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On codes over $\mathbb{R} = \mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$ where $u^3 = 0$ and its related parameters

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Abstract: In ring $\mathbb{R} = \mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$ where $u^3 = 0$, using Lee weight and generalized Lee weight, some lower bound and upper bound on the covering radius of codes is given and also to find the covering radius for various repetition codes with respect to same and different length in $\mathbb{R}$.

Keywords: Covering radius, codes over finite rings, generator matrix, generalized weight, Lee weight.

MSC: Primary 20G30, Secondary 20G35.

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1. Introduction

Codes over finite commutative rings have been studied for almost 50 years. The main motivation of studying codes over rings is that they can be associated with codes over finite fields through the Gray map. Recently, coding theory over finite commutative non-chain rings is a hot research topic and in last three decade, there are many researchers doing research on code over finite rings.

Researchers have more interest in codes over finite rings in recent years, especially the rings $\mathbb{Z}_{2k}$ where $2k$ denotes the ring of integers modulo $2k$. In [1–6], the authors have deeply studied codes over $\mathbb{F}_2 + u\mathbb{F}_2$. In gray map, binary linear and non-linear codes can be obtained from codes over $\mathbb{Z}_4$ and the covering radius of binary linear codes were studied [7,8]. Recently the covering radius of codes over $\mathbb{Z}_2 + u\mathbb{Z}_2$ has been investigated with respect to different distances [9,10]. In [11], the authors gave upper and lower bounds on the covering radius of a code over $\mathbb{Z}_4$ with different distances. In this paper, I obtain the covering radius of some particular codes over $\mathbb{R} = \mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$. I consider the finite ring $\mathbb{R} = \mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$ of integers modulo 2 in this paper.

2. Preliminaries

Let $\mathbb{R} = \mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$ be the finite ring. Consider the elements of ring are $\{0, 1, u, v, u^2, v^2, uv, v^3\}$, where $u^3 = 0$, $v = 1 + u, v^2 = 1 + u^2, v^3 = 1 + u + u^2, uv = u + u^2$.

Let $C \subseteq \mathbb{R}^n$ be a linear code of length n over $\mathbb{R}$ is an $\mathbb{R}$-submodule of $\mathbb{R}^n$. The element of C is called a *codeword* of C. The Hamming weight $wt_H(c)$ of a codeword c is the number of non-zero components. That is, $wt_H(c) = \{i|c_i \neq 0\}$. The *minimum Hamming weight* $wt_H(c) = \min\{i|c_i \neq 0\}$.

Let x_i and $y_i \in \mathbb{R}^n$, $d_H(x_i, y_i) = |\{i : x_i \neq y_i\}|$, where $i = 1, 2, \cdots, n$ is called *Hamming distance* between any distinct vectors $x, y \in \mathbb{R}^n$ and is denoted by $d_H(x, y) = wt_H(x - y)$. The minimum Hamming distance between distinct pairs of codewords of a code C is called *minimum distance* of C. It is denoted by $d_H(C) = wt_H(C)$.

The generalized Lee weight and Lee weight are the element $x \in \mathbb{R}$ is analogous to the definition of the generalized Lee weight and Lee weight of the elements of the ring $\mathbb{Z}_8$ [6,12,13].

Let $C \subseteq \mathbb{R}$ is permutation equivalent to a code C with generator matrix of the form

$$G = \begin{pmatrix} I_{k_0} & A_{01} & A_{02} & A_{03} \\ 0 & uI_{k_1} & uA_{12} & uA_{13} \\ 0 & 0 & u^2 I_{k_2} & u^2 A_{23} \end{pmatrix}$$

where A_{ij} are binary matrices for $i > 0$. A code with a generator matrix in this form is of type $\{k_0, k_1, k_2\}$ and has $8^{k_0} 4^{k_1} 2^{k_2}$ vectors[14].

In [15], the *generalized Lee weight* of the elements $x \in \mathbb{R}$ are given

$$wt_{GL}(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ 2 & \text{if} \quad x \neq u^2 \\ 4 & \text{if} \quad x = u^2 \end{cases}$$

and the *Lee weights* of the elements $0, \{1, v, v^2, v^3\}, \{u, uv\}, u^2$ of $\mathbb{R}$ are defined by $0, 1, 2, 2^2$ [13].

3. Covering Radius of Codes and Repetition Codes

Let $r_{d(C)} = \max_{u \in \mathbb{R}^n} \left\{ \min_{c \in C} \{d(u,c)\} \right\}$ be the covering of codes, where d with respect to Hamming distance, Lee distance and generalized distance.

Let $C = \{\bar{\alpha} | \alpha \in \mathbb{F}_q\}$, $\bar{\alpha} = \alpha\alpha \cdots \alpha$ be a q-ary repetition code over $\mathbb{F}_q$, with $[n, 1, n]$ code, where $\mathbb{F}_q$ is a finite field. The the covering radius of C is $\lfloor \frac{n(q-1)}{q} \rfloor$ [16].

