Analysis of the small oscillations of a heavy barotropic gas filling an elastic body with negligible density

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Abstract: In this work, we study the small oscillations of a system formed by an elastic container with negligible density and a heavy barotropic gas (or a compressible fluid) filling the container. By means of an auxiliary problem, that requires a careful mathematical study, we deduce the problem to a problem for a gas only. From its variational formulation, we prove that is a classical vibration problem.

Keywords: Barotropic gas, small oscillations, mixed boundary conditions, vibration problem, variational and spectral methods.

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1. Introduction

The problem of the small oscillations of a heavy homogeneous inviscid liquid in an open rigid container has been the subject, from the pioneering work by Moiseyev [1], of numerous papers, that are analyzed in the books [2–4].

The same problem in the case of an elastic container is studied in the book [5]. Recently, we have solved the problem of the small oscillations of an heterogeneous liquid in an elastic container [6].

In this work, we study the problem of the small oscillations of a system formed by a heavy barotropic gas (or a compressible fluid) and an elastic body with negligible density, circumstance that can happen in the transport of fluids. At first, we establish the equations of motion of the system body-gas and the boundaries conditions. Afterwards, introducing an auxiliary problem, that requires a careful mathematical discussion, and that is the problem of the motion of the body when the motion of the gas is known, we show a linear operator depending on the elasticity of the body, that permits us to reduce the problem to a problem for the gas only. From the variational equation of this last problem, we prove that it is a classical vibration problem.

2. Position of the problem

We consider, in the field of the gravity, an elastic body with negligible density, that occupies in the equilibrium position a domain Ω' bounded by a fixed external surface S and an internal surface Σ. The interior Ω of this surface is completely filled by a heavy barotropic gas.

We choose orthogonal axes Ox₁,Ox₂,Ox₃, Ox₃ vertical directed upwards and we denote by $\vec{n}$ the unit vector normal to the surfaces. We are going to study the small oscillations of the system elastic body-gas about its equilibrium position, in the framework of the linear theory.

3. The equations of the problem

3.1. The equations of the elastic body with negligible density

Let $\vec{u}'(x_i)$ the (small) displacement of the particle of the body from the natural state to the equilibrium position. The equilibrium equations are:

$$0 = \frac{\partial \sigma_{ij}'(\vec{u}')}{\partial x_j} \quad \text{in} \quad \Omega' \quad (i,j = 1,2,3) \quad (1)$$
and the boundary conditions are
\[ \vec{u}' |_{S} = 0 ; \quad \sigma'_{ij} (\vec{u}') n_j = -p_0 n_i \quad \text{on} \quad \Sigma , \]  
(2)

where \( p_0 \) is the pressure of the gas in the equilibrium position and where we have set:
\[ \sigma'_{ij}(\vec{u}') = \lambda' \delta_{ij} \text{div} \vec{u}' + 2 \mu' \epsilon'_{ij}(\vec{u}') \quad ; \quad \epsilon'_{ij}(\vec{u}') = \frac{1}{2} \left( \frac{\partial \hat{u}'_i}{\partial x_j} + \frac{\partial \hat{u}'_j}{\partial x_i} \right) \]
\( \lambda' \) and \( \mu' \) are the Lame's coefficients; \( \sigma'_{ij}(\vec{u}') \) and \( \epsilon'_{ij}(\vec{u}') \) are the components of the stress tensor and the strain tensor respectively.

Now, let \( \vec{u}'(x_i,t) \) the displacement of a particle from its equilibrium position to its position at the instant \( t \). We have
\[ 0 = \frac{\partial \sigma'_{ij}(\vec{u}' + \vec{u}'')} \frac{\partial x_j}{\partial x_j} \quad \text{in} \quad \Omega' \]
and consequently
\[ 0 = \frac{\partial \sigma'_{ij}(\vec{u}'')} \frac{\partial x_j}{\partial x_j} \quad \text{in} \quad \Omega' , \]  
(3)

and in the same manner
\[ \vec{u}' |_{S} = 0 . \]  
(4)

Let \( \vec{u}(x_i,t) \) the displacement of a particle of the gas from its equilibrium position to its position at the instant \( t \); we must have the kinematic condition:
\[ u'_{n|\Sigma} = u_{n|\Sigma} , \]  
(5)

where we have set \( u_{n} = \vec{u} \cdot \vec{n} \).

