Translation and homothetical TH-surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$ and Lorentzian-Minkowski space $\mathbb{E}_1^3$

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Received: 29 May 2019; Accepted: 9 July 2019; Published: 29 July 2019.

Abstract: In the 3-dimensional Euclidean space $\mathbb{E}^3$ and Lorentzian-Minkowski space $\mathbb{E}_1^3$, a translation and homothetical TH-surface is parameterized $z(u,v) = A(f(u) + g(v)) + Bf(u)g(v)$, where $f$ and $g$ are smooth functions and $A, B$ are non-zero real numbers. In this paper, we define TH-surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$ and Lorentzian-Minkowski space $\mathbb{E}_1^3$ and completely classify minimal or flat TH-surfaces.

Keywords: Translation surface, homothetical surface, minimal surface.

MSC: 51B20, 53A10, 53C45.

1. Introduction

The theory of minimal surfaces has found many applications in differential geometry and also in physics. In [1] and [2], H. Liu gave some classification results for translation surfaces. A minimal translation hypersurface in a Euclidean space is either locally a hyperplane or an open part of a cylinder on Scherk’s surfaces, as proved in Dillen et al. [3]. In [1] was generalized to translation surfaces with constant mean curvature and constant Gaussian curvature in $\mathbb{E}^3$. Sağlam and Sabuncuoğlu proved that every homothetical lightlike hypersurface in a semi-Euclidean $\mathbb{E}_q^{n+2}$ space is minimal [4]. Jiu and Sun studied $n$- dimensional minimal homothetical hypersurfaces and gave their classification [5]. R. López [6] studied translation surfaces in the 3-dimensional hyperbolic space and classified minimal translation surfaces. Meng and Liu [7] considered factorable surfaces along two lightlike directions and spacelike-lightlike directions in Minkowski 3-space $\mathbb{E}_1^3$ and they also gave some classification theorems. In [8], Yu and Liu studied the factorable minimal surfaces in $\mathbb{E}_1^3$ and $\mathbb{E}^3$, and gave some classification theorems. Güler et al. [9] defined by translation and homothetical TH-surfaces in the three dimensional Euclidean space.

2. Preliminaries

Let $\mathbb{E}_1^3$ be a 3-dimensional Minkowski space with the scalar product of index 1 given by

$$g_L = ds^2 = -dx^2 + dy^2 + dz^2,$$

where $(x, y, z)$ is a rectangular coordinate system of $\mathbb{E}_1^3$.

A vector $V$ of $\mathbb{E}_1^3$ is said to be timelike if $g_L(V, V) < 0$, spacelike if $g_L(V, V) > 0$ or $V = 0$ and lightlike or null if $g_L(V, V) = 0$ and $V \neq 0$. A surface in $\mathbb{E}_1^3$ is spacelike, timelike or lightlike if the tangent plane at any point is spacelike, timelike or lightlike, respectively.

The Lorentz scalar product of the vectors $V$ and $W$ is defined by $g_L(V, W) = -v_1w_1 + v_2w_2 + v_3w_3$, where $V = (v_1, v_2, v_3), W = (w_1, w_2, w_3) \in \mathbb{E}_1^3$.

For any $V, W \in \mathbb{E}_1^3$, the pseudo-vector product of $V$ and $W$ is defined as follows:

$$V \wedge L W = (-v_2w_3 + v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1).$$
We denote a surface $M^2$ in $\mathbb{E}^3_1$ by

$$r(u, v) = (r_1(u, v), r_2(u, v), r_3(u, v)).$$

**Definition 1** ([10]). A translation surface in Minkowski 3-space is a surface that is parameterized by either

$$r(u, v) = (u, v, f(u) + g(v)) \quad \text{if } L \text{ is timelike,}$$

$$r(u, v) = (f(u) + g(v), u, v) \quad \text{if } L \text{ is spacelike,}$$

$$r(u, v) = (u + v, g(v), f(u) + v) \quad \text{if } L \text{ is lightlike,}$$

with $L$ the intersection of the two planes that contain the curves that generate the surface.

**Theorem 2** ([11]).

i) The only translation surfaces with constant Gauss curvature $K = 0$ are cylindrical surfaces.

ii) There are no translation surfaces with constant Gauss curvature $K \neq 0$ if one of the generating curves is planar.

**Definition 3.** A homothetical (factorable) surface $M^2$ in the 3-dimensional Lorentzian space $\mathbb{E}^3_1$ is a surface that is a graph of a function

$$z(u, v) = f(u)g(v),$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ and $g : J \subset \mathbb{R} \to \mathbb{R}$ are two smooth functions.

**Theorem 4** ([11]). Planes and helicoids are the only minimal homothetical surfaces in Euclidean space.

Accordingly, we define an extended surface in $\mathbb{E}^3_1$ using definitions as above and called it TH-type surface as follows [9]:

**Definition 5.** A surface $M^2$ in the 3-dimensional Lorentzian space $\mathbb{E}^3_1$ is a TH-surface if it can be parameterized either by a patch

$$r(u, v) = (u, v, A(f(u) + g(v)) + Bf(u)g(v))$$

or

$$r(u, v) = (A(f(u) + g(v)) + Bf(u)g(v), u, v),$$

where $A$ and $B$ are non-zero real numbers.

