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FRACTIONAL INTEGRAL INEQUALITIES ON TIME SCALES

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Abstract. In this paper, we use the Delta Riemann-Liouville fractional integrals to establish some new integral inequalities for the Chebyshev functional in the case of two synchronous functions on time scales. Our results improve the inequalities for the discrete and continuous cases.

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1. Introduction

Integral inequalities and their extensions have received considerable attention in the theory of differential and difference equations. Recently, there has been much interest in the study of integral inequalities on time scales. Researchers have studied various aspects of inequalities on time scales [1, 2, 3]. New researches on dynamic inequalities using time scales was done by Agarwal in a monograph [4].
The fractional calculus is an extensions of derivatives and integrals to noninteger orders. This subject came to the attention of many researchers and fractional calculus on time scales has been studied [5]. In [6], some fractional integral inequalities have been studied in the real case. For more investigations, we refer [7, 8] to the readers.

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In this paper, we obtain some generalizations and refinements for some existing inequalities on time scales. We consider the functional [9]

$$T(f,g) := \frac{1}{t-a}\int_a^t f(x)\,g(x)\,\Delta x - \frac{1}{t-a}\left(\int_a^t f(x)\,\Delta x\right)\left(\frac{1}{t-a}\int_a^t g(x)\,\Delta x\right)$$

where f and g are two integrable functions which are synchronous on $[a,t]_{\mathbb{T}}$. (i.e. $(f(x)-f(y))(g(x)-g(y)) \geq 0$ *for any* $x,y \in [a,t]_{\mathbb{T}}$.) The intervals with $\mathbb{T}$ subscript are used to denote the intersection of the usual interval with $\mathbb{T}$; i.e., $[a,t]_{\mathbb{T}} := [a,t] \cap \mathbb{T}$ for convenience.
The main purpose of this paper is to establish some new fractional inequalities for synchronous functions using delta Reimann-Liouville fractional integrals on time scales. Our results unify and extend some continuous inequalities and their corresponding discrete analogues. In the following, we present some basic concepts about time scale calculus and refer the reader to resource [7] for more detailed information on subject.

2. Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. For $t \in \mathbb{T}$ we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T}\colon s > t\}$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T}\colon s < t\}.$$

If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$ we say that t is *left-scattered*. Also, if $\sigma(t) = t$, then t is called *right-dense, and if* $\rho(t) = t$, then t is called *left-dense*. The *graininess function* $\mu : \mathbb{T} \to [0,\infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

We introduce the set $\mathbb{T}^{\kappa}$ which is derived from the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has left-scattered maximum m, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}^{\kappa} = \mathbb{T}$.

Definition 2.1. Let f and g be two integrable functions defined on $[a,t]_{\mathbb{T}}$. If for any $x,y \in [a,t]_{\mathbb{T}}$

$$[f(x)-f(y)]\,[g(x)-g(y)] \geq 0,$$

then f and g are called synchronous functions on $[a,t]_{\mathbb{T}}$.

Definition 2.2. The rd continuous functions $h_{\alpha} : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ is called as generalized polynomials on time scales such that, for all $s,t \in \mathbb{T}$ and $\alpha \geq 0$;

$$\begin{aligned} h_0(t,s) &= 1 \\ h_{\alpha+1}(t,s) &= \int_s^t h_{\alpha}(\tau,s)\,\Delta\tau \end{aligned}$$

Definition 2.3. The function $f:\mathbb{T}\to\mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

Definition 2.4. The Delta-Riemann-Liouville fractional integral operator of order $\alpha\geq 1$ on time scales, for a function $f\in C_{rd}$ is defined as

$$\begin{aligned} D_a^\alpha f(t) &= \int_a^t h_{\alpha-1}(t,\sigma(\tau))\, f(\tau)\,\Delta\tau, \\ D_a^0 f &= f. \end{aligned}$$

3. Main Results

In this section we present our main results.