3.2. The equations of the barotropic gas

Let \( \rho^* \), \( P \) the density and the pressure of the gas that are related by
\[ P = \mathcal{P}(\rho^*) , \]  
(6)

where \( \mathcal{P} \) is a given smooth increasing function. If \( \rho_0 \) is the density in the equilibrium position, we have
\[ p_0 = \mathcal{P}(\rho_0) \]
and the equilibrium equation
\[ \overrightarrow{\text{grad}} p_0 = -\rho_0 g \hat{x}_3 \]  
(7)

Then, \( p_0 \) and \( \rho_0 \) are functions of \( x_3 \) only and we have
\[ \frac{dp_0(x_3)}{dx_3} = \rho_0(x_3) g . \]  
(8)
Setting classically
\[ c_0^2(x_3) = \mathcal{P}'(\rho_0(x_3)) \],
we obtain
\[ c_0^2(x_3) \rho_0'(x_3) = -\rho_0(x_3) g \] .

It is a differential equation of the first order that must be verified by \( \rho_0(x_3) \). The equation of the motion of the gas are, besides (6):
\[ \rho^* \ddot{\mathbf{u}} = -\nabla P - \rho^* g \mathbf{x}_3 \quad \text{(Euler's equation)} \quad \text{in} \quad \Omega ,
\]
\[ \frac{\partial \rho^*}{\partial t} + \text{div}(\rho^* \mathbf{u}) = 0 \quad \text{(continuity equation)} \quad \text{in} \quad \Omega .
\]

Since, we study the small motions of the gas about its equilibrium position, we set
\[ \rho^* = \rho_0(x_3) + \bar{\rho}(x, t) + \cdots ,
\]
\[ P = p_0(x_3) + p(x, t) + \cdots .
\]

The \( \bar{\rho} \) and the dynamic pressure \( p \) are of the first order with respect to the amplitude of the oscillations, the dots represent terms of order greater than one. We have, at the first order
\[ \frac{\partial \bar{\rho}}{\partial t} + \text{div}(\rho_0(x_3) \mathbf{u}) = 0 ;
\]
integrating between the datum of the equilibrium position and the instant \( t \), we have
\[ \bar{\rho} = -\text{div}[\rho_0(x_3) \mathbf{u}] . \] (13)

Using (6), we have
\[ p_0(x_3) + p + \cdots = \mathcal{P}(\rho_0(x_3) + \bar{\rho} + \cdots) \]
and then
\[ p = -c_0^2(x_3) \text{div}[\rho_0(x_3) \mathbf{u}] . \] (14)

The Euler’s Equation can be written
\[ \rho_0 \ddot{\mathbf{u}} + \cdots = -\nabla \left( p_0 + p + \cdots - \rho_0 - \text{div}(\rho_0 \mathbf{u}) + \cdots \right) g \mathbf{x}_3
\]
\[ = \nabla \left( c_0^2 \text{div}(\rho_0 \mathbf{u}) \right) + g \text{div}(\rho_0 \mathbf{u}) \mathbf{x}_3 + \cdots ,
\]
and, using the equation (10), finally we get
\[ \ddot{\mathbf{u}} = \nabla \left( \frac{c_0^2(x_3)}{\rho_0(x_3)} \text{div}(\rho_0(x_3) \mathbf{u}) \right) , \] (15)
which is the equation that contains \( \mathbf{u} \) only.

3.3. The dynamic conditions on the surface \( \Sigma_t \)

Let \( M \) a point of \( \Sigma \). We denote by \( M_g, M_s \) the particles of the gas and of the elastic body that are in \( M \) at the instant \( t = 0 \). These particles come in \( M_g', M_s' \) on \( \Sigma_t \) at the instant \( t \):
\[ \overrightarrow{MM_g'} = \bar{\mathbf{u}} \quad ; \quad \overrightarrow{MM_s'} = \bar{\mathbf{u}}' \]

In linear theory, we admit that the unit vectors normal to \( \Sigma_t \) in \( M_g' \) and \( M_s' \) are equipollent to the unit vector \( \bar{\mathbf{u}} \) normal in \( M \) to \( \Sigma \) and that the pressure of the gas \( P \) in \( M_g' \) is equal to the pressure of the gas in \( M' \), intersection of \( \Sigma_t \) with the normal in \( M \) to \( \Sigma \).
The dynamic conditions on $\Sigma_t$ are

$$\sigma'_{ij}(\vec{u} + \vec{u}') n_j = -P(M', t) n_i .$$

Or, using the second condition (2):

$$\sigma'_{ij}(\vec{u}') n_j = -\left[ P(M', t) - p_0(M) \right] \cdot n_i \quad \text{on } \Sigma .$$