**Remark 1.**

i) If $A \neq 0$ and $B = 0$ in (1), then surface is a translation surface.

ii) If $A = 0$ and $B \neq 0$ in (1), then surface is a homothetical (factorable) surface.

Let $N$ denotes the unit normal vector field of $M^2$ and put $g_L(N, N) = \varepsilon = \pm 1$, so that $\varepsilon = -1$ or $\varepsilon = 1$ according to $M^2$ is endowed with a Lorentzian or Riemannian metric, respectively.

The mean curvature and the Gauss curvature are

$$H = \frac{EN + GL - 2FM}{2|EG - F^2|}, \quad K = g_L(N, N)\frac{LN - M^2}{EG - F^2},$$

where $E, G, F$ are the coefficients of the first fundamental form, $L, M, N$ are the coefficients of the second fundamental form.

In this paper, we define TH-surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$ and Lorentzian-Minkowski space $\mathbb{E}^3_1$, and completely classify minimal or flat TH-surfaces.

3. **Minimal TH-surfaces in $\mathbb{E}^3_1$**

A surface $M^2$ in the 3-dimensional Lorentzian space $\mathbb{E}^3_1$ is called minimal when locally each point on the surface has a neighborhood which is the surface of least area with respect to its boundary [12]. In 1775, J. B. Meusnier showed that the condition of minimality of a surface in $\mathbb{E}^3_1$ is equivalent with the vanishing of its mean curvature function, $H = 0$.

Let $z = f(x, y)$ define a graph $M^2$ in the Euclidean 3-space $\mathbb{E}^3$. If $M^2$ is minimal, the function $f$ satisfies

$$(1 + f_y^2)f_{xx} - 2f_yf_xf_y + (1 + f_x^2)f_{yy} = 0, \quad (3)$$
which was obtained by J. L. Lagrange in 1760.

Let \( M^2 \) be a TH-surface in \( \mathbb{E}_1^3 \) parameterized by a patch

\[
r(u, v) = (u, v, A(f(u) + g(v)) + Bf(u)g(v)),
\]

where \( A \) and \( B \) are non-zero real numbers.

So

\[
r_u = (1, 0, f'\alpha), \quad r_v = (0, 1, g'\gamma),
\]

where \( \alpha = A + Bg \) and \( \gamma = A + Bf \).

After eliminating \( f' \) and \( g' \) we find

\[
E = \frac{\gamma'^2 \alpha^2 - B^2}{B^2}, \quad F = \frac{\alpha \gamma' \gamma' + B}{B^2}, \quad G = \frac{\gamma^2 \alpha^2 + B^2}{B^2}.
\]

The unit normal vector is given by

\[
N = \frac{1}{WB} (\alpha \gamma', -\gamma \alpha', B),
\]

where \( W^2 = B^{-2} g_L(N, N)(\gamma'^2 \alpha^2 - \alpha'^2 \gamma^2 - B^2) \) and

\[
g_L(N, N) = \varepsilon, \quad \varepsilon = \begin{cases} 1 & M^2 \text{ is spacelike } (\gamma'^2 \alpha^2 - \alpha'^2 \gamma^2 - B^2 > 0), \\ -1 & M^2 \text{ is timelike } (\gamma'^2 \alpha^2 - \alpha'^2 \gamma^2 - B^2 < 0). \end{cases}
\]

The constant \( \varepsilon \) is called the sign of \( M^2 \).

The coefficients of the second fundamental form are given by

\[
L = \frac{\alpha \gamma''}{BW}, \quad M = \frac{\alpha' \gamma'}{BW}, \quad N = \frac{\gamma \alpha''}{BW}.
\]

The expression of \( H \) is

\[
H = \frac{B^2(\alpha f''(1 + g'^2 \gamma^2) - 2B \alpha \gamma f'^2 \gamma + \gamma g''(f'^2 \alpha^2 - 1))}{2W^3} = \frac{\alpha \gamma''(B^2 + \alpha'^2 \gamma^2) - 2a \gamma a'^2 \gamma^2 + \gamma a''(\gamma^2 \alpha^2 - B^2)}{2B^3}.
\]

Then \( M^2 \) is a minimal surface if and only if

\[
\alpha \gamma''(B^2 + \alpha'^2 \gamma^2) - 2a \gamma a'^2 \gamma^2 + \gamma a''(\gamma^2 \alpha^2 - B^2) = 0.
\]

We distinguish the following cases.

**Case 1.** \( \gamma' = 0 \). In this case (5) gives \( \gamma a'' = 0 \).

i) If \( \gamma = 0 \), then \( f'(u) = -\frac{A}{B} \), \( M^2 \) is the horizontal plane of equation \( z = -\frac{A^2}{B} \).

ii) If \( \alpha'' = 0 \), then \( a(v) = a_1v + b_1, a_1, b_1 \in \mathbb{R} \), and \( \gamma(u) = c_1, c_1 \in \mathbb{R} \). \( M^2 \) is the plane of equation \( z = c_2v + c_3, c_2, c_3 \in \mathbb{R} \).

