Lacunary series expansions in hyperholomorphic $F^\alpha_G(p,q,s)$ spaces

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Abstract: In this paper we define a new class of hyperholomorphic functions, which is known as $F^\alpha_G(p,q,s)$ spaces. We characterize hyperholomorphic functions in $F^\alpha_G(p,q,s)$ space in terms of the Hadamard gap in Quaternion analysis.

Keywords: Quaternionic analysis, $F^\alpha_G(p,q,s)$ spaces, lacunary Series.

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1. Introduction

Quaternions were introduced for the first time by William Rowan Hamilton in 1843 [1]. The generalizations of the theory of holomorphic functions in one complex variable is known as Quaternion analysis [2–5]. Quaternions are also recognized as a powerful tool for modeling and solving problems in theoretical as well as applied mathematics [6]. The emergence of a large of software packages to perform computations in the algebra of the real quaternions [7], or more generally, Clifford algebra has been enhanced by the increasing interest in using quaternions and their applications in almost all applied sciences [8,9].

Definition 1. Let $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$ and let $f$ be an analytic function in $D$. If

$$
\|f\|_{F^\alpha(p,q,s)} = \sup_{a \in D} \int_D |f'(z)|^p (1 - |z|^2)^q \zeta^s(z,a) dA(z) < \infty,
$$

then $f \in F(p,q,s)$. Moreover, if

$$
\lim_{|a| \to 1} \int_D |f'(z)|^p (1 - |z|^2)^q \zeta^s(z,a) dA(z) = 0,
$$

then $f \in F_0(p,q,s)$.

To introduce the meaning of hyperholomorphic functions, let $\mathbb{H}$ be the skew field of quaternions. The element $w \in \mathbb{H}$ can be written in the form:

$$
w = w_0 + w_1i + w_2j + w_3k, \quad w_0, w_1, w_2, w_3 \in \mathbb{R},
$$

where $1, i, j, k$ are the basis elements of $\mathbb{H}$. For these elements we have the multiplication rules

$$
i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, kj = -jk = i, \quad ki = -ik = j.
$$

The conjugate element $\bar{w}$ is given by $\bar{w} = w_0 - w_1i - w_2j - w_3k$, and we have the property

$$
ww = \bar{w} = \|w\|^2 = w_0^2 + w_1^2 + w_2^2 + w_3^2.
$$
Moreover, we can identify each vector $\vec{x} = (x_0, x_1, x_2) \in \mathbb{R}^3$ with a quaternion $x$ of the form

$$x = x_0 + x_1 i + x_2 j.$$ 

We will work in the unit ball in the real three-dimensional space, $\mathbb{B}_1(0) \subset \mathbb{R}^3$. We will consider functions $f$ defined on $\mathbb{B}_1(0)$ with values in $\mathbb{H}$. We define a generalized Cauchy-Riemann operator $D$ and its conjugate $\overline{D}$ by

$$Df = \frac{\partial f}{\partial x_0} + i \frac{\partial f}{\partial x_1} + j \frac{\partial f}{\partial x_2},$$

and

$$\overline{D}f = \frac{\partial f}{\partial x_0} - i \frac{\partial f}{\partial x_1} - j \frac{\partial f}{\partial x_2}.$$ 

For these operators, we have

$$D \overline{D} = \overline{D} D = \Delta_3,$$

where $\Delta_3$ is the Laplacian for functions defined over domains in $\mathbb{R}^3$. We denote by $\varphi_a(x) = (a - x)(1 - \overline{a}x)^{-1}$, $|a| < 1$, the Möbius transform, which maps the unit ball onto itself.

Let $g(x, a) = \frac{1}{4\pi} \left( \frac{1}{|\varphi_a(x)|} - 1 \right)$ be the modified fundamental solution of the Laplacian in $\mathbb{R}^3$. Let $f : \mathbb{B} \mapsto \mathbb{H}$ be a hyperholomorphic function. Then [4]:

- $B(f) = \sup_{x \in \mathbb{B}} (1 - |x|^2)^{3/2} |Df(x)|$,
- $Q_p(f) = \sup_{a \in \mathbb{B}} \int_\mathbb{B} |Df(x)|^p g^p(x, a) d\mathbb{B}_x$.