We have

$$P(M', t) = P(M + u_n|\Sigma \vec{n}, t) = \mathcal{P}(M, t) + \vec{\text{grad}} P(M) \cdot u_n|\Sigma \vec{n} + \cdots ,$$

Since $u_n|\Sigma$ is of the first order, we can, in linear theory, replace $\vec{\text{grad}} P(M, t)$ by

$$\vec{\text{grad}} p_0 = -\rho_0 g \vec{x}_3 ,$$

so that

$$P(M', t) = P(M, t) - \rho_0 g u_n|\Sigma n_3|\Sigma + \cdots$$

and finally

$$\sigma'_{ij}(\vec{u}') n_j = \left[ -p(M, t) + \rho_0|\Sigma g n_3|\Sigma u_n|\Sigma \right] n_i \quad \text{on } \Sigma . \quad (16)$$

Let us call $\vec{T}_t (\vec{u}')|\Sigma$ the tangential stress and $T_n (\vec{u}')|\Sigma$ the normal stress; we have

$$\vec{T}_t (\vec{u}')|\Sigma = 0 ; \quad T_n (\vec{u}')|\Sigma = -p|\Sigma + \rho_0|\Sigma g n_3|\Sigma u_n|\Sigma . \quad (17)$$

4. The auxiliary problem

Step 1.

We introduce the following auxiliary problem:

$$- \frac{\partial \sigma'_{ij}(\vec{u}')}{\partial x_j} = 0 \quad \text{in } \Omega' ; \quad \vec{u}'|_{\partial \Omega} = 0 ; \quad u'_n|\Sigma = u_n|\Sigma ; \quad \vec{T}_t (\vec{u}')|\Sigma = 0 , \quad (18)$$

where $u_n|\Sigma$ is considered as a datum. It is the problem of the motion of an elastic body when the motion of the gas is known and we seek the solution of this auxiliary problem in the space.

$$\Xi^1(\Omega') \overset{\text{def}}{=} \left\{ \vec{u} \in \Xi^1(\Omega') \overset{\text{def}}{=} \left[ H^1(\Omega') \right]^3 ; \quad \vec{u}|_{\partial \Omega} = 0 \right\} .$$
Then \( u_{n|\Sigma} \in H^{1/2}(\Sigma) \) and consequently, we suppose that \( u_{n|\Sigma} \in H^{1/2}(\Sigma) \).

**Step 2.**

Let \( \Phi \) an element of \( \mathcal{H}^1(\Omega') \) such that \( \Phi_{n|\Sigma} = u_{n|\Sigma} \in H^{1/2}(\Sigma) \).

In the following, we will see the construction of such \( \Phi \). We denote by \( V_0 \) the subspace of \( \mathcal{H}^1(\Omega') \) defined by

\[
V_0 = \left\{ \tilde{v}_0 \in \mathcal{H}^1(\Omega') \mid v_{0|\Sigma} = 0 \right\}
\]

and we seek the solution \( \tilde{u}' \) of the auxiliary problem in the form

\[
\tilde{u}' = \tilde{\Phi} + \tilde{v}_0.
\]

The auxiliary problem (18) becomes a problem for \( \tilde{u}_0 \in V_0 \):

\[
-\frac{\partial \sigma'_{ij}(\tilde{u}_0)}{\partial x_j} = \frac{\partial \sigma'_{ij}(\tilde{\Phi})}{\partial x_j} \quad \text{in} \quad \Omega' \quad ; \quad u_{0|\Sigma} = 0 \quad ; \quad \bar{T}_e(\tilde{u}_0)_{|\Sigma} = -\bar{T}_e(\tilde{\Phi})_{|\Sigma}.
\]  

(19)