**Case 2.** \( \alpha' = 0 \). In this case (5) gives \( \gamma'' \alpha = 0 \).

i) If \( \alpha = 0 \), then \( g(v) = -\frac{A}{B} \), \( M^2 \) is the horizontal plane of equation \( z = -\frac{A^2}{B} \).

ii) If \( \gamma'' = 0 \), then \( \gamma(u) = a_2u + b_2, a_2, b_2 \in \mathbb{R} \), and \( a(v) = c_4, c_4 \in \mathbb{R} \). \( M^2 \) is the plane of equation \( z = c_5u + c_6, c_5, c_6 \in \mathbb{R} \).

**Case 3.** \( \gamma'' = 0 \) and \( \gamma' \neq 0 \). Then \( \gamma(u) = \lambda u + \delta, (\lambda, \delta) \in \mathbb{R} \setminus \{0\} \times \mathbb{R} \) and \( a \) is a solution of the following ODE

\[
-2\lambda^2 \alpha \alpha^2 + a''(\lambda^2 \alpha^2 - B^2) = 0.
\]

Then the general solution of (6) is given by

\[
a(v) = \frac{B}{\lambda} \coth(\lambda_1 v + \lambda_2), \quad \lambda_1, \lambda_2 \in \mathbb{R}.
\]
Then the general solution of (7) is given by
\[ u = \frac{B}{A} \tan(\lambda_1 u + \lambda_2), \quad \lambda_1, \lambda_2 \in \mathbb{R}. \]

**Case 4.** Let \( a'' = 0 \) and \( a' \neq 0 \). Then \( \alpha(v) = \lambda v + \delta, (\lambda, \delta) \in \mathbb{R} \setminus \{0\} \times \mathbb{R} \) and \( \gamma \) is a solution of the following ODE
\[ -2\lambda^2 \gamma \gamma'^2 + \gamma''(\lambda^2 \gamma^2 + B^2) = 0. \] (7)

Then the general solution of (7) is given by
\[ \gamma(u) = \frac{B}{A} \tan(\lambda_1 u + \lambda_2), \quad \lambda_1, \lambda_2 \in \mathbb{R}. \]

Hence
\[ f(u) = \frac{1}{A} \tan(\lambda_1 u + \lambda_2) - \frac{A}{B}, \quad \lambda_1, \lambda_2 \in \mathbb{R}. \]

**Case 5.** Let \( \gamma'' \neq 0 \). By symmetry in the discussion of the case, we also suppose \( a'' \neq 0 \). If we divide (5) by \( \alpha \gamma a'^2 \), we obtain
\[ \frac{B^2 \gamma''}{\alpha a'^2 \gamma^2} + \frac{\gamma' \gamma''}{\alpha a'^2} - \frac{B^2 a''}{\alpha a'^2} + \frac{a a''}{\alpha^2} = 2 = 0. \]

Thus, after a derivation with respect to \( u \), followed by a derivation with respect to \( v \), we obtain
\[ \left( \frac{\gamma''}{\gamma' \gamma^2} \right)_{,u} \left( \frac{1}{a'^2} \right)_{,v} - \left( \frac{a''}{\alpha a'^2} \right)_{,v} \left( \frac{1}{\gamma^2} \right)_{,u} = 0. \]

Hence we deduce the existence of a real number \( k \in \mathbb{R} \) such that
\[ \left\{ \begin{array}{l}
\left( \frac{\gamma''}{\gamma' \gamma^2} \right)_{,u} = k \left( \frac{1}{\gamma} \right)_{,u} \\
\left( \frac{a''}{\alpha a'^2} \right)_{,v} = k \left( \frac{1}{\gamma^2} \right)_{,v}.
\end{array} \right. \] (8)

The first equation of (8) can integrate obtaining
\[ \gamma'' = \gamma(k + c \gamma^2). \] (9)

From the second equation in (8), we obtain
\[ a'' = a(k + ba'^2). \] (10)

Substituting the above in (5), we get
\[ \alpha \gamma((k + c \gamma^2)(B^2 + a'^2 \gamma^2) - 2a'^2 \gamma^2 + (k + ba'^2)(\gamma'^2 a^2 - B^2)) = 0. \]

If we simplify by \( \alpha \gamma \) and then divide by \( a'^2 \gamma^2 \), we get
\[ \frac{b B^2 - k \gamma^2}{\gamma^2} - c \gamma^2 + 2 = \frac{c B^2 + k a^2}{\alpha^2} + ba^2. \]

Hence, we deduce the existence of a real number \( \lambda \in \mathbb{R} \) such that
\[ \left\{ \begin{array}{l}
\gamma^2 = \frac{b B^2 - k \gamma^2}{\lambda - 2 + \gamma^2} \\
a'^2 = \frac{c B^2 + k a^2}{\lambda - ba^2 - 1}.
\end{array} \right. \] (11)

Differentiating with respect to \( u \) and \( v \), respectively, we have
\[ \left\{ \begin{array}{l}
\gamma'' = -\frac{\gamma((\lambda - 2)k + bc B^2)}{(\lambda - 2 + \gamma^2)^2} \\
a'' = a((k + bc B^2) \frac{1}{\lambda - ba^2}).
\end{array} \right. \] (12)
Let us compare these expressions of $\alpha''$ and $\gamma''$ with those ones that appeared in (9) and (10) and replace the values of $\gamma^2$ and $\alpha^2$ obtained in (11).

