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Nonlocal initial value problems for Katugampola-Caputo type fractional differential equations on time scales

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Abstract: In this paper, we study Katugampola fractional differential equations (FDEs) with nonlocal conditions on time scales. By means of standard fixed point theorems, some new sufficient conditions for the existence of solutions are established.

Keywords: Nonlocal initial value problem, existence, fixed point, time scales.

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1. Introduction

The theory of FDEs has attracted attention of many researchers because of its wide applications in biology, medicine and in other applied fields, see for example [1–4] and references therein. Throughout this paper $(X, \|.\|)$ will be a Banach space, and $I = [0, T], T > 0$, a compact interval in $\mathbb{R}$. Let $C = C([0, T], X)$ be the Banach space of all continuous functions $[0, T] \to X$ endowed with the topology of uniform convergence (the norm in this space will be denoted by $\|.\|_c$).

In this work we consider the following Cauchy problem for the FDEs with nonlocal conditions on time scales

$$^{\rho}\Delta^{q}_{t_0} x(t) = f(t, x(t)), \quad t \in I, \tag{1}$$

$$x(0) + g(x) = x_0, \tag{2}$$

where $0 < q < 1$.

The system (1),(2) is equivalent to

$$x(t) = x_0 - g(x) + \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t (t^{\rho} - s^{\rho})^{q-1} s^{\rho-1} f(s, x(s))\Delta s$$

Recent studies of FDEs on time scales are done by Ahmadkhanlu in his papers [5]. The reader may also consult [6–8].

As indicated in Bashir's pioneering paper [2], the nonlocal condition $x(0) + g(0) = x_0$ can be applied in physics with better effect than the classical Cauchy problem with initial condition $x(0) = x_0$. For instance the author used

$$g(x) = \sum_{i=1}^{p} c_i x(t_i),$$

where $c_i = 1, 2, ..., p$ are given constants and $0 < t_1 < t_2 < ...t_p \leq T$. To describe the diffusion phenomenon of a small amount in a transparent tube, the Cauchy problem allows the additional measurement at $t_i, i = 1, 2, ...p$. We adopt some ideas from [9].

We investigate in our paper the Cauchy problem (1),(2) with the following assumptions:

(H1) $f : \mathbb{R} \times X \to X$ is jointly continuous.
(H2) $\|f(t,x) - f(t,y)\| \le L\|x-y\|, \forall t \in \mathbb{R}, x, y \in X$.
(H3) $g : C \to X$ is continuous and $\|g(x) - g(y)\| \le b\|x-y\|, \forall x, y \in C$.

2. Existence results

We are now ready to present our results.

Theorem 1. *Under assumptions (H1) and (H2), if $b < \frac{1}{2}$ and $L \le \frac{\Gamma(q+1)}{2T^q}$, then the equations (1) and (2) has a unique solution .*

Proof. Define $F : C \to C$ by

$$(Fx)(t) = x_0 - g(x) + \frac{\rho^{(1-q)}}{\Gamma(q)} \int_0^t (t^\rho - s^\rho)^{(q-1)} f(s, x(s)) \Delta s$$

choose $r \ge 2(\|x_0\| + G + \frac{MT^q}{\Gamma(q+1)})$,and let $sup_{t\in I}\|f(t,0)\| = M$. Then it is easy to see that $FB_r \subset B_r$ where $B_r = x \in C : \|x\| \le r$.

So let $x \in B_r$ and set $G = sup_{x \in C}\|g(x)\|$. Then we get

$$\begin{aligned}
\|Fx(t)\| &\le \|x_0\| + G + \frac{\rho^{(1-q)}}{\Gamma(q)} \int_0^t (t^\rho - s^\rho)^{q-1} s^{\rho-1} \|f(s, x(s))\| \Delta s \\
&\le \|x_0\| + G + \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t (t^\rho - s^\rho)^{q-1} s^{\rho-1} (\|f(s, x(s)) - f(s, o)\| + \|f(s, 0)\|) \Delta s \\
&\le \|x_0\| + G + (Lr + M) \frac{1}{\Gamma(q)} \int_0^t (t^\rho - s^\rho)^{q-1} \Delta s \\
&\le \|x_0\| + G + (Lr + M) \frac{T^q}{\Gamma(q+1)} \le r
\end{aligned}$$

by the choice of L and r. Now take $x, y \in C$, then we get

$$\begin{aligned}
\|(Fx)(t) - (Fx)(t)\| &\le \|g(x) - g(y)\| + \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t (t^\rho - s^\rho)^{q-1} s^{\rho-1} \|f(s, x(s)) - f(s, y(s))\| \Delta s \\
&\le \Omega_{b,L,T,q,\rho} \|x - y\|,
\end{aligned}$$

where $\Omega_{b,L,T,q,\rho} = (b + \frac{LT^{\rho q}}{\rho^q \Gamma(q+1)})$ depends only on the parameters of the problem and since $\Omega_{b,L,T,q,\rho} \le 1$, the result follows in view of the contraction mapping principle. $\square$

Theorem 2. *Let M be a nonempty convex subset of the Banach space X. Let A, B be two operators such that*

1. *$Ax + By \in M$ whenever $x, y \in M$,*
2. *A is compact and continuous,*
3. *B is a contraction mapping,*

then there exists $Z \in M$ such that $Z = Ax + Bz$.