Let us seek a variational formulation of this problem. We have, for each \( \tilde{v}_0 \in V_0 \):

\[
-\int_{\Omega'} \frac{\partial \sigma'_{ij}(\tilde{u}_0)}{\partial x_j} \cdot \tilde{v}_0 \, d\Omega' = \int_{\Omega'} \frac{\partial \sigma'_{ij}(\tilde{\Phi})}{\partial x_j} \cdot \tilde{v}_0 \, d\Omega'
\]

or

\[
-\int_{\Omega'} \left[ \frac{\partial}{\partial x_j} \left[ \sigma'_{ij}(\tilde{u}_0)\tilde{v}_0 \right] - \sigma'_{ij}(\tilde{\Phi})\tilde{v}_0 \right] \, d\Omega' = \int_{\Omega'} \left[ \frac{\partial}{\partial x_j} \left[ \sigma'_{ij}(\tilde{\Phi})\tilde{v}_0 \right] - \sigma'_{ij}(\tilde{\Phi})\tilde{v}_0 \right] \, d\Omega',
\]

or, using the Green’s formula and denoting by \( \bar{n}_e \), the external normal unit vector to \( \partial\Omega' \):

\[
-\int_S \sigma'_{ij}(\tilde{u}_0)n_e\tilde{v}_0 \, dS - \int_S \sigma'_{ij}(\tilde{\Phi})n_e\tilde{v}_0 \, dS + \int_{\Omega'} \sigma'_{ij}(\tilde{u}_0)\tilde{v}_0 \, d\Omega' = \int_S \sigma'_{ij}(\tilde{\Phi})n_e\tilde{v}_0 \, dS + \int_{\Omega'} \sigma'_{ij}(\tilde{\Phi})\tilde{v}_0 \, d\Omega' - \int_{\Omega'} \sigma'_{ij}(\tilde{\Phi})\tilde{v}_0 \, d\Omega'.
\]

The integrals on \( S \) disappear since \( \tilde{v}_0|_S = 0 \) and the integrals on \( \Sigma \) disappear by virtue of (19). The variational formulation of the problem for \( \tilde{u}_0 \) is to find \( \tilde{u}_0 \in V_0 \) such that

\[
\int_{\Omega'} \sigma'_{ij}(\tilde{u}_0)\tilde{v}_0 \, d\Omega' = -\int_{\Omega'} \sigma'_{ij}(\tilde{\Phi})\tilde{v}_0 \, d\Omega' \quad \forall \tilde{v}_0 \in V_0.
\]

(20)

Conversely, let \( \tilde{u}_0 \) a function of \( t \) with values in \( V_0 \) and verifying (20).

We have

\[
\int_{\Omega'} \frac{\partial \sigma'_{ij}(\tilde{u}_0)}{\partial x_j} \cdot \tilde{v}_0 \, d\Omega' = \int_{\Omega'} \left[ \frac{\partial}{\partial x_j} \left[ \sigma'_{ij}(\tilde{u}_0)\tilde{v}_0 \right] - \sigma'_{ij}(\tilde{\Phi})\tilde{v}_0 \right] \, d\Omega',
\]

and an analogous equation by replacing \( \tilde{u}_0 \) by \( \tilde{\Phi} \).

Using (20), we obtain

\[
-\int_{\Omega'} \frac{\partial \sigma'_{ij}(\tilde{u}_0)}{\partial x_j} \cdot \tilde{v}_0 \, d\Omega' + \int_{\Sigma} \sigma'_{ij}(\tilde{u}_0)n_e\tilde{v}_0 \, d\Sigma = \int_{\Omega'} \frac{\partial \sigma'_{ij}(\tilde{\Phi})}{\partial x_j} \cdot \tilde{v}_0 \, d\Omega' - \int_{\Sigma} \sigma'_{ij}(\tilde{\Phi})n_e\tilde{v}_0 \, d\Sigma.
\]

Taking \( \bar{v} \in [\mathcal{H}(\Omega')]^3 \subset V_0 \), we have

\[
-\frac{\partial \sigma'_{ij}(\tilde{u}_0)}{\partial x_j} = \frac{\partial \sigma'_{ij}(\tilde{\Phi})}{\partial x_j} \quad \text{in} \quad \mathcal{H}(\Omega').
\]
Taking into account of \( v_{0|\Sigma} = 0 \), we have

\[
\int_{\Sigma} \overline{T}_t(u_0) \cdot \check{\nu}_{0|\Sigma} \, d\Sigma = - \int_{\Sigma} \overline{T}_t(\Phi) \cdot \check{\nu}_{0|\Sigma} \, d\Sigma ,
\]

and, since \( \check{\nu}_{0|\Sigma} \) is arbitrary

\[
\overline{T}_t(u_0)_{|\Sigma} = - \overline{T}_t(\Phi)_{|\Sigma}
\]

and we find the auxiliary problem.