We get
\[
\begin{align*}
\lambda k + bcB^2 & = 0 \\
(\lambda - 2)k + bcB^2 & = 0,
\end{align*}
\]
We discuss all possibilities.

i) If
\[
\begin{align*}
\lambda k & = 0 \\
(\lambda - 2)k & = 0,
\end{align*}
\]
then $k = 0$ and $bc = 0$. Then (12) gives $\gamma'' = 0$ and $\alpha'' = 0$, a contradiction.

ii) If
\[
\begin{align*}
\lambda k & = 0 \\
c & = 0 \\
\lambda & = 1,
\end{align*}
\]
we obtain $k = 0$. Then (12) gives $\gamma'' = 0$ and $\alpha'' = 0$, a contradiction.

iii) If
\[
\begin{align*}
(\lambda - 2)k & = 0 \\
b & = 0 \\
\lambda & = 1,
\end{align*}
\]
we obtain $k = 0$. Then (12) gives $\gamma'' = 0$ and $\alpha'' = 0$, a contradiction.

iv) If
\[
\begin{align*}
\lambda - 1 - ba^2 & = 0 \\
\lambda - 1 + c\gamma^2 & = 0
\end{align*}
\]
we deduce that $\alpha, \gamma$ are both constant functions, and so, $\gamma'' = 0$ and $\alpha'' = 0$, a contradiction.

v) If $b = 0, c = 0$ and $\lambda = 1$, Equation (11) writes as
\[
\begin{align*}
\gamma^2 & = k\gamma^2 \\
\alpha^2 & = ka^2.
\end{align*}
\]

The equations (13) have the following solutions
\[
\begin{align*}
a(v) & = k_1e^{\sqrt{\gamma}v}, \quad \gamma(u) = k_2e^{\sqrt{\alpha}u}, \quad k > 0,
\end{align*}
\]
where $k_1, k_2 \in \mathbb{R}$ are integration constants.

Hence
\[
\begin{align*}
g(v) & = \lambda_1e^{\sqrt{\gamma}v} - \frac{A}{B}, \quad f(u) = \lambda_2e^{\sqrt{\alpha}u} - \frac{A}{B}, \quad k > 0.
\end{align*}
\]
Therefore, we have the following:

Theorem 6. Let $M^2$ be a TH-surface in $\mathbb{E}_3^3$. If $M^2$ is minimal surface, then $M^2$ can be parameterized as
\[
r(u, v) = (u, v, A(f(u) + g(v)) + Bf(u)g(v)),
\]
where
1) either $f(u) = -\frac{A}{B}$ and $g(v)$ is a smooth function in $v$.
2) $g(v) = -\frac{A}{B}$ and $f(u)$ is a smooth function in $u$.
3) $f(u) = \lambda_1u + \lambda_2$ and $g(v) = \lambda_3\coth(\lambda_4v + \lambda_5) - \lambda_6$, $\lambda_i \in \mathbb{R}$.
4) $f(u) = \frac{1}{\lambda_1u + \lambda_2} - \frac{A}{B}$, $\lambda_1, \lambda_2 \in \mathbb{R}$ and $g(v) = \delta_5v + \delta_6$, $\delta_i \in \mathbb{R}$.
5) $f(u) = \lambda_2e^{\sqrt{\alpha}u} - \frac{A}{B}$ and $g(v) = \lambda_1e^{\sqrt{\gamma}v} - \frac{A}{B}$.

Let $M^2$ be a TH-surface in $\mathbb{E}_3^3$ parameterized by a patch
\[
r(u, v) = (A(f(u) + g(v)) + Bf(u)g(v), u, v),
\]
where \( A \) and \( B \) are non-zero real numbers.

So
\[
r_u = (f'\alpha, 1, 0), \quad r_v = (g'\gamma, 0, 1),
\]
where \( \alpha = A + Bg \) and \( \gamma = A + Bf \).

We have
\[
E = -\frac{\gamma'^2\alpha^2 + B^2}{B^2}, \quad F = -\frac{\alpha\gamma'\gamma'}{B^2}, \quad G = -\frac{\gamma'^2\alpha^2 + B^2}{B^2}.
\]

The coefficients of the second fundamental form on \( M^2 \) are obtained by
\[
L = \frac{\alpha\gamma''}{BW}, \quad M = \frac{\alpha'\gamma'}{BW}, \quad N = \frac{\gamma''}{BW}.
\]

Then \( M^2 \) is a minimal surface if and only if
\[
\alpha\gamma''(B^2 - \alpha'^2\gamma^2) + 2\alpha\gamma\alpha'^2\gamma' - \gamma''(\gamma'^2\alpha^2 - B^2) = 0,
\]
where \( \alpha = A + Bg \) and \( \gamma = A + Bf \).