**Definition 2.** Let $0 < \alpha < \infty$. The hyperholomorphic $\alpha$-Bloch space is defined as follows (see [2]):

$$B^\alpha = \{ f \in \ker D : \sup_{x \in \mathbb{B}} (1 - |x|^2)^{3\alpha} |Df(x)| < \infty \}.$$ 

The little $\alpha$-Bloch type space $B^\alpha_0$ is a subspace of $B^\alpha$ consisting of all $f \in B^\alpha$ such that

$$\lim_{|x| \to 1} (1 - |x|^2)^{3\alpha} |Df(x)| = 0.$$ 

**Definition 3.** ([10]) Let $f$ be quaternion-valued function in $\mathbb{B}$. For $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. If

$$\|f\|_{F(p,q,s)}^p = \sup_{a \in \mathbb{B}} \int_\mathbb{B} |Df(x)|^p (1 - |x|^2)^{3q} \left( 1 - |\varphi_a(x)|^2 \right)^{s} d\mathbb{B}_x < \infty,$$

then $f \in F(p,q,s)$. Moreover, if

$$\lim_{|a| \to 1} \int_\mathbb{B} |Df(x)|^p (1 - |x|^2)^{3q} \left( 1 - |\varphi_a(x)|^2 \right)^{s} d\mathbb{B}_x = 0,$$

then $f \in F_0(p,q,s)$.

The green function in $\mathbb{R}^3$ is defined as (see [11]):

$$G(x, a) = \frac{1 - |\varphi_a(x)|^2}{|1 - \overline{a}x|}.$$ 

We introduce following new definition of so called hyperholomorphic $F^\alpha_C(p,q,s)$ spaces.

**Definition 4.** Let $1 < \alpha$, $p < \infty$, $-2 < q < \infty$, and $s > 0$. Assume that $f$ be hyperholomorphic function in the unit ball $\mathbb{B}_1(0)$. Then, $f \in F^\alpha_C(p,q,s)$, if
\[ F^p_G(s, q, s) = \left\{ f \in \ker D : \sup_{a \in B_1(0)} \int_{B_1(0)} |Df(x)|^p (1 - |x|^2)^{3d+2s} (G(x, a))^s d\mathbb{S}_x < \infty \right\}. \]

The space \( F^p_{G,q}(s, q, s) \) is a subspace of \( F^p_G(s, q, s) \) consisting of all functions \( f \in F^p_G(s, q, s) \), such that
\[
\lim_{|a| \to 1^-} \int_{B_1(0)} |Df(x)|^p (1 - |x|^2)^{3d+2s} (G(x, a))^s d\mathbb{S}_x = 0.
\]

Our objective in this article is twofold. First, we study the generalized quaternion space \( F^p_G(s, q, s) \) and characterize their relations to the quaternion \( B^p_G \). Second, characterizations \( F^p_G(s, q, s) \) function space by the coefficients of Hadamard gap expansions. The following lemma, we will need in the sequel:

**Lemma 5.** [12] Let \( 0 < R < 1, 1 < q, a \in B_1(0) \) and \( f : B_1(0) \to \mathbb{H} \) be a hyperholomorphic function. Then
\[
|Df(a)|^q \leq \frac{3 \cdot 4^{2+q}}{\pi R^3 (1 - R^2)^{2q} (1 - |a|^2)^q} \int_{\mathcal{M}(a, R)} |Df(x)|^q d\mathbb{S}_x.
\]

**2. Power series expansions in \( \mathbb{R}^3 \)**

The major difference to power series in the complex case consists in the absence of regularity of the basic variable \( x = x_0 + x_1 i + x_2 j \) and of all of its natural powers \( x^n, n = 2, 3, \ldots \). This means that we should expect other types of terms, which could be designated as generalized powers. We use a pair \( y = (y_1, y_2) \) of two regular variables given by
\[
y_1 = x_1 - ix_0 \quad \text{and} \quad y_2 = x_2 - jx_0,
\]
and a multi-index \( \nu = (\nu_1, \nu_2), |\nu| = (\nu_1 + \nu_2) \) to define the \( \nu \)-power of \( y \) by a \(|\nu|\)-ary product [5,13,14].