Let us return to its variational formulation (20). The left-hand side can be considered as a scalar product in \( V_0 \):

\[
\int_{\Omega'} c_{ij}'(\check{\nu}_0) c_{ij}'(\check{\nu}_0) \, d\Omega' = (\check{u}_0, \check{\nu}_0)_{V_0} ,
\]

The associated norm \( \| \check{u}_0 \|_{V_0} \) being classically equivalent in \( V_0 \) to the norm \( \| \check{u}_0 \|_{1} \) of \( \Xi^1(\Omega') \). Since \( \check{u}_0 \in V_0 \subset \Xi^1(\Omega') \), we have

\[
(\check{u}_0, \check{\nu}_0)_{V_0} = \int_{\Omega'} c_{ij}'(\check{u}_0) c_{ij}'(\check{\nu}_0) \, d\Omega' = (\check{u}_0, \check{\nu}_0)_{\Xi^1(\Omega')} .
\]

Setting \( \check{\nu}_0 = \check{u}_0 \), we have

\[
\| \check{u}_0 \|_{V_0} = \| \check{u}_0 \|_{\Xi^1(\Omega')} \quad \forall \check{u}_0 \in V_0 .
\]

The variational Equation (20) can be written as

\[
(\check{u}_0, \check{\nu}_0)_{V_0} = - (\hat{\Phi}, \check{\nu}_0)_{\Xi^1(\Omega')} \quad \forall \check{\nu}_0 \in V_0 . 
\]

But, we have

\[
\left| (\hat{\Phi}, \check{\nu}_0)_{\Xi^1(\Omega')} \right| \leq \| \hat{\Phi} \|_{\Xi^1(\Omega')} \| \check{\nu}_0 \|_{V_0} ,
\]

so that \( - (\hat{\Phi}, \check{\nu}_0)_{\Xi^1(\Omega')} \) is a continuous antilinear form on \( V_0 \).

Then, by the Lax-Milgram theorem, the precedent problem has one and only solution. Therefore, the problem (20) has one and one solution \( \check{u}_0 \in V_0 \) and the auxiliary problem has one and only one solution \( \check{u}' \) in \( \Xi^1(\Omega') \). The Equation (21) can be written

\[
(\check{u}', \check{\nu}_0)_{\Xi^1(\Omega')} = 0 \quad \forall \check{\nu}_0 \in V_0
\]

and the solution \( \check{u}' \) of the auxiliary problem belongs to the orthogonal of \( V_0 \) in \( \Xi^1(\Omega') \).

**Step 3.**

The solution \( \check{u}' \) of the auxiliary problem does not depend on \( \hat{\Phi} \), since \( \hat{\Phi} \) is not in the terms of the problem. We are going to use this remark for giving an estimate of \( \| \check{u}' \|_{\Xi^1(\Omega')} \).

We take, for \( \hat{\Phi} \), a continuous lifting of \( u_{n|\Sigma} \hat{\bar{n}} \) in \( \Xi^1(\Omega') \); we have

\[
\| \hat{\Phi} \|_{\Xi^1(\Omega')} \leq c \| u_{n|\Sigma} \|_{H^{1/2}(\Sigma)} \quad (c > 0).
\]

We have

\[
\left| (\check{u}_0, \check{\nu}_0)_{V_0} \right| \leq \| \hat{\Phi} \|_{\Xi^1(\Omega')} \| \check{\nu}_0 \|_{V_0}
\]

and then

\[
\| \check{u}_0 \|_{V_0} \leq \| \hat{\Phi} \|_{\Xi^1(\Omega')}
\]

and finally

\[
\| \check{u}_0 \|_{V_0} \leq c \| u_{n|\Sigma} \|_{H^{1/2}(\Sigma)}
\]
For the solution $\bar{u}'$ of the auxiliary problem, we have
\[ \bar{u}' = \bar{u}_0 + \Phi \]
and then
\[ \|\bar{u}'\|_{\mathcal{H}^1(\Omega')} \leq 2c \|u_n|\Sigma\|_{H^{1/2}(\Sigma)} \tag{22} \]