Using the same algebraic techniques as in the case of surfaces (1), we get:

**Theorem 7.** Let \( M^2 \) be a TH-surface in \( \mathbb{E}^3_1 \). If \( M^2 \) is minimal surface, then \( M^2 \) can be parameterized as
\[
r(u, v) = (f(u) + g(v)) + Bf(u)g(v), \quad u, v,
\]
where

1. either \( f(u) = \frac{1}{3}u + a \) and \( g(v) = -\frac{1}{3} u \coth(\lambda_3 v + \lambda_4) - \frac{4}{3} \).
2. \( f(u) = -\frac{4}{3} \) and \( g(v) \) is a smooth function in \( v \).
3. \( g(v) = -\frac{4}{3} \) and \( f(u) \) is a smooth function in \( u \).
4. or \( g(v) = \frac{1}{3}v + \mu \) and \( f(u) = -\frac{1}{3} u \coth(\lambda_1 u + \lambda_2) - \frac{4}{3} \).

4. **TH-surfaces with zero Gaussian curvature in \( \mathbb{E}^3_1 \)**

A non-degenerate surface in \( \mathbb{E}^3_1 \) is called flat, if its Gaussian curvature vanishes identically.

A surface in \( \mathbb{E}^3 \) parameterized by (1), after eliminating \( f, g \) and their derivatives, has Gaussian curvature
\[
K = \frac{\alpha'\gamma'' - \gamma'^2\alpha^2}{B^2 W^4}.
\]

Suppose that \( M^2 \) has zero Gaussian curvature. Then we have
\[
\alpha'\gamma'' - \gamma'^2\alpha^2 = 0.
\]

**Case 1.** Let \( \gamma' = 0 \). In this case \( \gamma \) is a constant function \( \gamma(u) = u_0 \) and the parametrization of (1) writes as
\[
r(u, v) = (u, v, \delta_1 g(v) + \delta_2); \quad \delta_1, \delta_2 \in \mathbb{R}.
\]

This means that \( M^2 \) is a cylindrical surface with base curve a plane curve in the \( vz \)- plane.

**Case 2.** Let \( \alpha' = 0 \). In this case \( \alpha \) is a constant function \( \alpha(v) = v_0 \) and the parametrization of (1) writes as
\[
r(u, v) = (u, v, \delta_3 f(u) + \delta_4); \quad \delta_3, \delta_4 \in \mathbb{R}.
\]

This means that \( M^2 \) is a cylindrical surface with base curve a plane curve in the \( uz \)- plane.

**Case 3.** Let \( \gamma'' = 0 \) and \( \gamma' \neq 0 \). Then \( \gamma(u) = \lambda_1 u + \lambda_2, \quad \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\} \times \mathbb{R} \). Moreover, (15) gives \( \alpha' = 0 \) and \( \alpha(v) = \delta_1 g(v) + \delta_2 \); \( \delta_1, \delta_2 \in \mathbb{R} \).

**Case 4.** Let \( \alpha'' = 0 \) and \( \alpha' \neq 0 \). Then \( \alpha(v) = \lambda v + \delta_1, \quad \lambda, \delta_1 \in \mathbb{R} \setminus \{0\} \times \mathbb{R} \). Moreover, (15) gives \( \gamma' = 0 \) and \( \gamma(u) = u_0 \) is a constant function. Now \( M^2 \) is the plane of equation \( z(u, v) = \lambda_3 u + \lambda_4; \lambda_5, \lambda_6 \in \mathbb{R} \).
Case 5. Let $\gamma'' \neq 0$ and $\lambda'' \neq 0$.
Equation (15) writes as
\[
\frac{\gamma'''}{\gamma'} = \frac{a'^2}{aa''}.
\]
Therefore, there exists a real number $\lambda \in \mathbb{R} \setminus \{0\}$ such that
\[
\frac{\gamma'''}{\gamma^2} = \lambda = \frac{a'^2}{aa''}.
\]
Integrate these equations
\[
\begin{cases}
\gamma' = k_1 \gamma^\lambda \\
a' = k_2 \alpha^\lambda,
\end{cases}
\] (16)
where $k_1$ and $k_2$ are constants of integration.

i) If $\lambda = 1$, the general solution of (16) is given by
\[
\begin{cases}
\gamma(u) = \lambda_1 e^{k_1 u} \\
a(v) = \lambda_2 e^{k_2 v}
\end{cases}
\]
where $\lambda_1$ and $\lambda_2$ are constants of integration.
Hence
\[
\begin{cases}
f(u) = \lambda_3 e^{k_1 u} + \lambda_4 \\
g(v) = \lambda_5 e^{k_2 v} + \lambda_6,
\end{cases}
\]
where $\lambda_3, \lambda_4, \lambda_5, \lambda_6 \in \mathbb{R}$.

ii) If $\lambda \neq 1$, the general solution of (16) is given by
\[
\begin{cases}
\gamma(u) = ((1 - \lambda)k_1 u + c_1)^{\frac{1}{\lambda}} \\
a(v) = ((\frac{k_1 - 1}{\lambda})k_2 v + c_2)^{\frac{1}{\lambda}},
\end{cases}
\]
where $c_1$ and $c_2$ are constants of integration.
Hence
\[
\begin{cases}
f(u) = c_3((1 - \lambda)k_1 u + c_1)^{\frac{1}{\lambda}} + c_4 \\
g(v) = c_5((\frac{k_1 - 1}{\lambda})k_2 v + c_2)^{\frac{1}{\lambda}} + c_6,
\end{cases}
\]
where $c_3, c_4, c_5, c_6 \in \mathbb{R}$.