Theorem 3.1. *Let f and g be two synchronous functions on $[0,\infty)_{\mathbb{T}}$. Then for all $t>a$, $\alpha\geq 1$, $a\geq 0$ we have*

$$D_a^\alpha (fg)(t) \geq (h_\alpha(t,a))^{-1} D_a^\alpha f(t)\, D_a^\alpha g(t). \tag{1}$$

Proof. Since f *and* g *are synchronous functions* on $[0.\infty)_{\mathbb{T}}$, then for all $\tau,\phi\geq 0$,

$$(f(\tau)-f(\phi))(g(\tau)-g(\phi))\geq 0 \tag{2}$$

and

$$f(\tau)g(\tau)-f(\tau)g(\phi)-f(\phi)g(\tau)+f(\phi)g(\phi)\geq 0. \tag{3}$$

Therefore,

$$f(\tau)g(\tau)+f(\phi)g(\phi)\geq f(\tau)g(\phi)+f(\phi)g(\tau). \tag{4}$$

For $\tau\in(a,t)$, multiplying both sides of (4) by $h_{\alpha-1}(t,\sigma(\tau))$, we have

$$\begin{aligned} & h_{\alpha-1}(t,\sigma(\tau))f(\tau)g(\tau)+h_{\alpha-1}(t,\sigma(\tau))f(\phi)g(\phi) \\ \geq\ & h_{\alpha-1}(t,\sigma(\tau))f(\tau)g(\phi)+h_{\alpha-1}(t,\sigma(\tau))g(\phi)g(\tau). \end{aligned} \tag{5}$$

Integrating (5) over (a,t), we get

$$\begin{aligned} & \int_a^t h_{\alpha-1}(t,\sigma(\tau))f(\tau)g(\tau)\Delta\tau+\int_a^t h_{\alpha-1}(t,\sigma(\tau))f(\phi)g(\phi)\Delta\tau \\ \geq & \int_a^t h_{\alpha-1}(t,\sigma(\tau))f(\tau)g(\phi)\Delta\tau+\int_a^t h_{\alpha-1}(t,\sigma(\tau))f(\phi)g(\tau)\Delta\tau. \end{aligned}$$

Since $f(\phi), g(\phi)$ and $f(\phi)g(\phi)$ are independent from τ, we take them out of the integral and we obtain

$$\int_a^t h_{\alpha-1}(t,\sigma(\tau))f(\tau)g(\tau)\Delta\tau+f(\phi)g(\phi)\int_a^t h_{\alpha-1}(t,\sigma(\tau))\Delta\tau$$

$$\geq g(\phi)\int_a^t h_{\alpha-1}(t,\sigma(\tau))\,f(\tau)\,\Delta\tau + f(\phi)\int_a^t h_{\alpha-1}(t,\sigma(\tau))\,g(\tau)\,\Delta\tau.$$

Using Definition 2.2 and Definition 2.4, we find

$$D_a^\alpha(fg)(t) + f(\phi)\,g(\phi)\,(h_\alpha(t,a)) \geq g(\phi)\,D_a^\alpha f(t) + f(\phi)\,D_a^\alpha g(t). \tag{6}$$

For $\phi \in (a,t)$, multiplying both sides of (6) by $h_{\alpha-1}(t,\sigma(\phi))$, we obtain

$$\begin{aligned} &h_{\alpha-1}(t,\sigma(\phi))\,D_a^\alpha(fg)(t) + h_{\alpha-1}(t,\sigma(\phi))\,f(\phi)\,g(\phi)\,(h_\alpha(t,a)) \\ &\geq h_{\alpha-1}(t,\sigma(\phi))\,g(\phi)\,D_a^\alpha f(t) + h_{\alpha-1}(t,\sigma(\phi))\,f(\phi)\,D_a^\alpha g(t). \end{aligned} \tag{7}$$

Integrating (7) over (a,t), we get

$$\int_a^t h_{\alpha-1}(t,\sigma(\phi))\,D_a^\alpha(fg)(t)\,\Delta\phi + \int_a^t h_{\alpha-1}(t,\sigma(\phi))\,f(\phi)\,g(\phi)\,(h_\alpha(t,a))\,\Delta\phi$$

$$\geq \int_a^t h_{\alpha-1}(t,\sigma(\phi))\,g(\phi)\,D_a^\alpha f(t)\,\Delta\phi + \int_a^t h_{\alpha-1}(t,\sigma(\phi))\,f(\phi)\,D_a^\alpha g(t)\,\Delta\phi.$$

