On codes over $\mathbb{R} = \mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$ where $u^3 = 0$ and its related parameters

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Received: 19 March 2019; Accepted: 30 August 2019; Published: 9 September 2019.

Abstract: In ring $\mathbb{R} = \mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$ where $u^3 = 0$, using Lee weight and generalized Lee weight, some lower bound and upper bound on the covering radius of codes is given and also to find the covering radius for various repetition codes with respect to same and different length in $\mathbb{R}$.

Keywords: Covering radius, codes over finite rings, generator matrix, generalized weight, Lee weight.

MSC: Primary 20G30, Secondary 20G35.

1. Introduction

Codes over finite commutative rings have been studied for almost 50 years. The main motivation of studying codes over rings is that they can be associated with codes over finite fields through the Gray map. Recently, coding theory over finite commutative non-chain rings is a hot research topic and in last three decades, there are many researchers doing research on code over finite rings.

Researchers have more interest in codes over finite rings in recent years, especially the rings $\mathbb{Z}_{2^k}$ where $2^k$ denotes the ring of integers modulo $2^k$. In [1–6], the authors have deeply studied codes over $\mathbb{F}_2 + u\mathbb{Z}_2$. In gray map, binary linear and non-linear codes can be obtained from codes over $\mathbb{Z}_4$ and the covering radius of binary linear codes were studied [7,8]. Recently the covering radius of codes over $\mathbb{Z}_2 + u\mathbb{Z}_2$ has been investigated with respect to different distances [9,10]. In [11], the authors gave upper and lower bounds on the covering radius of a code over $\mathbb{Z}_4$ with different distances. In this paper, I obtain the covering radius of some particular codes over $\mathbb{R} = \mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$, I consider the finite ring $\mathbb{R} = \mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$ of integers modulo 2 in this paper.

2. Preliminaries

Let $\mathbb{R} = \mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$ be the finite ring. Consider the elements of ring are $\{0, 1, u, v, u^2, v^2, uv, v^3\}$, where $u^3 = 0$, $v = 1 + u$, $v^2 = 1 + u^2$, $v^3 = 1 + u + u^2$, $uv = u + u^2$.

Let $C \subseteq \mathbb{R}^n$ be a linear code of length $n$ over $\mathbb{R}$ is an $\mathbb{R}$-submodule of $\mathbb{R}^n$. The element of $C$ is called a codeword of $C$. The Hamming weight $w_H(c)$ of a codeword $c$ is the number of non-zero components. That is, $w_H(c) = |\{i|c_i \neq 0\}|$. The minimum Hamming weight $w_H(c) = \min \{i|c_i \neq 0\}$.

Let $x_i$ and $y_i \in \mathbb{R}^n$, $d_H(x_i, y_i) = |\{i: x_i \neq y_i\}|$, where $i = 1, 2, \cdots, n$ is called Hamming distance between any distinct vectors $x, y \in \mathbb{R}^n$ and is denoted by $d_H(x, y)$ and is minimum Hamming distance $d_H(C)$ of $C$. The minimum Hamming distance between distinct pairs of codewords of a code $C$ is called $d_H(C)$.

The generalized Lee weight and Lee weight are the element $x \in \mathbb{R}$ is analogous to the definition of the generalized Lee weight and Lee weight of the elements of the ring $\mathbb{Z}_6$ [6,12,13].

Let $C \subseteq \mathbb{R}$ is permutation equivalent to a code $C$ with generator matrix of the form

$$G = \begin{pmatrix}
I_{k_0} & A_{01} & A_{02} & A_{03} \\
0 & uI_{k_1} & uA_{12} & uA_{13} \\
0 & 0 & u^2I_{k_2} & u^2A_{23}
\end{pmatrix}$$

where $A_{ij}$ are binary matrices for $i > 0$. A code with a generator matrix in this form is of type $\{k_0, k_1, k_2\}$ and has $8^{k_0}4^{k_1}2^{k_2}$ vectors [14].
In [15], the generalized Lee weight of the elements $x \in \mathbb{R}$ are given
\[
wt_{GL}(x) = \begin{cases} 
0 & \text{if } x = 0 \\
2 & \text{if } x \neq u^2 \\
4 & \text{if } x = u^2 
\end{cases}
\]
and the Lee weights of the elements $0, \{1, v, v^2, v^3\}, \{u, uv\}, u^2$ of $\mathbb{R}$ are defined by 0, 1, 2, 2\(^2\) [13].