Let $G = [\overbrace{11 \cdots 1}^{n} \overbrace{\alpha_2\alpha_2 \cdots \alpha_2}^{n} \cdots \overbrace{\alpha_{q-1}\alpha_{q-1} \cdots \alpha_{q-1}}^{n}]$ be a generator of code C with $[n(q-1), 1, n(q-1)]$ repetition code of block size is n. Use [16], that the code of the covering radius C is $\lfloor \frac{n(q-1)^2}{q} \rfloor$, since it will be equivalent to a repetition code of length $(q-1)n$.

In $\mathbb{R}$, there are two types of repetition codes of length n viz.

1. unit repetition code $C_I : [n, 1, d_H = n, d_L = n, d_{GL} = n]$ generated by $G_I = [\overbrace{11 \cdots 1}^{n}]$
2. zero repetition code $C_{II} : (n, 2, d_H = n, d_L = 4n, d_{GL} = 4n)$ generated by $G_{II} = [\overbrace{u^2u^2 \cdots u^2}^{n}]$ and $C_{III} : (n, 4, d_H = n, d_L = 2n, d_{GL} = 2n)$ generated by $G_{III} = [\overbrace{u\ uv\ u\ uv \cdots u\ uv}^{n}]$ or $[\overbrace{uv\ u\ uv\ u \cdots uv\ u}^{n}]$. The code generated by $[u\ u \cdots u]$ and $[uv\ uv \cdots uv]$ are similar to the code C_{III}.

Theorem 1. *1. $r_L(C_I) = \frac{3n}{2}$,*
2. $r_L(C_{II}) = 2n$ and
3. $n \leq r_L(C_{III}) \leq 2n$.

Proof. Lee weight of $\mathbb{R} : 0 \to 0, \{1, v, v^2, v^3\} \to 1, \{u, uv\} \to 2$ and $u^2 \to 4$. If $x \in \mathbb{R}^n$ with σ_i times i's, in x and $\sum_i \sigma_i = n$ and the code $c_{i, i=0 \text{ to } 7} \in \{\alpha(C_I) | \alpha \in \mathbb{R}\}$. Then

$$\begin{aligned} d_L(x, c_0) &= wt_L(x - 00 \cdots 0) \\ &= 0\sigma_0 + 1\sigma_1 + u\sigma_2 + v\sigma_3 + u^2\sigma_4 + uv\sigma_5 + v^2\sigma_6 + v^3\sigma_7 \\ d_L(x, c_0) &= n - \sigma_0 + \sigma_2 + 3\sigma_4 + \sigma_6. \\ d_L(x, c_1) &= wt_L(x - 11 \cdots 1) \\ &= v^3\sigma_0 + 0\sigma_1 + 1\sigma_2 + u\sigma_3 + v\sigma_4 + u^2\sigma_5 + uv\sigma_6 + v^2\sigma_7 \\ d_L(x, c_1) &= n - \sigma_1 + \sigma_3 + 3\sigma_5 + \sigma_7. \end{aligned}$$

Similarly, $d_L(x, c_2) = n - \sigma_2 + \sigma_0 + \sigma_4 + 3\sigma_6$, $d_L(x, c_3) = n - \sigma_3 + \sigma_5 + 3\sigma_7 + \sigma_1$, $d_L(x, c_4) = n - \sigma_4 + 3\sigma_0 + \sigma_2 + \sigma_6$, $d_L(x, c_5) = n - \sigma_5 + \sigma_7 + 3\sigma_1 + \sigma_3$, $d_L(x, c_6) = n - \sigma_6 + \sigma_0 + 3\sigma_2 + \sigma_4$ and $d_L(x, c_7) = n - \sigma_7 + \sigma_1 + 3\sigma_3 + \sigma_5$. Therefore, $d_L(x, C_I) = \min\{d_L(x, c_0), d_L(x, c_1), d_L(x, c_2) d_L(x, c_3), d_L(x, c_4), d_L(x, c_5), d_L(x, c_6), d_L(x, c_7)\}$. Thus, $r_L(C_I) \leq \frac{3n}{2}$. [$\because$, the minimum of data is less than or equal to the average of data and $\sum_i \sigma_i = n$, implies $d_L(x, C_I) \leq n + \frac{4n}{8} = \frac{3n}{2}$.]

Let $x = \overbrace{00 \cdots 0}^{g} \overbrace{11 \cdots 1}^{g} \overbrace{uu \cdots u}^{g} \overbrace{vv \cdots v}^{g} \overbrace{u^2u^2 \cdots u^2}^{g} \overbrace{v^2v^2 \cdots v^2}^{g} \overbrace{uvuv \cdots uv}^{g} \overbrace{v^3v^3 \cdots v^3}^{n-7g} \in \mathbb{R}^n$, where $g = \lfloor \frac{n}{2^3} \rfloor$, then $d_L(x, c_0) = n + 4g$, $d_L(x, c_1) = 2n - 4g$, $d_L(x, c_2) = n + 4g$, $d_L(x, c_3) = 4n - 20g$, $d_L(x, c_4) = n + 4g$, $d_L(x, c_5) = n + 4g$, $d_L(x, c_6) = n + 4g$ and $d_L(x, c_7) = n + 4g$. $\therefore$ $r_L(C_I) \geq \min\{n + 4g, 2n - 4g, 4n - 20g\} = n + 4g \geq \frac{3n}{2}$. Thus $r_L(C_I) = \frac{3n}{2}$.