Since $D_a^\alpha(fg)(t)$, $D_a^\alpha f(t), D_a^\alpha g(t)$ and $h_\alpha(t,a)$ are independent from ϕ, we obtain

$$D_a^\alpha(fg)(t)\int_a^t h_{\alpha-1}(t,\sigma(\phi))\,\Delta\phi + (h_\alpha(t,a))\int_a^t h_{\alpha-1}(t,\sigma(\phi))\,f(\phi)\,g(\phi)\,\Delta\phi$$

$$\geq D_a^\alpha f(t)\int_a^t h_{\alpha-1}(t,\sigma(\phi))\,g(\phi)\,\Delta\phi + D_a^\alpha g(t)\int_a^t h_{\alpha-1}(t,\sigma(\phi))\,f(\phi)\,\Delta\phi.$$

Therefore from Definition 2.2 and Definition 2.4, we can write

$$\begin{aligned} &D_a^\alpha(fg)(t)\,(h_\alpha(t,a)) + (h_\alpha(t,a))\,D_a^\alpha(fg)(t) \\ &\geq D_a^\alpha f(t)\,D_a^\alpha g(t) + D_a^\alpha g(t)\,D_a^\alpha f(t). \end{aligned}$$

So we have

$$\begin{aligned} 2D_a^\alpha(fg)(t)\,(h_\alpha(t,a)) &\geq 2D_a^\alpha f(t)\,D_a^\alpha g(t), \\ D_a^\alpha(fg)(t)\,(h_\alpha(t,a)) &\geq D_a^\alpha f(t)\,D_a^\alpha g(t), \end{aligned}$$

and finally we get

$$D_a^\alpha(fg)(t) \geq \frac{1}{(h_\alpha(t,a))}D_a^\alpha f(t)\,D_a^\alpha g(t).$$

□

Theorem 3.2. *Let f and g be two synchronous functions on $[0,\infty)_{\mathbb{T}}$. Then for all $t > a$, $\alpha, \beta \geq 1$, $a \geq 0$ we have:*

$$h_\alpha(t,a)\,D_a^\beta(fg)(t) + h_\beta(t,a)\,D_a^\alpha(fg)(t) \geq D_a^\alpha f(t)\,D_a^\beta g(t) + D_a^\alpha g(t)\,D_a^\beta f(t).$$

Proof. Since f *and* g *are synchronous functions* on $[0.\infty)_{\mathbb{T}}$, then for all $\tau,\phi\geq 0$, we have

$$(f(\tau)-f(\phi))(g(\tau)-g(\phi))\geq 0.$$

Therefore,

$$f(\tau)g(\tau)+f(\phi)g(\phi)\geq f(\tau)g(\phi)+f(\phi)g(\tau). \tag{8}$$

For $\tau\in(a,t)$, multiplying both sides of (8) by $h_{\alpha-1}(t,\sigma(\tau))$, we obtain

$$\begin{aligned} &h_{\alpha-1}(t,\sigma(\tau))f(\tau)g(\tau)+h_{\alpha-1}(t,\sigma(\tau))f(\phi)g(\phi)\\ &\geq h_{\alpha-1}(t,\sigma(\tau))f(\tau)g(\phi)+h_{\alpha-1}(t,\sigma(\tau))f(\phi)g(\tau). \end{aligned} \tag{9}$$

Integrating (9) over (a,t), we get

$$\begin{aligned} &\int_a^t h_{\alpha-1}(t,\sigma(\tau))f(\tau)g(\tau)\Delta\tau+\int_a^t h_{\alpha-1}(t,\sigma(\tau))f(\phi)g(\phi)\Delta\tau\\ &\geq \int_a^t h_{\alpha-1}(t,\sigma(\tau))f(\tau)g(\phi)\Delta\tau+\int_a^t h_{\alpha-1}(t,\sigma(\tau))f(\phi)g(\tau)\Delta\tau. \end{aligned}$$

Consequently

$$D_a^{\alpha}(fg)(t)+(fg)(\phi)(h_{\alpha}(t,a))\geq g(\phi)D_a^{\alpha}f(t)+f(\phi)D_a^{\alpha}g(t). \tag{10}$$