**Definition 6.** Let \( \nu_1 \) elements of the set \( a_1, \ldots, a_{|\nu|} \) be equal to \( y_1 \) and \( \nu_2 \) elements be equal to \( y_2 \). Then the \( \nu \)-power of \( y \) is defined by
\[
y := \frac{1}{|\nu|!} \sum_{(i_1, \ldots, i_{|\nu|}) \in \pi(1, \ldots, |\nu|)} a_{i_1} y_{i_2} \cdots a_{i_{|\nu|}},
\]
where the sum runs over all permutations of \((1, \ldots, |\nu|)\).

The general form of the Taylor series of left monogenic functions in the neighborhood of the origin is given as [14]:

\[
P(y) := \sum_{n=0}^{\infty} \left( \sum_{|\nu| = n} y^\nu c_{\nu} \right), \quad c_{\nu} \in \mathbb{H}.
\]

**Theorem 7.** [5,15] Let \( g(x) \) be left hyperholomorphic with the Taylor series (2). Then

\[
\left| \frac{1}{2} Dg(x) \right| \leq \sum_{n=1}^{\infty} n \left( \sum_{|\nu| = n} |c_{\nu}| \right) |x|^{n-1}.
\]

We introduce the notation \( H_n(x) := \sum_{|\nu| = n} y^\nu c_{\nu} \) and consider monogenic functions composed by \( H_n(x) \) in the following form:
\[
f(x) = \sum_{n=0}^{\infty} H_n(x)b_n, \quad b_n \in \mathbb{H}.
\]

Using (3), we have
\[
\left| \frac{1}{2} Df(x) \right| \leq \sum_{n=1}^{\infty} n \left( \sum_{|\nu| = n} |c_{\nu}| \right) |b_n||x|^{n-1}.
\]
This is the motivation for another notation,

$$a_n := \left( \sum_{|v|=n} |c_v| \right) |b_n| \quad (a_n \geq 0),$$  \tag{5}

finally, we have

$$\left| \frac{1}{2} \partial f(x) \right| \leq \sum_{n=1}^{\infty} na_n |x|^{n-1}. \quad \tag{6}$$

3. Lacunary series expansions in $F_{G}^{\alpha}(p, q, s)$ spaces

In this section, we give a sufficient and necessary condition for the hyperholomorphic function $f$ on $\mathbb{B}_1(0)$ of $\mathbb{R}^3$ with Hadamard gaps to belong to the weighted hyperholomorphic $F_{G}^{\alpha}(p, q, s)$ spaces. The function

$$f(r) = \sum_{k=0}^{\infty} a_k r^k \quad (n_k \in \mathbb{N}; \forall k \in \mathbb{N}) \quad \tag{7}$$

belong to the Hadamard gap class (Lacunary series) if there exists a constant $\lambda > 1$ such that $\frac{n_{k+1}}{n_k} \geq \lambda$, $\forall k \in \mathbb{N}$. Characterizations in higher dimensions using several complex variables and quaternion sense [16–18].

**Theorem 8.** Let $f(r) = \sum_{n=1}^{\infty} a_n r^n$, with $a_n \geq 0$. If $a > 0$, $p > 0$. Then

$$\int_0^1 (1 - r)^{a-1} (f(r))^p \, dr \approx \sum_{n=0}^{\infty} 2^{-n_k} t_n^p \quad \tag{8}$$

where $t_n = \sum_{k \in I_n} a_k$, $n \in \mathbb{N}$, $I_n = \{k: 2^n \leq k < 2^{n+1}; k \in \mathbb{N}\}$.