Let $x = \overbrace{u^2u^2 \cdots u^2}^{\frac{n}{2}} \overbrace{000 \cdots 0}^{\frac{n}{2}} \in \mathbb{R}^n$. The code $C_{II} = \{\alpha(u^2u^2 \cdots u^2) \mid \alpha \in \mathbb{R}^n\}$, that is $C_{II} = \{00 \cdots 0, u^2u^2 \cdots u^2\}$, generated by $[u^2u^2 \cdots u^2]$ is an $(n, 2, 4n)$ code. Then

$d_L(x, 00\cdots 0) = wt_L(\overbrace{u^2u^2\cdots u^2}^{\frac{n}{2}}\overbrace{00\cdots 0}^{\frac{n}{2}} - 00\cdots 0) = \frac{n}{2}wt_L(u^2) = 2n$ and $d_L(x, u^2u^2\cdots u^2) = wt_L(\overbrace{u^2u^2\cdots u^2}^{\frac{n}{2}}\overbrace{00\cdots 0}^{\frac{n}{2}} - u^2u^2\cdots u^2) = \frac{n}{2}wt_L(u^2) = 2n$. Therefore, $d_L(x, C_{II}) = \min\{2n, 2n\} = 2n$. Thus, by definition of covering radius $r_L(C_{II}) \geq 2n$.

Let x in $\mathbb{R}^n$ be any codeword. Let us take x has σ_i coordinates as i's, then $\sum_i \sigma_i = n$, where $i = 0$ to 7. Since $C_{II} = \{00\cdots 0, u^2u^2\cdots u^2\}$, then $d_L(x, 00\cdots 0) = n - \sigma_0 + \sigma_2 + 3\sigma_4 + \sigma_6$ and $d_L(x, u^2u^2\cdots u^2) = n - \sigma_4 + 3\sigma_0 + \sigma_2 + \sigma_6$. Thus $d_L(x, C_{II}) = \min\{n - \sigma_0 + \sigma_2 + 3\sigma_4 + \sigma_6, n - \sigma_4 + 3\sigma_0 + \sigma_2 + \sigma_6\}$ and $d_L(x, C_{II}) \leq n + n = 2n$. Therefore, $r_L(C_{II}) \leq 2n$. Hence, $r_L(C_{II}) = 2n$. For $x = \overbrace{uu\cdots u}^{\frac{n}{2}}\overbrace{00\cdots 0}^{\frac{n}{2}} \in \mathbb{R}^n$ and the code $c_{i,\, i=0 \text{ to } 7} \in \{\alpha(C_{III}) | \alpha \in \mathbb{R}\}$ generated by $[uu\cdots u]$ is an $(n, 4, 2n)$ code. Then $d_L(x, c_0) = wt_L(\overbrace{uu\cdots u}^{\frac{n}{2}}\overbrace{000\cdots 0}^{\frac{n}{2}} - 00\cdots 0) = \frac{n}{2}wt_L(u) = n$, $d_L(x, c_1) = wt_L(\overbrace{uu\cdots u}^{\frac{n}{2}}\overbrace{000\cdots 0}^{\frac{n}{2}} - uu\cdots u) = \frac{n}{2}wt_L(uv) = n$, $d_L(x, c_2) = wt_L(\overbrace{uu\cdots u}^{\frac{n}{2}}\overbrace{000\cdots 0}^{\frac{n}{2}} - u^2u^2\cdots u^2) = \frac{n}{2}wt_L(uv) + \frac{n}{2}wt_L(u^2) = n + 2n = 3n$ and $d_L(x, c_6) = wt_L(\overbrace{uu\cdots u}^{\frac{n}{2}}\overbrace{000\cdots 0}^{\frac{n}{2}} - uv\ uv\cdots uv) = \frac{n}{2}wt_L(u^2) + \frac{n}{2}wt_L(u) = 2n + n = 3n$. Therefore, $d_L(x, C_{III}) = \min\{n, n, 3n, 3n\} = n$. Thus, $r_L(C_{III}) \geq n$.

