \in (a, t)$, multiplying both sides of (10) by $h_{\beta-1}(t, \sigma(\phi))$, we have
\[h_{\beta-1}(t, \sigma(\phi)) D_a^\alpha (fg)(t) + h_{\beta-1}(t, \sigma(\phi))(fg)(\phi)(h_\alpha(t, a)) \geq h_{\beta-1}(t, \sigma(\phi))(\phi) D_a^\alpha f(t) + h_{\beta-1}(t, \sigma(\phi)) f(\phi) D_a^\alpha g(t). \tag{11}\]
Integrating (11) over $(a, t)$, we get
\[
\int_a^t h_{\beta-1}(t, \sigma(\phi)) D_a^\alpha (fg)(t) \Delta \phi + \int_a^t h_{\beta-1}(t, \sigma(\phi))(fg)(\phi)(h_\alpha(t, a)) \Delta \phi \\
\geq \int_a^t h_{\beta-1}(t, \sigma(\phi))(\phi) D_a^\alpha f(t) \Delta \phi + \int_a^t h_{\beta-1}(t, \sigma(\phi)) f(\phi) D_a^\alpha g(t) \Delta \phi.
\]
then
\[D_a^\alpha (fg)(t) \int_a^t h_{\beta-1}(t, \sigma(\phi)) \Delta \phi + (h_\alpha(t, a)) \int_a^t h_{\beta-1}(t, \sigma(\phi)) f(\phi) g(\phi) \Delta \phi \\
\geq D_a^\alpha f(t) \int_a^t h_{\beta-1}(t, \sigma(\phi)) g(\phi) \Delta \phi + D_a^\alpha g(t) \int_a^t h_{\beta-1}(t, \sigma(\phi)) f(\phi) \Delta \phi.
\]
Therefore, we write
\[D_a^\alpha (fg)(t) (h_\beta(t, a)) + (h_\alpha(t, a)) D_a^\beta (fg)(t) \geq D_a^\alpha f(t) D_a^\alpha g(t) + D_a^\alpha g(t) D_a^\alpha f(t). \]
\[\square\]
Theorem 3.3. Let \((f_i)_{i=1}^{n}\) be \(n\) positive increasing functions on \([0, \infty)\).
Then for all \(t > a\), \(\alpha \geq 1\), \(a \geq 0\) we have
\[
D_a^\alpha \left( \prod_{i=1}^{n} f_i \right) (t) \geq (h_\alpha (t, a))^{1-n} \prod_{i=1}^{n} D_a^\alpha f_i (t).
\]

Proof. We use induction method to prove our result. Clearly, for \(n = 1\), all \(t > a\), \(\alpha \geq 1\), \(a \geq 0\) we have
\[
D_a^\alpha (f_1) (t) \geq D_a^\alpha f_1 (t).
\]
For \(n = 2\), applying Theorem 3.1, for all \(t > a\), \(\alpha \geq 1\), \(a \geq 0\) we obtain
\[
D_a^\alpha \left( \prod_{i=1}^{2} f_i \right) (t) \geq (h_\alpha (t, a))^{-1} \sum_{i=1}^{2} D_a^\alpha f_i (t)
\]
then,
\[
D_a^\alpha (f_1 f_2) (t) \geq \frac{1}{h_\alpha (t, a)} D_a^\alpha f_1 (t) D_a^\alpha f_2 (t).
\]
Now for \(n - 1\), we assume that (induction hypothesis) the following inequality
\[
D_a^\alpha \left( \prod_{i=1}^{n-1} f_i \right) (t) \geq |h_\alpha (t, a)|^{1-n} \prod_{i=1}^{n-1} D_a^\alpha f_i (t) \tag{12}
\]
holds. We have to prove that the inequality
\[
D_a^\alpha \left( \prod_{i=1}^{n} f_i \right) (t) \geq |h_\alpha (t, a)|^{1-n} \prod_{i=1}^{n} D_a^\alpha f_i (t)
\]
holds for \(n\). Since \((f_i)_{i=1}^{n}\) are positive increasing functions, then \(\prod_{i=1}^{n-1} f_i \) \((t)\) is an increasing function. Hence we can apply Theorem 3.1 to the functions \(\prod_{i=1}^{n-1} f_i = g\), \(f_n = f\). We obtain
\[
\prod_{i=1}^{n} f_i = \prod_{i=1}^{n-1} f_i f_n = fg
\]
\[
D_a^\alpha (fg) (t) \geq |h_\alpha (t, a)|^{-1} D_a^\alpha (f) (t) D_a^\alpha (g) (t)
\]
\[
D_a^\alpha \left( \prod_{i=1}^{n} f_i \right) (t) = D_a^\alpha \left( \prod_{i=1}^{n-1} f_i f_n \right) (t) \geq |h_\alpha (t, a)|^{-1} D_a^\alpha \left( \prod_{i=1}^{n-1} f_i \right) (t) D_a^\alpha (f_n) (t).
\]
Multiplying both sides of (12) by \(|h_\alpha (t, a)|^{-1} D_a^\alpha (f_n) (t)\), we obtain
\[
|h_\alpha (t, a)|^{-1} D_a^\alpha (f_n) (t) D_a^\alpha \left( \prod_{i=1}^{n-1} f_i \right) (t)
\]
\[
\geq |h_\alpha (t, a)|^{-1} |h_\alpha (t, a)|^{2-n} \prod_{i=1}^{n-1} D_a^\alpha f_i (t) D_a^\alpha f_n (t)
\]
and
\[
[h_\alpha (t, a)]^{-1} D_\alpha^a (f_n) (t) D_\alpha^a \left( \prod_{i=1}^{n-1} f_i \right) (t) \geq [h_\alpha (t, a)]^{-1-n} \prod_{i=1}^{n} D_\alpha^a f_i (t) .
\]