**Theorem 8.** Let $M^2$ be a TH-surface in $\mathbb{E}^3$ with constant Gauss curvature $K$. If $M^2$ has zero Gaussian curvature, then $M^2$ can be parameterized as
\[
r(u, v) = (u, v, z(u, v) = A(f(u) + g(v)) + B f(u)g(v)),
\]
where
1) either $f(u) = \lambda_1 e^{k_1 u} + \lambda_2$ and $g(v) = \lambda_3 e^{k_2 v} + \lambda_4$,
2) or $f(u) = \mu_1 u + \mu_2$ and $g(v) = \mu_3$,
3) or $g(v) = v_1 v + v_2$ and $f(u) = v_3$,
4) or $f(u) = \xi_1((1 - \lambda)k_1 u + \xi_2)^{\frac{1}{\lambda}} + \xi_3$ and $g(v) = \xi_4((\frac{k_1 - 1}{\lambda})k_2 v + \xi_5)^{\frac{1}{\lambda}} + \xi_6$.

**5. Minimal TH-surfaces in $\mathbb{E}^3$**

Let $M^2$ be a TH-surface in the Euclidean 3-space $\mathbb{E}^3$. Then, $M^2$ is parameterized by
\[
r(u, v) = (u, v, A(f(u) + g(v)) + B f(u)g(v)),
\]
where $A$ and $B$ are non-zero real numbers.
We have the natural frame $\{e_u, e_v\}$ given by
\[
e_u = (1, 0, f'\alpha), \ e_v = (0, 1, g'\gamma),
\]
where \( a = A + Bg \) and \( \gamma = A + Bf \).

From this, the unit normal vector field \( N \) of \( M^2 \) is given by

\[
N = \frac{1}{W} (-af', -\gamma g', 1),
\]

where \( W = \sqrt{1 + f'^2a^2 + \gamma'^2} \).

The coefficients of the first fundamental form of \( M^2 \) are given by

\[
E = 1 + f'^2a^2, \quad G = 1 + \gamma'^2, \quad F = f'\gamma a\gamma.
\]

The coefficients of the second fundamental form of the surface are

\[
L = \frac{af''}{W}, \quad M = \frac{Bf'g'}{W}, \quad N = \frac{2f'g'}{W}.
\]

Hence, the mean curvature \( H \) and the Gaussian curvature \( K \) are given by, respectively

\[
H = \frac{af''(1 + \gamma'^2) - 2Ba\gamma f'^2a^2 + \gamma g''(1 + f'^2a^2)}{2W^3}, \quad K = \frac{a\gamma f''g'' - B^2f'^2a^2}{EG - F^2}.
\]

If the surface is minimal, that is, \( H = 0 \) on \( M^2 \), we have from (17)

\[
a\gamma''(B^2 + \gamma^2a^2) - 2a\gamma a'^2\gamma'^2 + \gamma\alpha''(B^2 + \gamma^2a^2) = 0.
\]

The previous equation may be rewritten as

\[
a\gamma''(B^2 + \gamma^2a^2) - 2a\gamma a'^2\gamma'^2 + \gamma\alpha''(B^2 + \gamma^2a^2) = 0.
\]

Since the roles of \( a \) and \( \gamma \) in (19) are symmetric, we only discuss the cases according to the function \( \gamma \). We distinguish cases.

**Case 1.** Let \( \gamma' = 0 \). In this case (19) gives \( B^2\gamma a'' = 0 \).

i) If \( \gamma = 0 \), then \( f(u) = -\frac{A}{B}, M^2 \) is the horizontal plane of equation \( z = -\frac{A^2}{B} \).

ii) If \( a'' = 0 \), then \( g'(v) = av + b, a, b \in \mathbb{R}, \) and \( f'(u) = c, c \in \mathbb{R}, M^2 \) is the plane of equation \( z = c_1v + c_2, c_1, c_2 \in \mathbb{R} \).

**Case 2.** Let \( \gamma'' = 0 \) and \( \gamma' \neq 0 \), and by symmetry, \( a' \neq 0 \). Then \( \gamma(u) = \lambda u + \delta, (\lambda, \delta) \in \mathbb{R}^* \times \mathbb{R} \) and \( a \) is a solution of the following ODE

\[
-2\lambda^2aa'^2 + a''(B^2 + \lambda^2a^2) = 0.
\]

Then the general solution of (20) is given by

\[
a(v) = \frac{B}{\lambda} \tan(\lambda_1v + \lambda_2), \ \lambda_1, \lambda_2 \in \mathbb{R}.
\]

Hence

\[
g(v) = \frac{1}{\lambda} \tan(\lambda_1v + \lambda_2) - \frac{A}{B}, \ \lambda_1, \lambda_2 \in \mathbb{R}.
\]

So, the parametrization of \( M^2 \) can be written in the form

\[
r(u, v) = (u, v, \lambda_3u + \delta_2 + \frac{A}{\lambda} \tan(\lambda_1v + \lambda_2) - \frac{A^2}{B} + B(\lambda_3u + \delta_2)(\frac{1}{\lambda} \tan(\lambda_1v + \lambda_2) - \frac{A}{B})),
\]

where \( (\lambda_3, \delta_2) \in \mathbb{R}^* \times \mathbb{R} \).