Let x be any word in $\mathbb{R}^n$. Let us take x has σ_i coordinates as i's, with $\sum_i \sigma_i = n$. Then $d_L(x, c_0) = n - \sigma_0 + \sigma_2 + 3\sigma_4 + \sigma_6$, $d_L(x, c_1) = n - \sigma_2 + \sigma_0 + \sigma_4 + 3\sigma_6$, $d_L(x, c_2) = n - \sigma_4 + 3\sigma_0 + \sigma_2 + \sigma_6$, $d_L(x, c_3) = n - \sigma_6 + \sigma_0 + 3\sigma_2 + \sigma_4$. Thus, $d_L(x, C_{III}) = \min\{d_L(x, c_0), d_L(x, c_1), d_L(x, c_2), d_L(x, c_3)\}$. Since the minimum of $\{d_L(x, c_0), d_L(x, c_1), d_L(x, c_2), d_L(x, c_3)\}$ is less than or equal to its average and $\sigma_0 + \sigma_2 + \sigma_4 \leq n$ implies $d_L(x, C_{III}) \leq n + \frac{4n}{4} = 2n$ and $r_L(C_{III}) \leq 2n$. Hence, $n \leq r_L(C_{III}) \leq 2n$. □

Theorem 2. *1. $r_{GL}(C_I) = 2n$,*
2. $r_{GL}(C_{II}) = 2n$ and
3. $n \leq r_{GL}(C_{III}) \leq 2n$.

Proof. Let $x \in \mathbb{R}^n$ with σ_i times i's, in x and $\sum_i \sigma_i = n$, $i = 0$ to 7.

In generalized Lee weight of $\mathbb{R}$: $0 \to 0$, $\{1, u, v, v^2, uv, v^3\} \to 2$ and $u^2 \to 4$ and $c_{i,\, i=0 \text{ to } 7} \in \{\alpha(C_I) | \alpha \in \mathbb{R}\}$. Then, $d_{GL}(x, c_0) = wt_{GL}(x - c_0) = 0\sigma_0 + 1\sigma_1 + u\sigma_2 + v\sigma_3 + u^2\sigma_4 + uv\sigma_5 + v^2\sigma_6 + v^3\sigma_7 = n - \sigma_0 + \sigma_1 + \sigma_2 + \sigma_3 + 3\sigma_4 + \sigma_5 + \sigma_6 + \sigma_7$, $d_{GL}(x, c_1) = wt_{GL}(x - c_1) = v^3\sigma_0 + 0\sigma_1 + 1\sigma_2 + u\sigma_3 + v\sigma_4 + u^2\sigma_5 + uv\sigma_6 + v^2\sigma_7 = n - \sigma_1 + \sigma_0 + \sigma_2 + \sigma_3 + \sigma_4 + 3\sigma_5 + \sigma_6 + \sigma_7$. Similarly, $d_{GL}(x, c_2) = n - \sigma_2 + \sigma_0 + \sigma_4 + 3\sigma_6 + \sigma_5 + \sigma_1 + \sigma_3 + \sigma_7$, $d_{GL}(x, c_3) = n - \sigma_3 + \sigma_5 + 3\sigma_7 + \sigma_1 + \sigma_0 + \sigma_2 + \sigma_4 + \sigma_6$, $d_{GL}(x, c_4) = n - \sigma_4 + 3\sigma_0 + \sigma_2 + \sigma_6 + \sigma_1 + \sigma_3 + \sigma_5 + \sigma_7$, $d_{GL}(x, c_5) = n - \sigma_5 + \sigma_7 + 3\sigma_1 + \sigma_3 + \sigma_2 + \sigma_5 + \sigma_4 + \sigma_0$, $d_{GL}(x, c_6) = n - \sigma_6 + \sigma_0 + \sigma_1 + \sigma_3 + 3\sigma_2 + \sigma_4 + \sigma_7 + \sigma_6$ and $d_{GL}(x, c_7) = n - \sigma_7 + \sigma_1 + \sigma_2 + 3\sigma_3 + \sigma_4 + \sigma_5 + \sigma_6 + \sigma_0$. Therefore, $d_{GL}(x, C_I) = \min\{d_{GL}(x, c_0), d_{GL}(x, c_1), d_{GL}(x, c_2), d_{GL}(x, c_3), d_{GL}(x, c_4), d_{GL}(x, c_5), d_{GL}(x, c_6), d_{GL}(x, c_7)\}$. Thus, $d_{GL}(x, C_I) \leq n + \frac{8n}{8} = 2n$, for all $x \in \mathbb{R}^n$ and the minimum of data is less than or equal to the average of data and $\sum_i \sigma_i = n$, $i = 0$ to 7. Hence, $r_{GL}(C_I) \leq 2n$.

Let $y = \overbrace{00\cdots 0}^{g}\overbrace{11\cdots 1}^{g}\overbrace{uu\cdots u}^{g}\overbrace{vv\cdots v}^{g}\overbrace{u^2u^2\cdots u^2}^{g}\overbrace{v^2v^2\cdots v^2}^{g}\overbrace{uv\ uv\cdots uv}^{g}\overbrace{v^3v^3\cdots v^3}^{n-7g} \in \mathbb{R}^n$, where $g = \lfloor \frac{n}{2^3} \rfloor$, then $d_{GL}(y, c_0) = 2n$, $d_{GL}(y, c_1) = 2n$, $d_{GL}(y, c_2) = 2n$, $d_{GL}(y, c_3) = 2n$, $d_{GL}(y, c_4) = 2n$, $d_{GL}(y, c_5) = 2n$, $d_{GL}(y, c_6) = 2n$ and $d_{GL}(y, c_7) = 2n$. Therefore, $r_{GL}(C_I) \geq \min\{2n, 2n, 2n\} = 2n \geq 2n$. Thus, $r_{GL}(C_I) = 2n$.