Therefore we obtain
\[
D_\alpha^a \left( \prod_{i=1}^{n} f_i \right) (t) \geq [h_\alpha (t, a)]^{-1-n} \prod_{i=1}^{n} D_\alpha^a f_i (t) .
\]

\[\Box\]

**Theorem 3.4.** Let \( f \) and \( g \) be two functions defined on \([0, \infty)_\mathbb{T}\), such that \( f \) is increasing and \( g \) is differentiable and there exist a real number \( m := \inf_{t \geq 0} g' (t) \). Then for all \( t > a, \alpha \geq 1, a \geq 0 \) we have
\[
D_\alpha^a (fg) (t) \geq [h_\alpha (t, a)]^{-1} D_\alpha^a f (t) D_\alpha^a g (t) - m [h_\alpha (t, a)]^{-1} D_\alpha^a f (t) D_\alpha^a (t) + mD_\alpha^a (tf (t)) .
\]

**Proof.** We consider the function \( k (t) := g (t) - mt \). It is evident that \( k \) is differentiable and it is increasing on \([0, \infty)_\mathbb{T}\). Then applying the Theorem 3.1 we have
\[
D_\alpha^a (kf) (t) \geq [h_\alpha (t, a)]^{-1} D_\alpha^a f (t) D_\alpha^a (k) (t) ,
\]
then
\[
D_\alpha^a ((g - mt) f (t)) \geq [h_\alpha (t, a)]^{-1} D_\alpha^a f (t) D_\alpha^a (g - mt) (t) .
\]
So we get
\[
D_\alpha^a (fg) (t) - mD_\alpha^a (tf (t)) \geq [h_\alpha (t, a)]^{-1} D_\alpha^a f (t) \left( D_\alpha^a g (t) - mD_\alpha^a (t) \right) = [h_\alpha (t, a)]^{-1} D_\alpha^a f (t) D_\alpha^a g (t) - m [h_\alpha (t, a)]^{-1} D_\alpha^a f (t) D_\alpha^a (t) ,
\]
and we obtain our desired inequality. \[\Box\]

**Corollary 3.5.** Let \( f \) and \( g \) be two functions defined on \([0, \infty)_\mathbb{T}\).

(1). Assume that \( f \) is decreasing, \( g \) is differentiable and there exist a real number \( M := \sup_{t \geq 0} g (t) \). Then for all \( t > a, \alpha \geq 1, a \geq 0 \), we have
\[
D_\alpha^a (fg) (t) \geq [h_\alpha (t, a)]^{-1} D_\alpha^a f (t) D_\alpha^a g (t) - M [h_\alpha (t, a)]^{-1} D_\alpha^a f (t) D_\alpha^a (t) + MD_\alpha^a (tf (t)) .
\]

(2). Assume that \( f \) and \( g \) are differentiable and there exist \( m_1 := \inf_{t \geq 0} f' (t), m_2 := \inf_{t \geq 0} g' (t) \). Then for all \( t > a, \alpha \geq 1, a \geq 0 \), we have
\[
D_\alpha^a (fg) (t) - m_2D_\alpha^a (tf (t)) - m_1D_\alpha^a (tg (t)) + m_1m_2D_\alpha^a t^2 \geq [h_\alpha (t, a)]^{-1} \left[ D_\alpha^a f (t) D_\alpha^a g (t) - m_2D_\alpha^a f (t) D_\alpha^a (t) - m_1D_\alpha^a (t) D_\alpha^a g (t) + m_1m_2 \left( D_\alpha^a (t) \right)^2 \right] .
\]
\textbf{(3). Assume that }f\text{ and }g\text{ are differentiable and there exist } M_1 := \sup_{t \geq 0} f'(t), \quad M_2 := \sup_{t \geq 0} g'(t). \text{ Then for all } t > a, \alpha \geq 1, a \geq 0, \text{ we have}

\[
D_a^\alpha (fg)(t) - M_2 D_a^\alpha (t g(t)) - M_1 D_2^\alpha t^2
\geq [h_\alpha(t,a)]^{-1} \left[ D_a^\alpha f(t) D_a^\alpha g(t) - M_2 D_a^\alpha f(t) D_a^\alpha (t) - M_1 D_2^\alpha (t) D_a^\alpha g(t)
\right.
\]
\[
+ M_1 M_2 (D_a^\alpha (t)^2) \].
\]