Theorem 3. *Assume the conditions (H1)and (H3) holds, $b < 1$ and*

(H4) $\|f(t,x)\| \le \mu(t), \quad \forall (t,x) \in I \times X$, where $\mu \in L^1(I, \mathbb{R}^+)$,

then the equations (1) and (2) has at least one solution on I.

Proof. Choose $r \ge \|x_0\| + G + \frac{T^{\rho q}\|\mu\|_{L1}}{\rho^q \Gamma(q+1)}$ and consider $B_r : x \in C : \|x\| \le r$. Now define on B_r the operators A and B as

$$\begin{aligned}
(Ax)(t) &= \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t (t^\rho - s^\rho)^{q-1} s^{\rho-1} f(s, x(s)) \Delta s, and \\
(Bx)(t) &= x_0 - g(x)
\end{aligned}$$

If $x, y \in B_r$ then $Ax + By \in B_r$. Indeed

$$\|Ax + By\| \leq \|x_0\| + G + \frac{T^{\rho q}\, \|\mu\|_{L^1}}{\rho^q \Gamma(q+1)} \leq r.$$

By $(H3)$, it is also clear that B is a contraction mapping for $b < 1$. Since x is continuous, then $(Ax)(t)$ is continuous in view of (H1). Note that A is uniformly bounded on B_r. Hence we have

$$\|(Ax)(t)\| \leq \frac{T^{\rho q}\, \|\mu\|_{L^1}}{\rho^q \Gamma(q+1)}.$$

Now, we prove that $(Ax)(t)$ is equicontinuous. Let $t_1, t_2 \in I$ and $x \in B_r$. Using the fact that f is bounded on the compact set $I \times B_r$ (thus $sup_{(t,x)\in I\times B_r} \|f(t,x)\| = c_0 < \infty$), we get

$$\begin{aligned}
&\|Ax(t_1) - Ax(t_2)\| \\
&= \left\| \frac{\rho^{1-q}}{\Gamma(q)} \int_0^{t_1} (t_1^\rho - s^\rho)^{q-1} s^{\rho-1} f(s, x(s))\Delta s - \frac{\rho^{1-q}}{\Gamma(q)} \int_0^{t_2} (t_2^\rho - s^\rho)^{q-1} s^{\rho-1} f(s, x(s))\Delta s \right\| \\
&\leq \frac{\rho^{1-q}}{\Gamma(q)} \left\| \int_0^{t_1} [(t_1^\rho - s^\rho)^{q-1} - (t_2^\rho - s^\rho)^{q-1}] s^{\rho-1} f(s, x(s))\Delta s + \int_{t_1}^{t_2} (t_2^\rho - s^\rho)^{q-1} s^{\rho-1} f(s, x(s))\Delta s \right\| \\
&\leq \frac{C_0 \rho^{1-q}}{\Gamma(q)} \left(\int_0^{t_1} \left| (t_1^\rho - s^\rho)^{q-1} - (t_2^\rho - s^\rho)^{q-1} \right| s^{\rho-1}\Delta s + \int_{t_1}^{t_2} (t_2^\rho - s^\rho)^{q-1} s^{\rho-1}\Delta s \right).
\end{aligned}$$

For $q < 1$, $(t_1^\rho - s^\rho)^{q-1} \geq (t_2^\rho - s^\rho)^{q-1}$, we have

$$\begin{aligned}
\int_{t_0}^{t_1} \left| (t_1^\rho - s^\rho)^{q-1} - (t_2^\rho - s^\rho)^{q-1} \right| s^{\rho-1}\Delta s &= \int_{t_0}^{t_1} [(t_1^\rho - s^\rho)^{q-1} - (t_2^\rho - s^\rho)^{q-1}] s^{q-1}\Delta s \\
&= \frac{1}{\rho q}(t_1^\rho q - t_2^\rho q) + \frac{1}{\rho q}(t_2^\rho - t_1^\rho)^q \\
&= \frac{1}{\rho q}(t_2^\rho - t_1^\rho)^q.
\end{aligned}$$

If $q > 1$, $(t_1^\rho - s^\rho)^{q-1} \leq (t_2^\rho - s^\rho)^{q-1}$, we have

$$\begin{aligned}
\int_{t_0}^{t_1} \left| (t_1^\rho - s^\rho)^{q-1} - (t_2^\rho - s^\rho)^{q-1} \right| s^{\rho-1}\Delta s &= \int_{t_0}^{t_1} [(t_1^\rho - s^\rho)^{q-1} - (t_1^\rho - s^\rho)^{q-1}] s^{\rho-1}\Delta s \\
&= \frac{1}{\rho q}(t_2^\rho q - t_1^\rho q) - \frac{1}{\rho q}(t_2^\rho - t_1^\rho)^q \\
&= \frac{1}{\rho q}(t_2^\rho q - t_1^\rho q)
\end{aligned}$$

$$\begin{aligned}
\|(Ax)(t_1) - (Ax)(t_2)\| &\leq \frac{2C_0}{\rho^q \Gamma(q+1)}(t_2^\rho - t_1^\rho)^q, \quad q \leq 1 \\
&\leq \frac{C_0}{\rho^q \Gamma(q+1)} \left[(t_2^\rho - t_1^\rho)^q + (t_2^{\rho q} - t_1^{\rho q}) \right], \quad q > 1,
\end{aligned}$$

which does not depend on x, so $A(B_r)$ is relatively compact. By the Arzela-Ascoli Theorem, A is compact. We now conclude the result of the theorem based on the Krasnoselkii's theorem above. □

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