Let $x = \overbrace{u^2u^2\cdots u^2}^{\frac{n}{2}}\overbrace{000\cdots 0}^{\frac{n}{2}} \in \mathbb{R}^n$. The code $C_{II} = \{\alpha(u^2u^2\cdots u^2) \mid \alpha \in \mathbb{R}^n\}$, that is $C_{II} = \{00\cdots 0, u^2u^2\cdots u^2\}$, generated by $[u^2u^2\cdots u^2]$. The parameter of C_{II} is $(n, 2, 4n)$ code. Then $d_{GL}(x, 00\cdots 0) = wt_{GL}(\overbrace{u^2u^2\cdots u^2}^{\frac{n}{2}}\overbrace{00\cdots 0}^{\frac{n}{2}} - 00\cdots 0) = \frac{n}{2}wt_{GL}(u^2) = 2n$ and $d_{GL}(x, u^2\cdots u^2) = wt_{GL}(\overbrace{u^2u^2\cdots u^2}^{\frac{n}{2}}\overbrace{00\cdots 0}^{\frac{n}{2}} - u^2u^2\cdots u^2) = \frac{n}{2}wt_{GL}(u^2) = 2n$. Therefore, $d_{GL}(x, C_{II}) = \min\{2n, 2n\} = 2n$. Thus, by definition of covering radius $r_{GL}(C_{II}) \geq 2n$.

For any word x in $\mathbb{R}^n$ and take x has σ_i coordinates as i's, with $\sum_i \sigma_i = n$, where $i = 1$to 7. Since $C_{II} = \{c_0 = 00\cdots 0, c_{u^2} = u^2u^2\cdots u^2\}$, then $d_{GL}(x, c_0) = wt_{GL}(x - 00\cdots 0) = 0\sigma_0 + 1\sigma_1 + u\sigma_2 + v\sigma_3 + u^2\sigma_4 + uv\sigma_5 + v^2\sigma_6 + v^3\sigma_7 = n - \sigma_0 + \sigma_1 + \sigma_2 + \sigma_3 + 3\sigma_4 + \sigma_5 + \sigma_6 + \sigma_7$ and $d_{GL}(x, c_{u^2}) = wt_{GL}(x - u^2u^2\cdots u^2) = u^2\sigma_0 + 1\sigma_1 + u\sigma_2 + 1\sigma_3 + 0\sigma_4 + 1\sigma_5 + u\sigma_6 + 1\sigma_7 = n - \sigma_4 + \sigma_1 + \sigma_2 + \sigma_3 + 3\sigma_0 + \sigma_5 + \sigma_6 + \sigma_7$. Thus $d_{GL}(x, C_{II}) = \min\{d_{GL}(x, c_0), d_{GL}(x, c_{u^2})\}$. Since the minimum of $\{d_{GL}(x, c_0), d_{GL}(x, c_{u^2})\} \leq n$, then $d_{GL}(x, C_{II}) \leq 2n$ and $r_{GL}(C_{II}) \leq 2n$. Hence, $r_{GL}(C_{II}) = 2n$.

If $x = \overbrace{uu\cdots u}^{\frac{n}{2}}\overbrace{00\cdots 0}^{\frac{n}{2}} \in \mathbb{R}^n$. Then $d_{GL}(x, c_0) = wt_{GL}(\overbrace{uu\cdots u}^{\frac{n}{2}}\overbrace{000\cdots 0}^{\frac{n}{2}} - 00\cdots 0) = \frac{n}{2}wt_{GL}(u) = n$, $d_{GL}(x, c_1) = wt_{GL}(\overbrace{uu\cdots u}^{\frac{n}{2}}\overbrace{000\cdots 0}^{\frac{n}{2}} - uu\cdots u) = \frac{n}{2}wt_{GL}(uv) = n$, $d_{GL}(x, c_2) = wt_{GL}(\overbrace{uu\cdots u}^{\frac{n}{2}}\overbrace{000\cdots 0}^{\frac{n}{2}} - u^2u^2\cdots u^2) = \frac{n}{2}wt_{GL}(uv) + \frac{n}{2}wt_{GL}(u^2) = n + 2n = 3n$, $d_{GL}(x, c_3) = wt_{GL}(\overbrace{uu\cdots u}^{\frac{n}{2}}\overbrace{000\cdots 0}^{\frac{n}{2}} - uv\ uv\cdots uv) = \frac{n}{2}wt_{GL}(u^2) + \frac{n}{2}wt_{GL}(u) = 2n + n = 3n$. Therefore, $d_{GL}(x, C_{III}) = \min\{n, n, 3n, 3n\} = n$. Thus, by definition of covering radius $r_{GL}(C_{III}) \geq n$.