3. Covering Radius of Codes and Repetition Codes

Let $r_d(C) = \max_{u \in \mathbb{R}^n} \left\{ \min_{c \in C} \{d(u,c)\} \right\}$ be the covering of codes, where $d$ with respect to Hamming distance, Lee distance and generalized distance.

Let $C = \{a | a \in \mathbb{F}_q\}$, $\alpha = \alpha \cdots \alpha$ be a $q$-ary repetition code over $\mathbb{F}_q$, with $[n,1,n]$ code, where $\mathbb{F}_q$ is a finite field. The the covering radius of $C$ is $\left\lfloor \frac{n(q-1)}{q} \right\rfloor$ [16].

Let $G = [1 \cdots 1 \alpha_2 \cdots \alpha_n \cdots \alpha_{q-1} \alpha_{q-1} \cdots \alpha_{q-1}]$ be a generator of code $C$ with $[n(q-1),1,n(q-1)]$ repetition code of block size is $n$. Use [16], that the code of the covering radius $C$ is $\left\lfloor \frac{n(q-1)^2}{q} \right\rfloor$, since it will be equivalent to a repetition code of length $(q-1)n$.

In $\mathbb{R}$, there are two types of repetition codes of length $n$ viz.

1. unit repetition code $C_I : [n,1,d_H = n,d_L = n,d_{GL} = n]$ generated by $G_I = [1 \cdots 1]$

2. zero repetition code $C_{II} : (n,2,d_H = n,d_L = 4n,d_{GL} = 4n)$ generated by $G_{II} = [u^2u^2 \cdots u^2]$ and $C_{III} : (n,4,d_H = n,d_L = 2n,d_{GL} = 2n)$ generated by $G_{III} = [uuuv uuuv uuuv uuuv]$ or $[uuuv uuuv uuuv uuuv]$. The code generated by $[u \ u \cdots \ u]$ and $[uuuv uuuv uuuv]$ are similar to the code $C_{III}$.

Theorem 1. 
1. $r_L(C_I) = \frac{3n}{2}$,
2. $r_L(C_{II}) = 2n$ and
3. $n \leq r_L(C_{III}) \leq 2n$.

Proof. Lee weight of $\mathbb{R} : 0 \to 0, \{1,v,v^2,v^3\} \to 1, \{u,uv\} \to 2$ and $u^2 \to 4$. If $x \in \mathbb{R}^n$ with $c_i$ times $i$'s, in $x$ and $\sum_i c_i = n$ and the code $c_{i=0 \to 7} \in \{a(C_i) | a \in \mathbb{R}\}$. Then
\[
\begin{align*}
d_L(x,c_0) &= \wt_L(x-0 \cdots 0) \\
&= 0c_0 + 1c_1 + uc_2 + vc_3 + u^2c_4 + uvc_5 + v^2c_6 + v^3c_7 \\
d_L(x,c_1) &= \wt_L(x-1 \cdots 1) \\
&= v^3c_0 + 0c_1 + 1c_2 + uc_3 + vc_4 + u^2c_5 + uvc_6 + v^2c_7 \\
d_L(x,c_2) &= \wt_L(x-1 \cdots 1) \\
&= v^3c_0 + 0c_1 + 1c_2 + uc_3 + vc_4 + u^2c_5 + uvc_6 + v^2c_7 \\
d_L(x,c_3) &= \wt_L(x-0 \cdots 0) \\
&= 0c_0 + 1c_1 + uc_2 + vc_3 + u^2c_4 + uvc_5 + v^2c_6 + v^3c_7 \\
d_L(x,c_4) &= \wt_L(x-0 \cdots 0) \\
&= 0c_0 + 1c_1 + uc_2 + vc_3 + u^2c_4 + uvc_5 + v^2c_6 + v^3c_7 \\
d_L(x,c_5) &= \wt_L(x-0 \cdots 0) \\
&= 0c_0 + 1c_1 + uc_2 + vc_3 + u^2c_4 + uvc_5 + v^2c_6 + v^3c_7 \\
d_L(x,c_6) &= \wt_L(x-0 \cdots 0) \\
&= 0c_0 + 1c_1 + uc_2 + vc_3 + u^2c_4 + uvc_5 + v^2c_6 + v^3c_7 \\
d_L(x,c_7) &= \wt_L(x-0 \cdots 0) \\
&= 0c_0 + 1c_1 + uc_2 + vc_3 + u^2c_4 + uvc_5 + v^2c_6 + v^3c_7
\end{align*}
\]