**Proof.** The prove of this theorem can be obtained easily from Theorem 2.5 of [19] with the same steps. \qed

**Theorem 9.** Let $\alpha$, $p \geq 1$, $-2 < q < \infty$, $s > 0$, and $I_n = \{k: 2^n \leq k < 2^{n+1}; k \in \mathbb{N}\}$. Suppose that $f(x) = \sum_{n=0}^{\infty} H_n(x)b_n$, $b_n \in \mathbb{H}$, where $H_n(x)$ be homogenous hyperholomorphic polynomials of degree $n$, and let $a_n$ be define as before in (5). If

$$\sum_{n=0}^{\infty} 2^{-n(\frac{1}{2}a+q+s+1)} \left( \sum_{k \in I_n} |a_k|^p \right)^p < \infty, \quad \tag{9}$$

then

$$\sup_{a \in \mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} \left| \frac{1}{2} \partial f(x) \right|^p \left( 1 - |x|^2 \right)^{\frac{3s+2}{2}} (G(x,a))^s d\mathbb{B}_x < \infty, \quad \tag{10}$$

and $f \in F_{G}^{\alpha}(p, q, s)$.

**Proof.** Suppose that (9) holds. Using the equality

$$G(x,a) = \frac{1 - |q_a(x)|^2}{|1 - \pi x|} = \frac{(1 - |a|^2)(1 - |x|^2)}{|1 - \pi x|^3}, \quad \tag{11}$$

where

$$1 - |x| \leq |1 - \pi x| \leq 1 + |x|, \quad 1 - |a| \leq |1 - \pi x| \leq 1 + |a| \leq 2. \quad \tag{12}$$

Then, we get
\[ \int_{B_1(0)} \left| \frac{1}{2} D f(x) \right|^p \left( 1 - |x|^2 \right)^{3q_0 + 2s} (G(x, a))^s d\mathbb{B}_x \]

\[ = \int_{B_1(0)} \left| \frac{1}{2} D \left( \sum_{n=0}^{\infty} H_n(x) b_n \right) \right|^p \left( 1 - |x|^2 \right)^{3q_0 + 2s} (1 - |a|^2)^s (1 - |x|^2)^s d\mathbb{B}_x \]

\[ \leq \int_{B_1(0)} \left( \sum_{n=0}^{\infty} n a_n x^{n-1} \right)^p \left( 1 - |x|^2 \right)^{3q_0 + 2s} (1 - |a|^2)^s (1 - |x|^2)^s d\mathbb{B}_x \]

\[ \leq 2^{3q_0 + 4s} \int_0^1 \left( \sum_{n=0}^{\infty} n a_n r^{n-1} \right)^p (1 - r)^{3q_0 + 2s} r^2 dr \]

\[ \leq \lambda \int_0^1 \left( \sum_{n=0}^{\infty} n a_n r^{n-1} \right)^p (1 - r)^{3q_0 + s} dr. \quad (13) \]

Using Theorem 8 in (13), we deduced that

\[ \int_{B_1(0)} \left| \frac{1}{2} D f(x) \right|^p \left( 1 - |x|^2 \right)^{3q_0 + 2s} (G(x, a))^s d\mathbb{B}_x \leq \lambda \int_0^1 \left( \sum_{n=0}^{\infty} n a_n r^{n-1} \right)^p (1 - r)^{3q_0 + s} dr \]

\[ \leq \lambda \sum_{n=0}^{\infty} 2^{-n(3q_0 + s + 1)} t_n^p. \quad (14) \]

Since

\[ t_n = \sum_{k \in I_n} k a_k < 2^{n+1} \sum_{k \in I_n} a_k, \]

we obtain that,

\[ \int_{B_1(0)} \left| \frac{1}{2} D f(x) \right|^p \left( 1 - |x|^2 \right)^{3q_0 + 2s} (G(x, a))^s d\mathbb{B}_x \leq \lambda_1 \sum_{n=0}^{\infty} 2^{-n(\frac{3q_0}{2} + s + 1)} (\sum_{k \in I_n} |a_k|)^p. \]

Therefore, we have

\[ \| f \|_{F^q_G(p,q,s)} \leq \lambda_1 \sum_{n=0}^{\infty} 2^{-n(\frac{3q_0}{2} + s + p + 1)} (\sum_{k \in I_n} |a_k|)^p < \infty, \]

where \( \lambda_1 \) is a constant. Then, the last inequality implies that \( f \in F^q_G(p,q,s) \) and the proof of our theorem is completed. \( \square \)

For the converse of Theorem 9, we consider the following theorem.