For $\phi\in(a,t)$, multiplying both sides of (10) by $h_{\beta-1}(t,\sigma(\phi))$, we have

$$\begin{aligned} &h_{\beta-1}(t,\sigma(\phi))D_a^{\alpha}(fg)(t)+h_{\beta-1}(t,\sigma(\phi))(fg)(\phi)(h_{\alpha}(t,a))\\ &\geq h_{\beta-1}(t,\sigma(\phi))g(\phi)D_a^{\alpha}f(t)+h_{\beta-1}(t,\sigma(\phi))f(\phi)D_a^{\alpha}g(t). \end{aligned} \tag{11}$$

Integrating (11) over (a,t), we get

$$\begin{aligned} &\int_a^t h_{\beta-1}(t,\sigma(\phi))D_a^{\alpha}(fg)(t)\Delta\phi+\int_a^t h_{\beta-1}(t,\sigma(\phi))(fg)(\phi)(h_{\alpha}(t,a))\Delta\phi\\ &\geq \int_a^t h_{\beta-1}(t,\sigma(\phi))g(\phi)D_a^{\alpha}f(t)\Delta\phi+\int_a^t h_{\beta-1}(t,\sigma(\phi))f(\phi)D_a^{\alpha}g(t)\Delta\phi. \end{aligned}$$

then

$$\begin{aligned} &D_a^{\alpha}(fg)(t)\int_a^t h_{\beta-1}(t,\sigma(\phi))\Delta\phi+(h_{\alpha}(t,a))\int_a^t h_{\beta-1}(t,\sigma(\phi))f(\phi)g(\phi)\Delta\phi\\ &\geq D_a^{\alpha}f(t)\int_a^t h_{\beta-1}(t,\sigma(\phi))g(\phi)\Delta\phi+D_a^{\alpha}g(t)\int_a^t h_{\beta-1}(t,\sigma(\phi))f(\phi)\Delta\phi. \end{aligned}$$

Therefore, we write

$$D_a^{\alpha}(fg)(t)\left(h_{\beta}(t,a)\right)+(h_{\alpha}(t,a))D_a^{\beta}(fg)(t)\geq D_a^{\alpha}f(t)D_a^{\beta}g(t)+D_a^{\alpha}g(t)D_a^{\beta}f(t).$$

□

Theorem 3.3. *Let $(f_i)_{=1,\ldots,n}$ be n positive increasing functions on $[0,\infty)_{\mathbb{T}}$. Then for all $t>a$, $\alpha\geq 1$, $a\geq 0$ we have*

$$D_a^{\alpha}\left(\prod_{i=1}^{n} f_i\right)(t) \geq (h_{\alpha}(t,a))^{1-n}\prod_{i=1}^{n} D_a^{\alpha} f_i(t).$$

Proof. We use induction method to prove our result. Clearly, for $n=1$, all $t>a$, $\alpha\geq 1$, $a\geq 0$ we have

$$D_a^{\alpha}(f_1)(t) \geq D_a^{\alpha} f_1(t).$$

For $n=2$, applying Theorem 3.1, for all $t>a$, $\alpha\geq 1$, $a\geq 0$ we obtain

$$D_a^{\alpha}\left(\prod_{i=1}^{2} f_i\right)(t) \geq [h_{\alpha}(t,a)]^{-1}\prod_{i=1}^{2} D_a^{\alpha} f_i(t)$$

then,

$$D_a^{\alpha}(f_1 f_2)(t) \geq \frac{1}{h_{\alpha}(t,a)} D_a^{\alpha} f_1(t)\, D_a^{\alpha} f_2(t).$$