Similarly, $d_L(x,c_2) = n - c_2 + c_3 + 3c_6$, $d_L(x,c_3) = n - c_3 + c_3 + 3c_6 + c_1$, $d_L(x,c_4) = n - c_4 + 3c_0 + c_2 + 3c_6$, $d_L(x,c_5) = n - c_5 + c_0 + 3c_2 + c_3 + 3c_6$, $d_L(x,c_6) = n - c_6 + 3c_0 + 3c_2 + c_3 + 3c_6$, $d_L(x,c_7) = n - c_7 + c_0 + 3c_2 + c_3 + 3c_6$.

Therefore, $d_L(x,C_i) = \min\{d_L(x,c_0),d_L(x,c_1),d_L(x,c_2)d_L(x,c_3),d_L(x,c_4),d_L(x,c_5),d_L(x,c_6),d_L(x,c_7)\}$. Thus, $r_L(C_I) \leq \frac{3n}{2}$ [16], the minimum of data is less than or equal to the average of data and $\sum_i c_i = n$, implies $d_L(x,C_I) \leq n + \frac{4n}{3} = \frac{3n}{2}$.

Let $x = \underbrace{0 \cdots 0}_{n-7g} \underbrace{1u \cdots u}_{g} \underbrace{uvuuv \cdots uuv}_{g} \underbrace{u^2u^2 \cdots u^2}_{g} \underbrace{v^2v^2 \cdots v^2}_{g} \underbrace{uvuvuuv \cdots uuuv}_{g} \in \mathbb{R}^n$, where $g = \left\lfloor \frac{g}{2} \right\rfloor$, then $d_L(x,c_0) = n + 4g$, $d_L(x,c_1) = 2n - 4g$, $d_L(x,c_2) = n + 4g$, $d_L(x,c_3) = 4n - 20g$, $d_L(x,c_4) = n + 4g$, $d_L(x,c_5) = n + 4g$, $d_L(x,c_6) = n + 4g$ and $d_L(x,c_7) = n + 4g$... $r_L(C_I) \geq \min\{n + 4g, 2n - 4g, 4n - 20g\} = n + 4g \geq \frac{3n}{2}$. Thus $r_L(C_I) = \frac{3n}{2}$.

Let $x = \underbrace{u^2u^2 \cdots u^2}_2 \underbrace{000 \cdots 0}_8 \in \mathbb{R}^n$. The code $C_{II} = \{a(u^2u^2 \cdots u^2) | a \in \mathbb{R}\}$, that is $C_{II} = \{000 \cdots 0, u^2u^2 \cdots u^2\}$, generated by $[u^2u^2 \cdots u^2]$ is an $(n,2,4n)$ code. Then
\[ d_L(x,00\ldots0) = wt_L(u^2v^2\ldotsu^200\ldots00\ldots0) = \frac{n}{2}wt_L(u^2) = 2n \text{ and } d_L(x,u^2u^2\ldotsu^2) = wt_L(u^2u^2\ldotsu^200\ldots0u^2\ldotsu^2) = \frac{n}{2}wt_L(u^2) = 2n. \]

Therefore, \( d_L(x,C_{II}) = \min\{2n,2n\} = 2n. \) Thus, by definition of covering radius \( r_L(C_{II}) \geq 2n. \)

Let \( x \in \mathbb{R}^n \) be any codeword. Let us take \( x \) has \( \sigma_i \) coordinates as \( i's, \) then \( \sum \sigma_i = n, \) where \( i = 0 \) to 7.