**Proposition 10.** (see [5]) Let \( a = (a_1, a_2), a_1 \in \mathbb{R}, i = 1, 2 \) be the vector of real coefficients defining \( H_{n,a}(x) = (y_1 a_1 + y_2 a_2)^n \). Suppose that \( |a|^2 = a_1^2 + a_2^2 \neq 0 \). Then,

\[ \| H_{n,a} \|_{L_p(\partial B_1)} = 2\pi \sqrt{\pi} |a|^n \frac{\Gamma(\frac{n}{2} + 1)}{\Gamma(\frac{n}{2} + \frac{1}{p})}, \quad \text{where } 0 < p < \infty. \quad (15) \]

Moreover, we have (see [5])

\[ \frac{\| - \frac{1}{2} D H_{n,a} \|_{L_p(\partial B_1)}}{\| H_{n,a} \|_{L_p(\partial B_1)}} = n^p \frac{B \left(\frac{1}{2}, \frac{n-1}{2} p + 1\right)}{B \left(\frac{1}{2}, \frac{n-1}{2} p + 1\right)} \geq \lambda n^p > 0, \quad (16) \]

where, \( B \left(\frac{1}{2}, \frac{n-1}{2} p + 1\right) = \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{n-1}{2} + 1)}{\Gamma(\frac{n}{2} + \frac{1}{2})}, \) and \( \lim_{n \to \infty} B \left(\frac{1}{2}, \frac{n-1}{2} p + 1\right) = 1. \)
Corollary 11. [5] Assume that \( p \geq 2 \), then
\[
\left\| -\frac{1}{2} \overline{\Delta} H_{n,a} \right\|_{L^2_2(\partial B_1)}^2 \geq \lambda n^{2+3q}.
\] (17)

Theorem 12. Let \( a \geq 1, 2 \leq p < \infty, -2 < q < s, > 0, \) and \( 0 < |x| = r < 1 \). If
\[
f(x) = \left( \sum_{n=0}^{\infty} \frac{H_{n,a}}{(1 - |x|^2)^{q + p/2}} a_n \right) \in F_0^p(p,q,s).
\] (18)

Then,
\[
\sum_{n=0}^{\infty} \left| \left( \sum_{k \in I_n} |a_k| \right)^2 \right| < \infty.
\] (19)

Proof. Since
\[
\|f\|_{F_0^p(p,q,s)} = \sup_{a \in B_1(0)} \int_{B_1(0)} |\overline{D} f(x)|^p (1 - |x|^2)^{q + p/2} (G(x,a))^s d\mathbb{B}_x
\]
\[
= \sup_{a \in B_1(0)} \int_{B_1(0)} |\overline{D} f(x)|^p (1 - |x|^2)^{q + p/2} \left( \frac{1 - |x|^2 (1 - |a|^2)}{1 - ax^2} \right)^s d\mathbb{B}_x
\]
\[
\geq \sup_{a \in B_1(0)} \int_{B_1(0)} |\overline{D} f(x)|^p (1 - |x|^2)^{q + p/2} d\mathbb{B}_x \quad \text{(where } a = 0). \] (20)

Hence, we have
\[
\|f\|_{F_0^p(p,q,s)} \geq \int_{B_1(0)} |\overline{D} f(x)|^p (1 - |x|^2)^{q + p/2} d\mathbb{B}_x \quad \text{(where } a = 0).
\]
\[
= \int_{B_1(0)} \left[ \frac{-\frac{1}{2} \overline{\Delta} H_{n,a}}{\|H_{n,a}\|_{L^2_2(\partial B_1)}} \right] a_n \left( 1 - |x|^2 \right)^{q + p/2} d\mathbb{B}_x. \] (21)

where \( \left[ \frac{-\frac{1}{2} \overline{\Delta} H_{n,a}}{\|H_{n,a}\|_{L^2_2(\partial B_1)}} \right] \) is a homogeneous hyperholomorphic polynomial of degree \( n-1 \) and it can be written in the form
\[
\left[ \frac{-\frac{1}{2} \overline{\Delta} H_{n,a}}{\|H_{n,a}\|_{L^2_2(\partial B_1)}} \right] = r^{n-1} \Phi_n(\phi_1, \phi_2), \] (22)

where
\[
\Phi_n(\phi_1, \phi_2) := \left[ \frac{-\frac{1}{2} \overline{\Delta} H_{n,a}}{\|H_{n,a}\|_{L^2_2(\partial B_1)}} \right]_{\partial B_1}. \] (23)