**Step 4.**
Finally, we study $T_n(\bar{u}')|\Sigma$ that is in the second dynamic condition (17) of the problem. We are going to show that it can be expressed by means of $u_n|\Sigma$. The solution $\bar{u}'$ of our problem verifies:
\[ \frac{\partial \sigma^i_j(\bar{u}')}{\partial x_j} = 0 \quad \text{in} \quad \Omega' \]

Let $\bar{v}'$ an element of $\mathbb{E}^1(\Omega')$. We have, by Green’s formula and $\bar{v}'|\partial\Omega = 0$:
\[ 0 = - \int_{\Omega'} \frac{\partial \sigma^i_j(\bar{u}')}{\partial x_j} \cdot \bar{v}' \, d\Omega' = - \int_{\Sigma} \bar{v}' n_j \bar{v}_i \, d\Sigma + \int_{\Omega'} \sigma^i_j(\bar{u}') \bar{v}'_j(\bar{v}') \, d\Omega' \]

Since the solution $\bar{u}'$ of the initial problem verifies $\bar{T}_n(\bar{u}')|\Sigma = 0$, the precedent equation gives:
\[ \int_{\Omega'} \sigma^i_j(\bar{u}') \bar{v}'_j(\bar{v}') \, d\Omega' = - \int_{\Sigma} T_n(\bar{u}')|\Sigma \bar{v}_i n_j \, d\Sigma, \quad \forall \bar{v}' \in \mathbb{E}^1(\Omega') \tag{23} \]

On the other hand, if $\bar{v}' \in [\mathcal{Q}(\Omega')]^3$, we have
\[ 0 = - \left< \frac{\partial \sigma^i_j(\bar{u}')}{\partial x_j}, \bar{v}'_i \right> = \int_{\Omega'} \sigma^i_j(\bar{u}') \bar{v}'_j(\bar{v}') \, d\Omega' \]
by virtue of the definition of the distributional derivatives. Therefore, we have
\[ \int_{\Omega'} \sigma^i_j(\bar{u}') \bar{v}'_j(\bar{v}') \, d\Omega' = 0 \quad \forall \bar{v}' \in [\mathcal{Q}(\Omega')]^3 \]
and by density
\[ \int_{\Omega'} \sigma^i_j(\bar{u}') \bar{v}'_j(\bar{v}') \, d\Omega' = 0 \quad \forall \bar{v}' \in \mathbb{E}^1(\Omega') \]

Now, we are going to particularize $\bar{v}'$. Let call $w_n|\Sigma$ a function defined on $\Sigma$ and belonging to $H^{1/2}(\Sigma)$ and let take for $\bar{v}'$ a lifting of $w_n|\Sigma\bar{n}$ in $\mathbb{E}^1(\Omega')$ (so that we have $\bar{w}'_n|\Sigma = \bar{w}_n|\Sigma$). We set
\[ \ell(\bar{v}') = \int_{\Omega'} \sigma^i_j(\bar{u}') \bar{v}'_j(\bar{v}') \, d\Omega' \tag{24} \]

Since the difference between lifting belongs to $\mathbb{E}^1(\Omega')$, the right-hand side doesn’t depend on the lifting $\bar{v}'$. Therefore, $\ell$ depends on $w_n|\Sigma$. Let choose for $\bar{v}'$ a continuous lifting of $w_n|\Sigma\bar{n}$; for this lifting, we have
\[ \|\bar{v}'\|_{\mathbb{E}^1(\Omega')} \leq \alpha \|w_n|\Sigma\bar{n}\|_{H^{1/2}(\Sigma)}^3, \quad (\alpha > 0) \]
and, if the components of $\bar{n}$ are sufficiently smooth:
\[ \|\bar{v}'\|_{\mathbb{E}^1(\Omega')} \leq \beta \|w_n|\Sigma\|_{H^{1/2}(\Sigma)}, \quad (\beta > 0) \]