Since \( C_{II} = \{00\ldots0,u^2u^2\ldotsu^2\}, \) then \( d_L(x,00\ldots0) = n - \sigma_0 + \sigma_2 + 3\sigma_4 + \sigma_6 \) and \( d_L(x,u^2u^2\ldotsu^2) = n - \sigma_0 + 3\sigma_0 + \sigma_2 + \sigma_6. \) Thus \( d_L(x,C_{II}) = \min\{n - \sigma_0 + \sigma_2 + 3\sigma_4 + \sigma_6, n - \sigma_0 + 3\sigma_0 + \sigma_2 + \sigma_6\} \) and \( d_L(x,C_{II}) \leq n + n = 2n. \) Therefore, \( r_L(C_{II}) \leq 2n. \) Hence, \( r_L(C_{II}) = 2n. \)

For \( x = u^2u^2\ldotsu00\ldots0 \in \mathbb{R}^n \) and the code \( c_i, i = 0 \) to 7 \( \in \{C_{II} \} \) generated by \( [uu\ldotsu] \) is an \( (n,4,2n) \) code. Then \( d_L(x,c_0) = wt_L(uu\ldotsu00\ldots00\ldots0) = \frac{n}{2}wt_L(u) = n, \) \( d_L(x,c_1) = wt_L(uu\ldotsu00\ldots00\ldots00u\ldotsu) = \frac{n}{2}wt_L(uv) = n, \) \( d_L(x,c_2) = wt_L(uu\ldotsu00\ldots00\ldots00u\ldotsu) = \frac{n}{2}wt_L(u^2) + \frac{n}{2}wt_L(u) = 2n + n = 3n. \)

Therefore, \( d_L(x,C_{III}) = \min\{n,n,3n,3n\} = n. \) Thus, \( r_L(C_{III}) \geq n. \)

Let \( x \) be any word in \( \mathbb{R}^n. \) Let us take \( x \) has \( \sigma_i \) coordinates as \( i's, \) with \( \sum \sigma_i = n. \) Then \( d_L(x,c_0) = n - \sigma_0 + \sigma_2 + 3\sigma_4 + \sigma_6, \)

\( d_L(x,c_1) = n - \sigma_2 + \sigma_0 + \sigma_3 + 3\sigma_6, \)

\( d_L(x,c_2) = n - \sigma_0 + \sigma_2 + 3\sigma_0 + \sigma_4 + \sigma_6, \)

\( d_L(x,c_3) = n - \sigma_0 + \sigma_2 + 3\sigma_0 + \sigma_4 + \sigma_6. \)

Thus, \( d_L(x,C_{III}) = \min\{d_L(x,c_0),d_L(x,c_1),d_L(x,c_2),d_L(x,c_3)\}. \) Therefore, \( d_L(x,C_{III}) \) is less than or equal to its average and \( \sigma_0 + \sigma_2 + \sigma_4 \leq n \) implies \( d_L(x,C_{III}) \leq n + \frac{4n}{2} = 2n \) and \( r_L(C_{III}) \leq 2n. \) Hence, \( n \leq r_L(C_{III}) \leq 2n. \)

**Theorem 2.**

1. \( r_{GL}(C_1) = 2n, \)
2. \( r_{GL}(C_{II}) = 2n \) and
3. \( n \leq r_{GL}(C_{III}) \leq 2n. \)

**Proof.** Let \( x \in \mathbb{R}^n \) with \( \sigma_i \) times \( i's, \) in \( x \) and \( \sum \sigma_i = n, \) \( i = 0 \) to 7.

In generalized Lee weight of \( \mathbb{R} : 0 \rightarrow 0, \{1,u,v,u^2,uv,v^3\} \rightarrow 2 \) and \( u^2 \rightarrow 4 \) and \( c_i, i = 0 \) to 7 \( \in \{C_i\} \) \( \in \mathbb{R} \). Then, \( d_L(x,c_0) = w_{GL}(x-c_0) = 0c_0 + 1c_1 + uc_2 + vc_3 + u^2c_4 + uvv_5 + v^2c_6 + v^3c_7 = n - \sigma_0 + \sigma_1 + \sigma_2 + 3\sigma_4 + \sigma_5 + \sigma_6 + \sigma_7. \)

Similarly, \( d_L(x,c_1) = w_{GL}(x-c_1) = v^3c_0 + v\sigma_1 + v\sigma_2 + v\sigma_3 + v^2c_4 + vuvv_5 + v^2c_6 + v^3c_7 = n - \sigma_0 + \sigma_1 + \sigma_2 + 3\sigma_4 + \sigma_5 + \sigma_6 + \sigma_7. \)