Let $x \in \mathbb{R}^n$ be any codeword and take x has σ_i coordinates as i's, then $\sum_i \sigma_i = n$ and $c_{i, i = 0 \text{ to } 7(v^3)} \in \{\alpha(C_{III}) | \alpha \in \mathbb{R}\}$. Then $d_{GL}(x, c_0) = n - \sigma_0 + \sigma_1 + \sigma_2 + \sigma_3 + 3\sigma_4 + \sigma_5 + \sigma_6 + \sigma_7$, $d_{GL}(x, c_1) = n - \sigma_2 + \sigma_0 + \sigma_1 + \sigma_3 + \sigma_4 + \sigma_5 + 3\sigma_6 + \sigma_7$, $d_{GL}(x, c_2) = n - \sigma_4 + 3\sigma_0 + \sigma_1 + \sigma_2 + \sigma_3 + \sigma_5 + \sigma_6 + \sigma_7$ and $d_{GL}(x, c_3) = n - \sigma_6 + \sigma_0 + \sigma_1 + 3\sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 + \sigma_7$. Thus $d_{GL}(x, C_{III}) = \min\{d_{GL}(x, c_0), d_{GL}(x, c_1), d_{GL}(x, c_2), d_{GL}(x, c_3)\}$. Hence $r_{GL}(C_{III}) \leq 2n$ for, the minimum of $\{d_{GL}(x, c_0), d_{GL}(x, c_1), d_{GL}(x, c_2), d_{GL}(x, c_3)\} \leq 2n$. □

4. Block Repetition Code for Same Size in $\mathbb{R}$

In this section, give the covering radius of repetition code C with respect to Lee and generalized Lee weight.

Let $G_1 = [\overbrace{11\cdots 1}^{m}\overbrace{vv\cdots v}^{m}\overbrace{v^2v^2\cdots v^2}^{m}\overbrace{v^3v^3\cdots v^3}^{m}]$ be a generator matrix with same block size(m) of repetition code and the parameters of repetition code $BRep^{4m} : [4m, 1, 4m, 4m, 8m]$.

Theorem 3. *Let C be a code and G_1, be a generator matrix of C over $\mathbb{R}$. Then $r_L(BRep^{4m}) = 6m$ and $r_{GL}(BRep^{4m}) = 8m$.*

Proof. In Theorem 1, (Proposition(mattson) [7]) and the given generator matrix G_1, then

$$r_L(BRep^{4m}) \geq 6m \tag{1}$$

Let $x = (u_1 \mid u_2 \mid u_3 \mid u_4) \in \mathbb{R}^{4m}$ where $u_1, u_2, u_3, u_4 \in \mathbb{R}^n$. Let us take in u_1, i appears r_i times, in u_2, i appears s_i times, in u_3, i appears t_i times, in u_4, i appears v_i times, with $\sum_i r_i = \sum_i s_i = \sum_i t_i = \sum_i v_i = m$ and $c_i \in \{\alpha(G_1) | \alpha \in \mathbb{R}\}$, $i = 0$ to 7. Then $d_L(x, c_0) = 4m - r_0 + r_2 + 3r_4 + r_6 - s_0 + s_2 + 3s_4 + s_6 - t_0 + s_t + 3t_4 + t_6 - v_0 + v_2 + 3v_4 + v_6$, $d_L(x, c_1) = 4m - r_1 + r_3 + 3r_5 + r_7 - s_3 + s_5 + 3s_7 + s_1 - t_5 + t_7 + 3t_1 + t_3 - v_7 + v_1 + 3v_3 + v_5$, $d_L(x, c_2) = 4m - r_2 + r_0 + r_4 + 3r_6 - s_6 + s_0 + 3s_2 + s_4 - t_2 + t_0 + t_4 + 3t_6 - v_6 + v_0 + 3v_2 + v_4$, $d_L(x, c_3) = 4m - r_3 + r_5 + 3r_7 + r_1 - s_1 + s_3 + 3s_5 + s_7 - t_7 + t_1 + 3t_3 + t_5 - v_5 + v_7 + 3v_1 + v_3$, $d_L(x, c_4) = 4m - r_4 + 3r_0 + r_2 + r_6 - s_4 + 3s_0 + s_2 + s_6 - t_4 + 3t_0 + t_2 + t_6 - v_4 + 3v_0 + v_2 + v_6$, $d_L(x, c_5) = 4m - r_5 + r_7 + 3r_1 + r_3 - s_7 + s_1 + 3s_3 + s_5 - t_1 + t_3 + 3t_5 + t_7 - v_3 + v_5 + 3v_7 + v_1$, $d_L(x, c_6) = 4m - r_6 + r_0 + 3r_2 + r_4 - s_2 + s_0 + s_4 + 3s_6 - t_6 + t_0 + 3t_2 + t_4 - v_2 + v_0 + v_4 + 3v_6$ and $d_L(x, c_7) = 4m - r_7 + r_1 + 3r_3 + r_5 - s_5 + s_7 + 3s_1 + s_3 - t_3 + t_5 + 3t_7 + t_1 - v_1 + v_3 + 3v_5 + v_7$. Therefore, $d_L(x, BRep^{4m}) = \min\{d_L(x, c_0), d_L(x, c_1), d_L(x, c_2), d_L(x, c_3), d_L(x, c_4), d_L(x, c_5), d_L(x, c_6), d_L(x, c_7)\}$ is less than or equal to $6m$. Then $d_L(x, BRep^{4m}) \leq 6m$ and hence