But, we have
\[ |\ell(\bar{v}')| \leq \|\bar{u}'\|_{\mathbb{E}^1(\Omega')} \cdot \|\bar{v}'\|_{\mathbb{E}^1(\Omega')} \]
and consequently
Then, since \( \ell \) depends on \( w'_{n|\Sigma} \), it is a continuous antilinear functional on \( H^{1/2}(\Sigma) \), i.e., an element of \( \left[H^{1/2}(\Sigma)\right]' \). Taking into account of \( \bar{w'}_{n|\Sigma} = w'_{n|\Sigma} \), the equation (23) can be written

\[
\int_{\Sigma} T_n(\bar{w}')_{\Sigma} \cdot \bar{w'}_{n|\Sigma} \, d\Sigma = -\ell(\bar{w}') ,
\]

so that the normal stress \( T_n(\bar{w}')_{\Sigma} \) can be considered as an element of \( \left[H^{1/2}(\Sigma)\right]' \). Therefore, we have

\[
\left| \left< T_n(\bar{w}')_{\Sigma}, w'_{n|\Sigma} \right>_{\left[H^{1/2}(\Sigma)\right]',\left[H^{1/2}(\Sigma)\right]} \right| \leq \beta \left\| \bar{w}' \right\|_{\Sigma,\Omega} \cdot \left\| w'_{n|\Sigma} \right\|_{H^{1/2}(\Sigma)} ,
\]

\( \forall w'_{n|\Sigma} \in H^{1/2}(\Sigma) \),

and then

\[
\left\| T_n(\bar{w}') \right\|_{\left[H^{1/2}(\Sigma)\right]}' \leq \beta \left\| \bar{w}' \right\|_{\Sigma,\Omega} .
\]

Using (22), we obtain finally

\[
\left\| T_n(\bar{w}') \right\|_{\left[H^{1/2}(\Sigma)\right]}' \leq \delta \left\| u_{n|\Sigma} \right\|_{H^{1/2}(\Sigma)} \quad (\delta = 2c\beta) .
\]

Consequently, there exists a continuous linear operator \( \hat{T} \) from \( H^{1/2}(\Sigma) \) into \( \left[H^{1/2}(\Sigma)\right]' \) such that

\[
\hat{T}u_{n|\Sigma} = -T_n(\bar{w}')_{\Sigma} .
\]  

So, we have expressed linearly \( T_n(\bar{w}')_{\Sigma} \) by means of \( u_{n|\Sigma} \). The linear operator \( \hat{T} \) depends on the elasticity of the body. We are going to prove that it has properties of symmetry and positivity. We introduce the analogous problem: to find \( \bar{u}' \in \tilde{H}^1(\Omega') \) verifying

\[
- \frac{\partial \sigma_{ij}^e(\bar{u}')}{\partial x_j} = 0 \text{ in } \Omega'; \quad \bar{u}'|_{\Sigma} = 0; \quad \bar{u}'_{n|\Sigma} = \tilde{u}_{n|\Sigma} \in H^{1/2}(\Sigma); \quad \overrightarrow{T}(\bar{u}')_{\Sigma} = 0 .
\]  

In (23), we replace \( \bar{w}' \) by \( \bar{u}' \) and we have

\[
\int_{\Omega'} \sigma_{ij}^e(\bar{u}') e_{ij}^e(\bar{u}') \, d\Omega' = - \int_{\Sigma} T_n(\bar{u}')_{\Sigma} \tilde{u}_{n|\Sigma} \, d\Sigma = \left< \hat{T}u_{n|\Sigma}, \tilde{u}_{n|\Sigma} \right> ,
\]

and since \( \tilde{u}_{n|\Sigma} = \bar{u}_{n|\Sigma} \):

\[
\int_{\Omega'} \sigma_{ij}^e(\bar{u}') e_{ij}^e(\bar{u}') \, d\Omega' = \left< \hat{T}u_{n|\Sigma}, u_{n|\Sigma} \right> .
\]

Inverting roles of \( \bar{u}' \) and \( \bar{u}' \), we obtain

\[
\int_{\Omega'} \sigma_{ij}^e(\bar{u}') e_{ij}^e(\bar{u}') \, d\Omega' = \left< \hat{T}u_{n|\Sigma}, u_{n|\Sigma} \right> .
\]

By virtue of the classical symmetry of the left-hand side, we obtain the property of hermitian symmetry

\[
\left< \hat{T}u_{n|\Sigma}, \bar{u}_{n|\Sigma} \right> = \overline{\left< \hat{T}u_{n|\Sigma}, u_{n|\Sigma} \right>} .
\]