Thus, \( d_L(x,C_{III}) = \min\{d_L(x,c_0),d_L(x,c_1),d_L(x,c_2),d_L(x,c_3)\}. \) Therefore, \( d_L(x,C_{III}) \) is less than or equal to its average and \( \sigma_0 + \sigma_2 + \sigma_4 \leq n \) implies \( d_L(x,C_{III}) \leq n + \frac{4n}{2} = 2n \) and \( r_L(C_{III}) \leq 2n. \) Hence, \( r_L(C_1) \leq 2n. \)

Let \( y = 00\ldots01^g11\ldots1u^2u^2\ldotsu^200\ldots0u^2\ldotsu^2v^2v^2\ldotsv^2uvuvuvuvuvuvuvu^2v^2v^2v^2v^2v^2v^2v^2v^2v^2 \in \mathbb{R}^n, \) where \( g = \lfloor \frac{n}{2} \rfloor, \) then \( d_L(y,c_0) = 2n, \) \( d_L(y,c_1) = 2n, \) \( d_L(y,c_2) = 2n, \) \( d_L(y,c_3) = 2n, \) \( d_L(y,c_4) = 2n, \) \( d_L(y,c_5) = 2n, \) \( d_L(y,c_6) = 2n, \) \( d_L(y,c_7) = 2n. \) Therefore, \( r_L(C_I) \geq \min\{2n,2n,2n\} = 2n \geq 2n. \) Thus, \( r_L(C_1) = 2n. \)

Let \( x = u^2u^2\ldotsu^200\ldots0 \in \mathbb{R}^n. \) The code \( C_{II} = \{\alpha(u^2\ldotsu^2) \mid \alpha \in \mathbb{R}^n\}, \) that is \( C_{II} = \{00\ldots0,u^2u^2\ldotsu^2\}, \) generated by \( [u^2u^2\ldotsu^2]. \) The parameter of \( C_{II} \) is \( (n,2,4n) \) code. Then \( d_L(x,00\ldots0) = wt_{GL}(u^2u^2\ldotsu^200\ldots00\ldots0) = \frac{n}{2}wt_{GL}(u^2) = 2n \) and \( d_L(x,u^2u^2\ldotsu^2) = wt_{GL}(u^2u^2\ldotsu^200\ldots00\ldots0u^2\ldotsu^2) = \frac{n}{2}wt_{GL}(u^2) = 2n. \) Therefore, \( d_L(x,C_{II}) = \min\{2n,2n\} = 2n. \) Thus, by definition of covering radius \( r_L(C_{II}) \geq 2n. \)
For any word $x$ in $\mathbb{R}^n$ and take $x$ has $s_i$ coordinates as $i$'s, with $\sum_i s_i = n$, where $i = 1$ to 7. Since $C_{I_1} = \{c_0 = 00 \cdots 0, c_{a^t} = u^2 u^2 \cdots u^2\}$, then $d_{GL}(x, c_0) = wt_{GL}(x - 00 \cdots 0) = 0v_0 + 1v_1 + uv_2 + uv_3 + u^2 v_4 + uv_5 + u^2 v_6 + v^8 v_7 = n - c_0 + c_1 + c_2 + 3v_3 + c_5 + c_6 + c_7$ and $d_{GL}(x, c_{a^t}) = wt_{GL}(x - u^2 u^2 \cdots u^2) = u^2 v_0 + 1v_1 + uv_2 + uv_3 + 3v_4 + 4v_5 + 5v_6 + 6v_7 = n - c_0 + c_1 + c_2 + 3v_3 + c_5 + c_6 + c_7$. Thus $d_{GL}(x, C_{I_1}) = \min\{d_{GL}(x, c_0), d_{GL}(x, c_{a^t})\}$. Since the minimum of $\{d_{GL}(x, c_0), d_{GL}(x, c_{a^t})\} \leq n$, then $d_{GL}(x, C_{I_1}) \leq 2n$ and $r_{GL}(C_{I_1}) \leq 2n$. Hence, $r_{GL}(C_{I_1}) = 2n$.