$$r_L(BRep^{4m}) \leq 6n \tag{2}$$

By (1) and (2), then $r_L(BRep^{4m}) = 6m$. Similarly, the Generalized Lee weight of $\mathbb{R} : 0 \to 0$, $\{1, u, v, v^2, uv, v^3\} \to 2$ and $u^2 \to 4$. Thus, $r_{GL}(BRep^{4m}) = 8m$. □

The three block repetition code $BRep^{3n}$: $(3m, 4, 2m, 6m, 8m)$ generated by $G_2* = [\overbrace{uu\cdots u}^{m}\overbrace{u^2u^2\cdots u^2}^{m}\overbrace{uv\,uv\cdots uv}^{m}]$.

Theorem 4. *Let C be a code over $\mathbb{R}$ generated by the matrix G_2*. Then $r_L(BRep^{3m}) = 4m$ and $r_{GL}(BRep^{3m}) = 4m$.*

Proof. In Theorem 1, (Proposition(mattson) [7]) and G_2*, than $r_L(BRep^{3m}) \geq 4m$. Let $x = (u_1 \mid u_2 \mid u_3) \in \mathbb{R}^{3m}$ where $u_1, u_2, u_3 \in \mathbb{R}^m$ and $c_{i,_{i=0 \text{ to } v^3}} \in \{\alpha(G_2)|\alpha \in \mathbb{R}\}$. Then, $d_L(x, c_0) = 3m - r_0 + r_2 + 3r_4 + r_6 - s_0 + s_2 + 3s_4 + s_6 - t_0 + s_t + 3t_4 + t_6$, $d_L(x, c_1) = 3m - r_2 + r_0 + r_4 + 3r_6 - s_4 + 3s_0 + s_2 + s_6 - t_6 + t_0 + 3t_2 + t_4$, $d_L(x, c_2) = 3m - r_4 + 3r_0 + r_2 + r_6 - s_0 + s_2 + 3s_4 + s_6 - t_4 + 3t_0 + t_2 + t_6$, $d_L(x, c_3) = 3m - r_6 + r_0 + 3r_2 + r_4 - s_4 + 3s_0 + s_2 + s_6 - t_2 + t_0 + t_4 + 3t_6$. Therefore, $d_L(x, BRep^{3m}) \leq 4m$ and $r_L(BRep^{3m}) \leq 4m$. Hence $r_L(BRep^{3m}) = 4m$. Similarly, the Generalized Lee weight of $\mathbb{R}$: 0 is 0, $\{1, u, v, v^2, uv, v^3\}$ is 2 and u^2 is 4. Then, $r_{GL}(BRep^{3n}) = 4m$. □

Corollary 5. *Let C_i be a code over $\mathbb{R}$ and the G_i be the generator matrices of code C_i, $i = 1, 2, 3$. Then*

1. $G_1 = [\overbrace{11\cdots 1}^{m}\overbrace{u^2u^2\cdots u^2}^{m}]$, *prove that* $r_L(BRep^{2m}) = \frac{7m}{2}$ *and* $r_{GL}(BRep^{2m}) = 4m$.
2. $G_2 = [\overbrace{11\cdots 1}^{m}\overbrace{uu\cdots u}^{m}]$, *show that* $r_L(BRep^{2m}) = \frac{5m}{2}$ *and* $r_{GL}(BRep^{2m}) = 3m$.
3. $G_3 = [\overbrace{11\cdots 1}^{m}\overbrace{uu\cdots u}^{m}\overbrace{vv\cdots v}^{m}\overbrace{u^2u^2\cdots u^2}^{m}\overbrace{v^2v^2\cdots v^2}^{m}\overbrace{uv\,uv\cdots uv}^{m}\overbrace{v^3v^3\cdots v^3}^{m}]$, *to find* $r_L(BRep^{7m}) = 10m$ *and* $r_{GL}(BRep^{7m}) = 12m$.

Proof. In Theorem 1 and Theorem 2. □

5. Different size of blocks repetition code

In this section, only detail that the covering radius for different size of blocks repetition code with respect to Lee and generalized Lee weight.

Let m_1 and m_2 be two different size of block repetition code C_2 is define in $\mathbb{R}$, $BRep^{m_1+m_2}$: $[m_1 + m_2, 1, \min\{2m_1, 2m_1 + 2m_2\}, \min\{4m_1, 3m_1 + 3m_2\}]$ generated by $G_2 = [\overbrace{11\cdots 1}^{m_1}\overbrace{u^2u^2\cdots u^2}^{m_2}]$ and also obtain the following theorem.