Now, by the inner product \( \langle f, g \rangle_{\partial B_1(0)} = \int_{\partial B_1(0)} f(x)g(x)d\Gamma_x \), the orthogonality of the spherical monogenic \( \Phi_n(\phi_1, \phi_2) \) (see [20]) in \( L^2_2(\partial B_1(0)) \). From (22) and (23) to (21), we have
\[
\int_{B_1(0)} \left[ \sum_{n=0}^{\infty} \frac{-\frac{1}{2} \overline{\Delta} H_{n,a}}{\|H_{n,a}\|_{L^2_2(\partial B_1)}} \right] a_n \left( 1 - |x|^2 \right)^{q + p/2} d\mathbb{B}_x
\]
\[
= \int_0^1 \int_{\partial B_1(0)} \left( \sum_{n=0}^{\infty} \frac{-\frac{1}{2} \overline{\Delta} H_{n,a}}{\|H_{n,a}\|_{L^2_2(\partial B_1)}} \Phi_n(\phi_1, \phi_2) a_n \right)^2 \frac{r^2}{r^2(1 - r^2)^{3q/2 + p}} d\Gamma_x dr
\]
\[
= \int_0^1 \int_{\partial B_1(0)} \left( \sum_{n=0}^{\infty} \frac{-\frac{1}{2} \overline{\Delta} H_{n,a}}{\|H_{n,a}\|_{L^2_2(\partial B_1)}} \Phi_n(\phi_1, \phi_2) \Phi_j(\phi_1, \phi_2) a_j \right)^2 \frac{r^2}{r^2(1 - r^2)^{3q/2 + p}} d\Gamma_x dr = L. \] (24)

From Hölder’s inequality, we have
\[
\int_{\partial B_1(0)} |f(x)|^p d\Gamma_x \geq (4\pi)^{1-p} \left| \int_{\partial B_1(0)} f(x) d\Gamma_x \right|^p, \quad \text{(where } 1 \leq p < \infty). \tag{25}
\]

From (25), for \(2 \leq p < \infty\), we have
\[
L \geq (4\pi)^{1-\frac{p}{4}} \int_0^1 \left( \sum_{n=0}^{\infty} \left| a_n \right|^2 r^{2n-2} \left\| \Phi_n(\phi_1, \phi_2) \right\|_{L^2(\partial B_1)}^2 \right)^{\frac{p}{4}} r^2 (1 - r^2)^{\frac{3s}{2} + \frac{3s}{4} + \frac{p}{4}} dr
\]
\[
\geq (4\pi)^{1-\frac{p}{4}} \int_0^1 \left( \sum_{n=0}^{\infty} \left| a_n \right|^2 r^{2n-2} \left\| \Phi_n(\phi_1, \phi_2) \right\|_{L^2(\partial B_1)}^2 \right)^{\frac{p}{4}} r^3 (1 - r^2)^{\frac{3s}{2} + \frac{3s}{4} + \frac{p}{4}} dr \tag{26}
\]

From Corollary 11, we have
\[
\left\| \Phi_n(\phi_1, \phi_2) \right\|_{L^2(\partial B_1)}^2 = \frac{\| - \frac{1}{4} \mathcal{D}H_{n, \lambda} \|_{L^2(\partial B_1)}}{\| H_{n, \lambda} \|_{L^p(\partial B_1)}} \geq \lambda n^{\frac{2+3p}{2p}} \geq \lambda n^{\frac{3}{2}}.
\]