If $x = \frac{\bar{x}}{u \bar{u} \cdots u 0 \cdots 0} \in \mathbb{R}^n$. Then $d_{GL}(x, c_0) = wt_{GL}(\frac{\bar{x}}{u \bar{u} \cdots u 0 \cdots 0} - 0u^0 \cdots 0) = \frac{4}{2}wt_{GL}(u) = n$, $d_{GL}(x, c_1) = wt_{GL}(\frac{\bar{x}}{u \bar{u} \cdots u 0 \cdots 0} - uu \cdots u) = \frac{4}{2}wt_{GL}(uv) = n$, $d_{GL}(x, c_2) = wt_{GL}(\frac{\bar{x}}{u \bar{u} \cdots u 0 \cdots 0} - u^2 u^2 \cdots u^2) = \frac{4}{2}wt_{GL}(u^2) = n + 2n = 3n$, $d_{GL}(x, c_3) = wt_{GL}(\frac{\bar{x}}{u \bar{u} \cdots u 0 \cdots 0} - uv \bar{u} \cdots u) = \frac{4}{2}wt_{GL}(u^2) + \frac{4}{2}wt_{GL}(u) = 2n + n = 3n$. Therefore, $d_{GL}(x, C_{III}) = \min\{n, n, 3n, 3n\} = n$. Thus, by definition of covering radius $r_{GL}(C_{III}) \geq n$.

Let $x = \frac{\bar{x}}{u \bar{u} \cdots u} \in \mathbb{R}^n$ be any codeword and take $x$ has $s_i$ coordinates as $i$'s, then $\sum_i s_i = n$ and $c_i = 0$ to 7(a$^t$) $\in \{a(C_{III})|a \in \mathbb{R}\}$. Then $d_{GL}(x, c_0) = n - c_0 + c_1 + c_2 + c_3 + 3c_4 + c_5 + c_6 + c_7$, $d_{GL}(x, c_1) = n - c_2 + c_0 + c_1 + c_3 + c_4 + c_5 + c_6 + c_7$, $d_{GL}(x, c_2) = n - c_4 + 3c_0 + c_1 + c_2 + c_3 + c_5 + c_6 + c_7$, and $d_{GL}(x, c_3) = n - c_0 + c_0 + c_1 + 3c_2 + c_3 + c_4 + c_5 + c_7$. Thus $d_{GL}(x, C_{III}) = \min\{d_{GL}(x, c_0), d_{GL}(x, c_1), d_{GL}(x, c_2), d_{GL}(x, c_3)\}$. Hence $r_{GL}(C_{III}) \leq 2n$ for the minimum of $\{d_{GL}(x, c_0), d_{GL}(x, c_1), d_{GL}(x, c_2), d_{GL}(x, c_3)\} \leq 2n$. □

4. Block Repetition Code for Same Size in $\mathbb{R}$

In this section, give the covering radius of repetition code $C$ with respect to Lee and generalized Lee weight.

Let $G_1 = \begin{bmatrix} m & m & m \\ \cdots & \cdots & \cdots \\ \bar{v} & \bar{v} & \bar{v} \end{bmatrix}$ be a generator matrix with same block size($m$) of repetition code and the parameters of repetition code $BRep^{4m} : [4m, 1, 4m, 4m, 8m]$.

Theorem 3. Let $C$ be a code and $G_1$, be a generator matrix of $C$ over $\mathbb{R}$. Then $r_L(BRep^{4m}) = 6m$ and $r_{GL}(BRep^{4m}) = 8m$.

Proof. In Theorem 1, (Proposition(mattson) [7]) and the given generator matrix $G_1$, then

\[ r_L(BRep^{4m}) \geq 6m \]

(1)