\textbf{Proof. \textbf{(1).}} Let } G(t) := g(t) - Mt \text{ be a function such that it is decreasing and differentiable on } [0, \infty)_T. \text{ Then for all } t > a, \alpha \geq 1, a \geq 0, \text{ we have}

\[
D_a^\alpha (Gf)(t) \geq [h_\alpha(t,a)]^{-1} D_a^\alpha f(t) D_a^\alpha (g - Mt)(t).
\]

\text{Then we have}

\[
D_a^\alpha (fg)(t) - M_2 D_a^\alpha (t f(t)) \geq [h_\alpha(t,a)]^{-1} D_a^\alpha f(t) (D_a^\alpha g(t) - M D_a^\alpha (t))
\]
\[
= [h_\alpha(t,a)]^{-1} D_a^\alpha f(t) D_a^\alpha g(t) - M [h_\alpha(t,a)]^{-1} D_a^\alpha f(t) D_a^\alpha (t),
\]

and so

\[
D_a^\alpha (fg)(t) \geq [h_\alpha(t,a)]^{-1} D_a^\alpha f(t) D_a^\alpha g(t) - M [h_\alpha(t,a)]^{-1} D_a^\alpha f(t) D_a^\alpha (t) + M D_a^\alpha (tf(t)).
\]

\textbf{(2).} Let } F(t) := f(t) - m_1 t, G(t) := g(t) - m_2 t \text{ be two functions such that they are increasing and differentiable on } [0, \infty)_T. \text{ Then for all } t > a, \alpha \geq 1, a \geq 0 \text{ and by Theorem 3.1 we have}

\[
D_a^\alpha (FG)(t) \geq [h_\alpha(t,a)]^{-1} D_a^\alpha F(t) D_a^\alpha G(t),
\]

and

\[
D_a^\alpha ((f(t) - m_1 t)(g(t) - m_2 t))(t)
\geq [h_\alpha(t,a)]^{-1} D_a^\alpha (f(t) - m_1 t)(t) D_a^\alpha (g(t) - m_2 t)(t).
\]

\text{Then we get}

\[
D_a^\alpha ((fg)(t) - m_2 t f(t) - m_1 t g(t) + m_1 m_2 t^2)
\geq [h_\alpha(t,a)]^{-1} [D_a^\alpha f(t) - m_1 D_a^\alpha t] [D_a^\alpha g(t) - m_2 D_a^\alpha t]
\]
\[
= [h_\alpha(t,a)]^{-1} \left[ D_a^\alpha f(t) D_a^\alpha g(t) - m_2 D_a^\alpha f(t) D_a^\alpha t - m_1 D_a^\alpha g(t) D_a^\alpha t
\right.
\]
\[
+ m_1 m_2 (D_a^\alpha t)^2 \].
\]

\text{Therefore we obtain}

\[
D_a^\alpha ((fg)(t) - m_2 t f(t) - m_1 t g(t) + m_1 m_2 t^2)
\geq [h_\alpha(t,a)]^{-1} \left[ D_a^\alpha f(t) D_a^\alpha g(t) - m_2 D_a^\alpha f(t) D_a^\alpha t - m_1 D_a^\alpha g(t) D_a^\alpha t
\right.
\]
\[
+ m_1 m_2 (D_a^\alpha t)^2 \].
(3). Let $F(t) := f(t) - M_1 t$, $G(t) := g(t) - M_2 t$ be two functions such that they are decreasing and differentiable on $[0, \infty)$. Then for all $t > a$, $\alpha \geq 1$, $a \geq 0$ and by Theorem 3.1, we have

$$D_\alpha^a (FG)(t) \geq [h_\alpha (t,a)]^{-1} D_\alpha^a F(t), D_\alpha^a G(t)$$

so

$$D_\alpha^a ((f(t) - M_1 t)(g(t) - M_2 t))(t) \geq [h_\alpha (t,a)]^{-1} D_\alpha^a (f(t) - M_1 t)(t) D_\alpha^a (g(t) - M_2 t)(t).$$

Hence we obtain

$$D_\alpha^a ((fg)(t) - M_2 tf(t) - M_1 t g(t) + M_1 M_2 t^2) \geq [h_\alpha (t,a)]^{-1} [D_\alpha^a f(t) - M_1 D_\alpha^a t][D_\alpha^a g(t) - M_2 D_\alpha^a t]$$

$$= [h_\alpha (t,a)]^{-1} [D_\alpha^a f(t) D_\alpha^a g(t) - M_2 D_\alpha^a f(t) D_\alpha^a t - M_1 D_\alpha^a g(t) D_\alpha^a t + M_1 M_2 (D_\alpha^a t)^2].$$

□

4. Conclusion

In this paper, we have studied some fractional integral inequalities and have extended results for time scale calculus. The theorems in this work improves previously results and this presents a new approach to use new definitions of fractional integrals for integral inequalities on time scales.

Competing Interests

The author(s) do not have any competing interests in the manuscript.

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