Then, from above we have
\[
L \geq (4\pi)^{1-\frac{p}{4}} \lambda \int_0^1 \left( \sum_{n=0}^{\infty} \left| a_n \right|^2 r^{2n-2} \left\| \Phi_n(\phi_1, \phi_2) \right\|_{L^2(\partial B_1)}^2 \right)^{\frac{p}{4}} r^3 (1 - r^2)^{\frac{3s}{2} + \frac{3s}{4} + \frac{p}{4}} dr
\]
\[
\quad = \lambda_1 \int_0^1 \left( \sum_{n=0}^{\infty} \left| a_n \right|^2 r^{2n-2} \right)^{\frac{p}{4}} (1 - r^2)^{\frac{3s}{2} + \frac{3s}{4} + \frac{p}{4}} dr
\]
\[
\quad = \frac{\lambda_2}{2} \int_0^1 \left( \sum_{n=0}^{\infty} \left| a_n \right|^2 r^{2n-2} \right)^{\frac{p}{4}} \left(1 - r^2\right)^{\frac{3s}{2} + \frac{3s}{4} + \frac{p}{4}} r dr
\]
\[
\quad \geq \lambda_3 \int_0^1 \left( \sum_{n=0}^{\infty} \left| a_n \right|^2 r^{2n-2} \right)^{\frac{p}{4}} \left(1 - r^2\right)^{\frac{3s}{2} + \frac{3s}{4} + \frac{p}{4}} r dr.
\tag{27}
\]

where \(\lambda_j, j = 1, 2, 3\), are constants do not depending on \(n\).

Now, by applying Theorem 8 in equation (27), we deduced that
\[
\| f \|_{L^p_1(p,q,s)} \geq \frac{\lambda_3}{k} \sum_{n=0}^{\infty} 2^{-n(\frac{3s}{2} + \frac{3s}{4} + \frac{p}{4} + 1)} \left( \sum_{k \in I_n} k^2 |a_k|^2 \right)^{\frac{p}{4}}, \tag{28}
\]
where
\[
\sum_{k \in I_n} k^2 |a_k|^2 \geq \left( 2^{n} \right)^{\frac{2}{p}} \left( \sum_{k \in I_n} |a_k|^2 \right)^{\frac{p}{4}}.
\]

Then,
\[
\| f \|_{L^p_1(p,q,s)} \geq \frac{\lambda_3}{k} \sum_{n=0}^{\infty} 2^{-n(\frac{3s}{2} + \frac{3s}{4} + \frac{p}{4} + 1)} \left( \sum_{k \in I_n} |a_k|^2 \right)^{\frac{p}{4}},
\tag{29}
\]

From [21], we have
\[
\sum_{n=0}^{N} a_n^p \leq \left( \sum_{n=0}^{N} a_n^p \right)^p \leq N^{p-1} \sum_{n=0}^{N} a_n^p.
\]

Then, we have
\[
\| f \|_{L^p_1(p,q,s)} \geq \lambda_1 \sum_{n=0}^{\infty} 2^{-n(\frac{3s}{2} + \frac{3s}{4} + \frac{p}{4} + 1)} \left( \sum_{k \in I_n} |a_k|^2 \right)^{\frac{p}{4}}, \tag{30}
\]
where \(C_1\) be a constants which do not depend on \(n\). Then,
\[
\sum_{n=0}^{\infty} 2^{-n(\frac{3s}{2} + \frac{3s}{4} + \frac{p}{4} + 1)} \left( \sum_{k \in I_n} |a_k|^2 \right)^{p} < \infty. \tag{31}
\]
This completes the proof of theorem. □

Theorem 13. Let $\alpha \geq 1$, $2 \leq p < \infty$, $-2 < q < \infty$, and $s > 0$, then we have

$$f(x) = \left( \sum_{n=0}^{\infty} \frac{H_{n,\alpha}}{(1 - |x|^2)^{\frac{n+p}{2}}} \|H_{n,\alpha}\|_{L_p(\partial B_1)}^p a_n \right) \in F_\alpha^G(p,q,s),$$

(32)

if and only if,

$$\sum_{n=0}^{\infty} 2^{-n(\frac{3}{2}a q + s - p + 1)} \left( \sum_{k \in I_n} |a_k| \right)^p < \infty.$$ (33)

Proof. This theorem can be proved directly from Theorem 9 and Theorem 12. □

4. Conclusion

We have introduce a new class of hyperholomorphic functions, which is also called $F_\alpha^G(p,q,s)$ spaces. For this class, we give some characterizations of the hyperholomorphic $F_\alpha^G(p,q,s)$ functions by the coefficients of certain lacunary series expansions in quaternion analysis.

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References


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