Let $x = (u_1 | u_2 | u_3 | u_4) \in \mathbb{R}^{4m}$ where $u_1, u_2, u_3, u_4 \in \mathbb{R}^n$. Let us take in $u_1$, $i$ appears $r_1$ times, in $u_2$, $i$ appears $s_1$ times, in $u_3$, $i$ appears $t_1$ times, in $u_4$, $i$ appears $v_1$ times, with $\sum_i r_i = \sum_i s_i = \sum_i t_i = \sum_i v_i = m$ and $c_i \in \{a(G_1)|a \in \mathbb{R}\}$, $i = 0$ to 7. Then $d_L(x, c_0) = 4m - r_1 + r_2 + 3r_4 + r_6 - s_0 + s_2 + 3s_4 + s_6 - t_0 + s_1 + 3t_4 + t_6 - v_0 + v_2 + 3v_4 + v_6$, $d_L(x, c_1) = 4m - r_1 + r_3 + 3r_5 + r_7 - s_3 + s_5 + 3s_7 + s_1 - t_5 + t_7 + 3t_1 + t_3 - v_1 + 3v_3 + v_5$, $d_L(x, c_2) = 4m - r_2 + r_0 + r_4 + 3r_6 - s_6 + s_8 + 3s_2 + s_4 - t_2 + t_0 + t_4 + 3t_6 - v_6 + v_0 + 3v_2 + v_4$, $d_L(x, c_3) = 4m - r_3 + 3r_7 + r_1 - s_1 + s_3 + 3s_5 + s_7 - t_1 + t_3 + t_5 - v_5 + v_7 + 3v_3 + v_9$, $d_L(x, c_4) = 4m - r_4 + 3r_0 + r_2 + r_6 - s_4 + 3s_0 + s_2 + s_6 - t_4 + t_0 + t_2 + t_4 + v_0 + v_2 + v_6$, $d_L(x, c_5) = 4m - r_5 + r_7 + 3r_1 + r_3 - s_9 + s_1 + 3s_3 + s_5 + s_7 + t_1 + 3t_3 + t_7 - v_3 + v_5 + 3v_7 + v_1$, $d_L(x, c_6) = 4m - r_6 + r_0 + 3r_2 + r_4 - s_0 + s_4 + 3s_6 - t_6 + t_0 + 3t_2 + 4v_2 + v_0 + v_4 + 3v_6$, and $d_L(x, c_7) = 4m - r_7 + r_1 + 3r_3 + r_5 - s_5 + s_7 + 3s_1 + s_3 - t_3 + 3t_5 + t_7 + t_1 - v_1 + v_3 + 3v_5 + v_7$. Therefore, $d_L(x, BRep^{4m}) = \min\{d_L(x, c_0), d_L(x, c_1), d_L(x, c_2), d_L(x, c_3), d_L(x, c_4), d_L(x, c_5), d_L(x, c_6), d_L(x, c_7)\}$ is less than or equal to $6m$. Then $d_L(x, BRep^{4m}) \leq 6m$ and hence

\[ r_L(BRep^{4m}) \leq 6m \]

(2)

By (1) and (2), then $r_L(BRep^{4m}) = 6m$. Similarly, the Generalized Lee weight of $\mathbb{R} : 0 \rightarrow 0$, $\{1, u, v, v^2, uv, v^3\} \rightarrow 2$ and $u^2 \rightarrow 4$. Thus, $r_{GL}(BRep^{4m}) = 8m$. □
The three block repetition code $BRep^{3m} : (3m, 4, 2m, 6m, 8m)$ generated by $G_2^* = \left[ \begin{array}{ccc} m & \cdot & \cdot \\ m & \cdot & \cdot \\ m & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \right]$.  

Theorem 4. Let $C$ be a code over $\mathbb{R}$ generated by the matrix $G_2^*$. Then $r_L(BRep^{3m}) = 4m$ and $r_{GL}(BRep^{3m}) = 4m$.

Proof. In Theorem 1, (Proposition (matsson) [7]) and $G_2^*$, than $r_L(BRep^{3m}) \geq 4m$. Let $x = (u_1 | u_2 | u_3) \in \mathbb{R}^{3m}$ where $u_1, u_2, u_3 \in \mathbb{R}^m$ and $c_{u_1-u_3} \in \{a(x) | x \in \mathbb{R}\}$. Then, $d_L(x, c_0) = 3m - r_0 + r_2 + 3r_4 + r_6 - s_0 + s_2 + 3s_4 + s_6 - t_0 + s_1 + 3t_4 + t_6, d_L(x, c_1) = 3m - r_0 + r_2 + 4 + 3r_6 - s_4 + 3s_0 + s_2 + s_6 - t_0 + t_1 + 3t_2 + t_4, d_L(x, c_2) = 3m - r_0 + r_2 + 6 - s_0 + s_2 + 3s_4 + s_6 - t_0 + t_1 + t_2 + t_6, d_L(x, c_3) = 3m - r_0 + r_2 + 4 + s_4 + 3s_0 + s_2 + s_6 - t_0 + t_1 + t_2 + t_3 + t_4. Therefore, $d_L(x, BRep^{3m}) \leq 4m$ and $r_L(BRep^{3m}) \leq 4m$. Hence $r_L(BRep^{3m}) = 4m$. Similarly, the Generalized Lee weight of $\mathbb{R} : 0$ is 0, $\{1, u, v, u^2, uv, v^3\}$ is 2 and $u^2$ is 4. Then, $r_{GL}(BRep^{3m}) = 4m$. $\square$