**Case 3.** Let \( \gamma'' \neq 0 \). By symmetry in the discussion of the case, we also suppose \( \alpha'' \neq 0 \). If we divide (19) by \( a\gamma a'^2\gamma'^2 \), we obtain

\[
\frac{B^2\gamma''}{\gamma a'^2\gamma'^2} + \frac{\gamma\gamma''}{aa'^2\gamma'^2} + \frac{aa''}{a'^2} = \frac{2}{B^2} = 2 = 0.
\]
Thus, after a derivation with respect to \( u \), followed by a derivation with respect to \( v \), we obtain

\[
\left( \frac{\gamma''}{\gamma''', \gamma''} \right)_{\mu} \left( \frac{1}{a'''} \right)_{\nu} + \left( \frac{a'''}{\gamma'''^2} \right)_{\mu} \left( \frac{1}{\gamma'''^2} \right)_{\nu} = 0.
\]

Hence we deduce the existence of a real number \( k \in \mathbb{R} \) such that

\[
\left\{ \begin{array}{l}
\left( \frac{\gamma''}{\gamma''', \gamma''} \right)_{\mu} = k \left( \frac{1}{\gamma''} \right)_{\mu} \\
\left( \frac{a'''}{\gamma'''^2} \right)_{\mu} = -k \left( \frac{1}{\gamma''} \right)_{\nu}.
\end{array} \right.
\]

(21)

The first equation of (21) can integrate obtaining

\[
\gamma'' = \gamma (k + b_1 \gamma^2).
\]

(22)

From the second equation in (21), we obtain

\[
a'' = -a (k + b_2 \gamma^2).
\]

(23)

Substituting the above in (19), we get

\[
a \gamma ((k + b_1 \gamma^2)(B^2 + a^2 \gamma^2) - 2a^2 \gamma^2 - (k + b_2 \gamma^2)(B^2 + \gamma'^2 a^2)) = 0.
\]

If we simplify by \( a \gamma \) and then we divide by \( a^2 \gamma^2 \), we get

\[
\frac{k \gamma^2 - b_2 B^2}{\gamma^2} - 2 + b_1 \gamma^2 = \frac{k a^2 - b_1 B^2}{a^2} + b_2 a^2.
\]

Hence, we deduce the existence of a real number \( \lambda \in \mathbb{R} \) such that

\[
\left\{ \begin{array}{l}
\gamma'^2 = \frac{2 \gamma^2 - b_2 B^2}{\lambda + 2 - b_1 B} \\
a'^2 = \frac{\alpha a^2 - b_1 B^2}{\lambda - b_2 a^2}.
\end{array} \right.
\]

(24)

Differentiating with respect to \( u \) and \( v \), respectively, we have

\[
\left\{ \begin{array}{l}
\gamma'' = \frac{2(\lambda k + 2k - b_1 b_2 B^2)}{(\lambda + 2 - b_1 B)^2} \\
a'' = \frac{a a'''}{(\lambda - b_2 a^2)^2}.
\end{array} \right.
\]

(25)

Let us compare these expressions of \( a'' \) and \( \gamma'' \) with those ones that appeared in (22) and (23) and replace the value of \( \gamma'^2 \) and \( a'^2 \) obtained in (24). We get

\[
(\lambda k + 2k - b_1 b_2 B^2)(1 + \lambda - b_1 \gamma^2) = 0,
\]

\[
(\lambda k - b_1 b_2 B^2)(\lambda - 1 - b_2 a^2) = 0.
\]

We discuss all possibilities.

i) If \( \lambda k + 2k - b_1 b_2 B^2 = 0 \) and \( \lambda k - b_1 b_2 B^2 = 0 \), then \( k = 0 \) and \( b_1 b_2 = 0 \). Then (25) gives \( \gamma'' = 0 \) and \( a'' = 0 \), a contradiction.

ii) If \( \lambda k + 2k - b_1 b_2 B^2 = 0, \lambda = 1 \) and \( b_2 = 0 \), we obtain \( k = 0 \). Then (25) gives \( \gamma'' = 0 \) and \( a'' = 0 \), a contradiction.

iii) If \( \lambda k - b_1 b_2 B^2 = 0, \lambda = -1 \) and \( b_1 = 0 \), we obtain \( k = 0 \). Then (25) gives \( \gamma'' = 0 \) and \( a'' = 0 \), a contradiction.

iv) If \( 1 + \lambda - b_1 \gamma^2 = 0 \) and \( \lambda - 1 - b_2 a^2 = 0 \), we deduce that \( a, \gamma \) are both constant functions, and so, \( \gamma'' = 0 \) and \( a'' = 0 \), a contradiction.