Now for $n-1$, we assume that (induction hypothesis) the following inequality

$$D_a^{\alpha}\left(\prod_{i=1}^{n-1} f_i\right)(t) \geq [h_{\alpha}(t,a)]^{2-n}\prod_{i=1}^{n-1} D_a^{\alpha} f_i(t) \tag{12}$$

holds. We have to prove that the inequality

$$D_a^{\alpha}\left(\prod_{i=1}^{n} f_i(t)\right) \geq [h_{\alpha}(t,a)]^{1-n}\prod_{i=1}^{n} D_a^{\alpha} f_i(t)$$

holds for n. Since $(f_i)_{i=1,2,\ldots,n}$ are positive increasing functions, then $\left(\prod_{i=1}^{n-1} f_i\right)(t)$ is an increasing function. Hence we can apply Theorem 3.1 to the functions $\prod_{i=1}^{n-1} f_i = g$, $f_n = f$. We obtain

$$\prod_{i=1}^{n} f_i = \prod_{i=1}^{n-1} f_i f_n = fg$$

$$D_a^{\alpha}(fg)(t) \geq [h_{\alpha}(t,a)]^{-1} D_a^{\alpha}(f)(t)\, D_a^{\alpha}(g)(t)$$

$$D_a^{\alpha}\left(\prod_{i=1}^{n} f_i\right)(t) = D_a^{\alpha}\left(\prod_{i=1}^{n-1} f_i f_n\right)(t) \geq [h_{\alpha}(t,a)]^{-1} D_a^{\alpha}\left(\prod_{i=1}^{n-1} f_i\right)(t)\, D_a^{\alpha}(f_n)(t).$$

Multiplying both sides of (12) by $[h_{\alpha}(t,a)]^{-1} D_a^{\alpha}(f_n)(t)$, we obtain

$$\begin{aligned}&[h_{\alpha}(t,a)]^{-1} D_a^{\alpha}(f_n)(t)\, D_a^{\alpha}\left(\prod_{i=1}^{n-1} f_i\right)(t)\\ &\geq [h_{\alpha}(t,a)]^{-1}[h_{\alpha}(t,a)]^{2-n}\prod_{i=1}^{n-1} D_a^{\alpha} f_i(t)\, D_a^{\alpha} f_n(t)\end{aligned}$$

and

$$[h_\alpha(t,a)]^{-1} D_a^\alpha (f_n)(t) D_a^\alpha \left(\prod_{i=1}^{n-1} f_i\right)(t) \geq [h_\alpha(t,a)]^{1-n} \prod_{i=1}^{n} D_a^\alpha f_i(t).$$

Therefore we obtain

$$D_a^\alpha \left(\prod_{i=1}^{n} f_i\right)(t) \geq [h_\alpha(t,a)]^{1-n} \prod_{i=1}^{n} D_a^\alpha f_i(t).$$

□

Theorem 3.4. *Let f and g be two functions defined on $[0,\infty)_{\mathbb{T}}$, such that f is increasing and g is differentiable and there exist a real number $m := inf_{t\geq 0} g^{'}(t)$. Then for all $t > a$, $\alpha \geq 1$, $a \geq 0$ we have*

$$\begin{aligned} D_a^\alpha (fg)(t) \quad \geq \quad & [h_\alpha(t,a)]^{-1} D_a^\alpha f(t) D_a^\alpha g(t) - m [h_\alpha(t,a)]^{-1} D_a^\alpha f(t) D_a^\alpha (t) \\ & + m D_a^\alpha (tf(t)). \end{aligned}$$

Proof. We consider the function $k(t) := g(t) - mt$. It is evident that k is differentiable and it is increasing on $[0,\infty)_{\mathbb{T}}$. Then applying the Theorem 3.1 we have

$$D_a^\alpha (kf)(t) \geq [h_\alpha(t,a)]^{-1} D_a^\alpha f(t) D_a^\alpha k(t),$$

then

$$D_a^\alpha ((g-mt) f(t)) \geq [h_\alpha(t,a)]^{-1} D_a^\alpha f(t) D_a^\alpha (g-mt)(t).$$