Corollary 5. Let $C_i$ be a code over $\mathbb{R}$ and the $G_i$ be the generator matrices of code $C_i$, $i = 1, 2, 3$. Then

1. $G_1 = \left[ \begin{array}{ccc} m & \cdot & \cdot \\ m & \cdot & \cdot \\ m & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \right]$, prove that $r_L(BRep^{2m}) = \frac{2m}{3}$ and $r_{GL}(BRep^{2m}) = 4m$.

2. $G_2 = \left[ \begin{array}{ccc} m & \cdot & \cdot \\ m & \cdot & \cdot \\ m & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \right]$, show that $r_L(BRep^{2m}) = \frac{5m}{3}$ and $r_{GL}(BRep^{2m}) = 3m$.

3. $G_3 = \left[ \begin{array}{ccc} m & \cdot & \cdot \\ m & \cdot & \cdot \\ m & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \right]$, to find $r_L(BRep^{3m}) = 10m$ and $r_{GL}(BRep^{3m}) = 12m$.

Proof. In Theorem 1 and Theorem 2. $\square$

5. Different size of blocks repetition code

In this section, only detail that the covering radius for different size of blocks repetition code with respect to Lee and generalized Lee weight.

Let $m_1$ and $m_2$ be two different size of block repetition code $C_2$ is define in $\mathbb{R}$, $BRep^{m_1 + m_2} : [m_1 + m_2, 1, \min\{2m_2, 2m_1 + 2m_2\}, \min\{4m_1, 3m_1 + 3m_2\}]$ generated by $G_2 = \left[ \begin{array}{ccc} m_1 & \cdot & \cdot \\ m_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \right]$ and also obtain the following theorem.

Theorem 6. In $\mathbb{R}$, let $C_2$ be a code and $G_2$ is the generator matrix of $C_2$. Then the covering radius of code $C_2$ is $\left( \frac{3m_1}{2} + 2m_2 \right)$ and $\left( 2m_1 + 2m_2 \right)$ with respect to Lee and generalized Lee weight.

Proof. Use to Corollary 5 and apply the two different size of length (That is, $m_1$ and $m_2$). $\square$

In four different block of repetition code of size $m_1$, $m_2$, $m_3$ and $m_4$ in $\mathbb{R}$, is $BRep^{m_1 + m_2 + m_3 + m_4} : [m_1 + m_2 + m_3 + m_4, 1, \min\{m_1 + m_2 + m_3 + m_4, 2(m_1 + m_2 + m_3 + m_4), \min\{2(m_1 + m_2 + m_3 + m_4), 4(m_1 + m_2 + m_3 + m_4)\}]$ generated by $G_4 = \left[ \begin{array}{ccc} m_1 & \cdot & \cdot \\ m_2 & \cdot & \cdot \\ m_3 & \cdot & \cdot \\ m_4 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \right]$. I obtain the following theorem

Theorem 7. Let $C_4$ be a code and $G_4$ is a generator matrix of $C_4$ in $\mathbb{R}$. Then

$r_L(BRep^{m_1 + m_2 + m_3 + m_4}) = \frac{3m}{2} (m_1 + m_2 + m_3 + m_4)$

and

$r_{GL}(BRep^{m_1 + m_2 + m_3 + m_4}) = 2(m_1 + m_2 + m_3 + m_4)$.

Acknowledgments: The author wishes to express his profound gratitude to the reviewers for their useful comments on the manuscript.

Conflicts of Interest: “The author declare no conflict of interest.”
References


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