Therefore, we have the following:
Theorem 9. Let $M^2$ be a TH-surface in $\mathbb{E}^3$. If $M^2$ is minimal, then $M^2$ is plane or parameterized as

$$r(u, v) = (u, v, A(f(u) + g(v)) + Bf(u)g(v)),$$

where

i) either $f(u) = \frac{\lambda_1}{B}u + \frac{\lambda_2 - A}{B}$ and $g(v) = \frac{1}{\lambda_1} \tan(\lambda_3 v + \lambda_4) - \frac{A}{B}$ or

ii) $f(u) = \frac{1}{\lambda_1} \tan(\lambda_2 u + \lambda_3) - \frac{A}{B}$ and $g(v) = \frac{\lambda_1}{B}v + \frac{\lambda_4 - A}{B}$.

6. **TH-surfaces with zero Gaussian curvature in $\mathbb{E}^3$**

A surface in Euclidean 3-space parameterized by (1) has Gaussian curvature

$$K = \frac{a\gamma'^2 g'' - B^2f'^2g''}{EG - F^2}.$$

Hence that if $K = 0$, then

$$a\gamma'^2g'' - \gamma'^2a'' = 0. \tag{26}$$

Since the roles of the function $\gamma$ and $\alpha$ are symmetric in (26), we discuss the different cases according the function $\gamma$.

**Case 1.** Let $\gamma' = 0$. In this case $\gamma$ is a constant function $\gamma(u) = u_0$ and the parametrization of (1) writes as

$$r(u, v) = (u, v, \delta_1g(v) + \delta_2).$$

This means that $M^2$ is a cylindrical surface with base curve a plane curve in the $vz-$ plane.

**Case 2.** Let $\gamma'' = 0$ and $\gamma' \neq 0$. Then $\gamma(u) = \lambda u + \delta_1$, $(\lambda, \delta) \in \mathbb{R}^+ \times \mathbb{R}$. Moreover, (26) gives $a' = 0$ and $a(v) = v_0$ is a constant function. Now $M^2$ is the plane of equation $z(u, v) = \lambda u + \delta_1$, $\lambda, \delta \in \mathbb{R}$.

**Case 3.** Let $\gamma'' \neq 0$. By the symmetry on the arguments, we also suppose $\alpha'' \neq 0$.

Equation (26) writes as

$$\frac{\gamma''}{\gamma'^2} = \frac{a'}{a''}.$$

Therefore, there exists a real number $\lambda \in \mathbb{R}^+$ such that

$$\frac{\gamma''}{\gamma'^2} = \lambda = \frac{a'}{a''}. \tag{27}$$

Integrate these equations

$$\begin{cases} \gamma' = k_1 \gamma^\lambda \\ \alpha' = k_2 \alpha^{\frac{1}{\lambda}} \end{cases} \tag{27}$$

where $k_1$ and $k_2$ are constants of integration.

i) If $\lambda = 1$, the general solution of (27) is given by

$$\begin{cases} \gamma(u) = \lambda_1 e^{k_1 u} \\ \alpha(v) = \lambda_2 e^{k_2 v} \end{cases},$$

where $\lambda_1$ and $\lambda_2$ are constants of integration.

Hence

$$\begin{cases} f(u) = \lambda_3 e^{\frac{k_1 u}{\lambda}} + \lambda_4 \\ g(v) = \lambda_5 e^{k_2 v} + \lambda_6 \end{cases},$$

where $\lambda_3, \lambda_4, \lambda_5, \lambda_6 \in \mathbb{R}$.

i) If $\lambda \neq 1$, the general solution of (27) is given by

$$\begin{cases} \gamma(u) = (1 - \lambda)k_1 u + c_1 \frac{1}{\lambda} \\ \alpha(v) = ((\frac{1}{\lambda} - 1)k_2 v + c_2) \frac{1}{\lambda} \end{cases}.$$
where $c_1$ and $c_2$ are constants of integration.

Hence

$$\begin{align*}
f(u) &= c_3((1-\lambda)k_1u + c_1)^{\frac{1}{1-\lambda}} + c_4 \\
g(v) &= c_5((\frac{1-\lambda}{\lambda})k_2v + c_2)^{\frac{1}{\lambda-1}} + c_6,
\end{align*}$$

where $c_3, c_4, c_5, c_6 \in \mathbb{R}$.

**Theorem 10.** Let $M^2$ be a TH-surface in Euclidean 3-space $E^3$ with constant Gauss curvature $K$. Then $K = 0$. Furthermore, the surface is plane or is a cylindrical surface over a plane curve or parameterized as

$$r(u,v) = (u, v, A(f(u) + g(v)) + Bf(u)g(v)),$$

where

i) either $f(u) = \lambda_3 e^{\lambda_4 u} + \lambda_4$ and $g(v) = \lambda_5 e^{\lambda_6 v} + \lambda_6$ or

ii) $f(u) = c_3((1-\lambda)k_1u + c_1)^{\frac{1}{1-\lambda}} + c_4$ and $g(v) = c_5((\frac{1-\lambda}{\lambda})k_2v + c_2)^{\frac{1}{\lambda-1}} + c_6$.

**Acknowledgments:** The authors would like to express their thanks to the referee for his useful remarks.

**Author Contributions:** All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

**Conflicts of Interest:** “The authors declare no conflict of interest.”

**References**