So we get

$$\begin{aligned} & D_a^\alpha (fg)(t) - m D_a^\alpha (tf(t)) \geq [h_\alpha(t,a)]^{-1} D_a^\alpha f(t) (D_a^\alpha g(t) - m D_a^\alpha (t)) \\ & \quad = [h_\alpha(t,a)]^{-1} D_a^\alpha f(t) D_a^\alpha g(t) - m [h_\alpha(t,a)]^{-1} D_a^\alpha f(t) D_a^\alpha (t), \end{aligned}$$

and we obtain our desired inequality. □

Corollary 3.5. *Let f and g be two functions defined on $[0,\infty)_{\mathbb{T}}$.*
(1). *Assume that f is decreasing, g is differentiable and there exist a real number $M := \sup_{t\geq 0} g^{'}(t)$. Then for all $t > a$, $\alpha \geq 1$, $a \geq 0$, we have*

$$\begin{aligned} D_a^\alpha (fg)(t) \quad \geq \quad & [h_\alpha(t,a)]^{-1} D_a^\alpha f(t) D_a^\alpha g(t) - M [h_\alpha(t,a)]^{-1} D_a^\alpha f(t) D_a^\alpha (t) \\ & + M D_a^\alpha (tf(t)). \end{aligned}$$

(2). *Assume that f and g are differentiable and there exist $m_1 := \inf_{t\geq 0} f^{'}(t)$, $m_2 := \inf_{t\geq 0} g^{'}(t)$. Then for all $t > a$, $\alpha \geq 1, a \geq 0$, we have*

$$\begin{aligned} & D_a^\alpha (fg)(t) - m_2 D_a^\alpha (tf(t)) - m_1 D_a^\alpha (tg(t)) + m_1 m_2 D_a^\alpha t^2 \\ & \geq [h_\alpha(t,a)]^{-1} \left[D_a^\alpha f(t) D_a^\alpha g(t) - m_2 D_a^\alpha f(t) D_a^\alpha (t) - m_1 D_a^\alpha (t) D_a^\alpha g(t) \right. \\ & \left. + m_1 m_2 (D_a^\alpha (t))^2 \right]. \end{aligned}$$

(3). *Assume that f and g are differentiable and there exist $M_1 := \sup_{t\geq 0} f^{'}(t)$, $M_2 := \sup_{t\geq 0} g^{'}(t)$. Then for all $t > a$, $\alpha \geq 1$, $a \geq 0$, we have*

$$\begin{aligned}&D_a^{\alpha}(fg)(t) - M_2 D_a^{\alpha}(tf(t)) - M_1 D_a^{\alpha}(tg(t)) + M_1 M_2 D_a^{\alpha} t^2\\ &\geq [h_{\alpha}(t,a)]^{-1}\left[D_a^{\alpha} f(t) D_a^{\alpha} g(t) - M_2 D_a^{\alpha} f(t) D_a^{\alpha}(t) - M_1 D_a^{\alpha}(t) D_a^{\alpha} g(t)\right.\\ &\left.+M_1 M_2 (D_a^{\alpha}(t))^2\right].\end{aligned}$$

Proof. **(1).** Let $G(t) := g(t) - Mt$ be a function such that it is decreasing and differentiable on $[0,\infty)_{\mathbb{T}}$. Then for all $t > a$, $\alpha \geq 1$, $a \geq 0$, we have

$$D_a^{\alpha}(Gf)(t) \geq [h_{\alpha}(t,a)]^{-1} D_a^{\alpha} G(t) D_a^{\alpha} f(t)$$

by Theorem 3.1. Hence, we get

$$D_a^{\alpha}((g - Mt) f(t)) \geq [h_{\alpha}(t,a)]^{-1} D_a^{\alpha} f(t) D_a^{\alpha}(g - Mt)(t).$$

Then we have

$$\begin{aligned}D_a^{\alpha}(fg)(t) - M D_a^{\alpha}(tf(t)) &\geq [h_{\alpha}(t,a)]^{-1} D_a^{\alpha} f(t) (D_a^{\alpha} g(t) - M D_a^{\alpha}(t))\\ &= [h_{\alpha}(t,a)]^{-1} D_a^{\alpha} f(t) D_a^{\alpha} g(t) - M [h_{\alpha}(t,a)]^{-1} D_a^{\alpha} f(t) D_a^{\alpha}(t),\end{aligned}$$

and so

$$\begin{aligned}D_a^{\alpha}(fg)(t) \quad \geq \quad & [h_{\alpha}(t,a)]^{-1} D_a^{\alpha} f(t) D_a^{\alpha} g(t) - M [h_{\alpha}(t,a)]^{-1} D_a^{\alpha} f(t) D_a^{\alpha}(t)\\ &+M D_a^{\alpha}(tf(t)).\end{aligned}$$