Theorem 6. *In $\mathbb{R}$, let C_2 be a code and G_2 is the generator matrix of C_2. Then the covering radius of code C_2 is $(\frac{3m_1}{2} + 2m_2)$ and $(2m_1 + 2m_2)$ with respect to Lee and generalized Lee weight.*

Proof. Use to Corollary 5 and apply the two different size of length (That is, m_1 and m_2). □

In four different block of repetition code of size m_1, m_2, m_3 and m_4 in $\mathbb{R}$, is $BRep^{m_1+m_2+m_3+m_4}$: $[m_1 + m_2 + m_3 + m_4, 1, \min\{(m_1 + m_2 + m_3 + m_4), 2(m_1 + m_2 + m_3 + m_4)\}, \min\{2(m_1 + m_2 + m_3 + m_4), 4(m_1 + m_2 + m_3 + m_4)\}]$ generated by

$$G_4 = [\overbrace{11\cdots 1}^{m_1}\overbrace{vv\cdots v}^{m_2}\overbrace{v^2v^2\cdots v^2}^{m_3}\overbrace{v^3v^3\cdots v^3}^{m_4}].$$

I obtain the following theorem

Theorem 7. *Let C_4 be a code and G_4 is a generator matrix of C_4 in $\mathbb{R}$. Then*

$$r_L(BRep^{m_1+m_2+m_3+m_4}) = \frac{3m}{2}(m_1 + m_2 + m_3 + m_4)$$

and

$$r_{GL}(BRep^{m_1+m_2+m_3+m_4}) = 2(m_1 + m_2 + m_3 + m_4).$$

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References

[1] Bachoc, C. (1997). Applications of coding theory to the construction of modular lattices. *Journal of Combinatorial Theory, Series A, 78*(1), 92-119.

[2] Bonnecaze, A., & Udaya, P. (1999). Cyclic codes and self-dual codes over $F_2 + uF_2$,. *IEEE Transactions on Information Theory, 45*(4), 1250-1255.

[3] Bonnecaze, A., & Udaya, P. (1999). Decoding of cyclic codes over $F_2 + uF_2$,. *IEEE Transactions on Information Theory, 45*(6), 2148-2156.

[4] Dougherty, S. T., Gaborit, P., Harada, M., & Solé, P. (1999). Type II Codes Over $F_2 + uF_2$,. *IEEE Transactions on Information Theory, 45*(1), 32-45.

[5] Al-Ashker, & Al-Ashker, M. (2003). Linear codes over the ring $F_2 + uF_2$. *Islamic University Journal for natural Science,11*(2), 53-68.

[6] Sadek, S., El-Atrash, M., & Naji, A. Codes of Constant Lee or Euclidean weight over the ring $F_2 + uF_2$. *Journal of Al-Aqsa University of Gaza*(To appear).

[7] Cohen, G., Karpovsky, M., Mattson, H., & Schatz, J. (1985). Covering radius—Survey and recent results. *IEEE Transactions on Information Theory, 31*(3), 328-343.

[8] Cohen, G. E. R., Lobstein, A., & Sloane, N. (1986). Further results on the covering radius of codes. *IEEE Transactions on Information Theory, 32*(5), 680-694.

[9] Pandian, P. C., & Durairajan, C. (2014). On the covering radius of some code over $R = \mathbb{Z}_2 + u\mathbb{Z}_2$,, where $u^2 = 0$. *International Journal of Research in Applied, 2*(1), 61-70.

[10] Pandian, P. C. (2017). On Covering Radius of Codes Over $R = \mathbb{Z}_2 + u\mathbb{Z}_2$, where $u^2 = 0$, Using Bachoc Distance. *International Journal of Mathematics And its Applications, 5*(4-C), 277-282.

[11] Aoki, T., Gaborit, P., Harada, M., Ozeki, M., & Solé, P. (1999). On the covering radius of $\mathbb{Z}_4$ codes and their lattices. *IEEE Transactions on Information Theory,45*(6), 2162-2168.

[12] Pandian, P. C. (2018). On codes over $\mathbb{Z}_{2^3}$ and its covering radius for Lee weight and Homogeneous weight. *Journal of Information and Optimization Sciences, 39*(8), 1705-1715.

[13] Ling, S., & Blackford, J. T. (2002).$\mathbb{Z}_{p^k+1}$-Lineer Codes. *IEEE Transactions on Information Theory, 45*(9), 2592-2605.

[14] Qian, J., Zhang, L., & Yin, Z. (2006, October). Type II code over $F_2 + uF_2 + u2F_2$. In *2006 IEEE Information Theory Workshop-ITW'06 Chengdu* (pp. 21-23). IEEE.

[15] Al-Ashker, M. M. (2005). Simplex codes over the ring $\sum_{n=0}^{s} u^n F_2$. *Turkish Journal of Mathematics, 29*(3), 221-234.

[16] Durairajan, C. (1996). *On Covering Codes and Covering radius of some optimal codes.* (Doctoral dissertation, Ph. D. Thesis, Department of Mathematics, IIT Kanpur).