(2). Let $F(t) := f(t) - m_1 t$, $G(t) := g(t) - m_2 t$ be two functions such that they are increasing and differentiable on $[0,\infty)_{\mathbb{T}}$. Then for all $t > a$, $\alpha \geq 1$, $a \geq 0$ and by Theorem 3.1 we have

$$D_a^{\alpha}(FG)(t) \geq [h_{\alpha}(t,a)]^{-1} D_a^{\alpha} F(t) D_a^{\alpha} G(t),$$

and

$$\begin{aligned}&D_a^{\alpha}((f(t) - m_1 t)(g(t) - m_2 t))(t)\\ &\geq [h_{\alpha}(t,a)]^{-1} D_a^{\alpha}(f(t) - m_1 t)(t) D_a^{\alpha}(g(t) - m_2 t)(t).\end{aligned}$$

Then we get

$$\begin{aligned}&D_a^{\alpha}\left((fg)(t) - m_2 t f(t) - m_1 t g(t) + m_1 m_2 t^2\right)\\ &\geq [h_{\alpha}(t,a)]^{-1} [D_a^{\alpha} f(t) - m_1 D_a^{\alpha} t][D_a^{\alpha} g(t) - m_2 D_a^{\alpha} t]\\ &= [h_{\alpha}(t,a)]^{-1}\left[D_a^{\alpha} f(t) D_a^{\alpha} g(t) - m_2 D_a^{\alpha} f(t) D_a^{\alpha} t - m_1 D_a^{\alpha} g(t) D_a^{\alpha} t\right.\\ &\left.+m_1 m_2 (D_a^{\alpha} t)^2\right].\end{aligned}$$

Therefore we obtain

$$\begin{aligned}&D_a^{\alpha}\left((fg)(t) - m_2 t f(t) - m_1 t g(t) + m_1 m_2 t^2\right)\\ &\geq [h_{\alpha}(t,a)]^{-1}\left[D_a^{\alpha} f(t) D_a^{\alpha} g(t) - m_2 D_a^{\alpha} f(t) D_a^{\alpha} t - m_1 D_a^{\alpha} g(t) D_a^{\alpha} t\right.\\ &\left.+m_1 m_2 (D_a^{\alpha} t)^2\right].\end{aligned}$$

(3). Let $F(t) := f(t) - M_1 t$, $G(t) := g(t) - M_2 t$ be two functions such that they are decreasing and differentiable on $[0,\infty)_{\mathbb{T}}$. Then for all $t > a$, $\alpha \geq 1$, $a \geq 0$ and by Theorem 3.1, we have

$$D_a^\alpha (FG)(t) \geq [h_\alpha(t,a)]^{-1} D_a^\alpha F(t), D_a^\alpha G(t)$$

so

$$\begin{aligned} &D_a^\alpha ((f(t) - M_1 t)(g(t) - M_2 t))(t) \\ &\geq [h_\alpha(t,a)]^{-1} D_a^\alpha (f(t) - M_1 t)(t) D_a^\alpha (g(t) - M_2 t)(t). \end{aligned}$$

Hence we obtain

$$\begin{aligned} &D_a^\alpha \left((fg)(t) - M_2 t f(t) - M_1 t g(t) + M_1 M_2 t^2\right) \\ &\geq [h_\alpha(t,a)]^{-1} [D_a^\alpha f(t) - M_1 D_a^\alpha t][D_a^\alpha g(t) - M_2 D_a^\alpha t] \\ &= [h_\alpha(t,a)]^{-1} \big[D_a^\alpha f(t) D_a^\alpha g(t) - M_2 D_a^\alpha f(t) D_a^\alpha t - M_1 D_a^\alpha g(t) D_a^\alpha t \\ &+ M_1 M_2 (D_a^\alpha t)^2 \big]. \end{aligned}$$

□

4. Conclusion

In this paper, we have studied some fractional integral inequalities and have extended results for time scale calculus. The theorems in this work improves previously results and this presents a new approach to use new definitions of fractional integrals for integral inequalities on time scales

Competing Interests

The author(s) do not have any competing interests in the manuscript.

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