Now, setting \( \bar{u}' = u' \), we have

\[
\left< \hat{T}u_{n|\Sigma}, u_{n|\Sigma} \right> = \int_{\Omega'} \sigma_{ij}^e(\bar{u}') e_{ij}^e(\bar{u}') \, d\Omega' = \left\| u' \right\|_{\Sigma,\Omega}^2 .
\]
By virtue of a trace theorem, we have
\[
\left\| u_n|\xi_1 \right\|_{H^{1/2}(\Sigma)} \leq C \left\| u' \right\|_{L^1(\Omega)} \quad (C > 0) .
\]
so that we have
\[
\left\langle \hat{T} u_n|\Sigma, u_n|\Sigma \right\rangle \geq C^{-2} \left\| u'_n|\Sigma \right\|_{H^{1/2}(\Sigma)}^2
\]
and, since \( u'_n|\Sigma = u_n|\Sigma \), the relation of positivity
\[
\left\langle \hat{T} u_n|\Sigma, u_n|\Sigma \right\rangle \geq C^{-2} \left\| u_n|\Sigma \right\|_{H^{1/2}(\Sigma)}^2 .
\]
The second dynamic condition (17) can be written as:
\[
p|\Sigma = \hat{T} u_n|\Sigma + p_0|\Sigma \cdot n|\Sigma \cdot u_n|\Sigma .
\]
So, we have reduced our problem to a problem for a gas only:
\[
\ddot{\vec{u}} = \text{grad} \left( \frac{c_0^2(x_3) \text{div} [\rho_0(x_3)\vec{u}]}{\rho_0(x_3)} \right) .
\]
\[ - c_0^2 \text{div} (\rho_0\vec{u})|\Sigma = \hat{T} u_n|\Sigma + p_0|\Sigma \cdot n|\Sigma \cdot u_n|\Sigma . \tag{29}
\]
Afterwards, the auxiliary problem gives \( \vec{u}' \), i.e. the motion of the elastic body.

5. Variational formulation of the problem

We consider a field of admissible displacements \( \vec{u}(x_i) \), smooth in \( \Omega \) and such that \( \vec{u} = \text{grad} \, \phi \). We have
\[
\int \rho_0 \vec{u} \cdot \vec{u} \, d\Omega = \int \rho_0 \text{grad} \left( \frac{c_0^2 \text{div} (\rho_0\vec{u})}{\rho_0} \right) \cdot \vec{u} \, d\Omega
\]
and then
\[
\int \rho_0 \vec{u} \cdot \vec{u} \, d\Omega + \int \rho_0 \text{div} (\rho_0\vec{u}) \text{div} (\rho_0\vec{u}) \, d\Omega + \int \vec{u} \left( \hat{T} u_n|\Sigma + p_0|\Sigma \cdot n|\Sigma \cdot u_n|\Sigma \right) \, d\Sigma = 0 . \tag{31}
\]
Conversely, let \( \vec{u} \) a function of \( t \) with values in the field of the admissible displacements and verifying (31). We obtain easily from (31)
\[
0 = \int \rho_0 \left[ \vec{u} - \text{grad} \left( \frac{c_0^2 \text{div} (\rho_0\vec{u})}{\rho_0} \right) \right] \cdot \text{grad} \, \phi \, d\Omega
\]
or
\[
0 = - \int \phi \cdot \rho_0 \left[ \vec{u} - \text{grad} \left( \frac{c_0^2 \text{div} (\rho_0\vec{u})}{\rho_0} \right) \right] \cdot n|\Sigma \, d\Sigma \tag{32}
\]
Taking \( \phi \in \mathcal{D}(\Omega) \) and setting
\[
\Phi_0 = \rho_0 \nabla \left[ \phi - \frac{c_0^2}{\rho_0} \text{div} \left( \rho_0 \nabla \phi \right) \right],
\]
we have
\[
\text{div} \Phi_0 = 0 \quad \text{in} \quad \Omega. \tag{33}
\]

Taking \( \phi|_{\Sigma} \) arbitrary and \( \frac{\partial \phi}{\partial n}|_{\Sigma} = 0 \), we obtain
\[
\rho_0 \left[ \tilde{u} - \nabla \left( \frac{c_0^2}{\rho_0} \text{div} \left( \rho_0 \tilde{u} \right) \right) \right] \cdot \vec{n} = 0 \quad \text{on} \quad \Sigma
\]
or
\[
\Phi_0 \cdot \vec{n} = 0 \quad \text{on} \quad \Sigma. \quad \tag{34}
\]

Finally, taking \( \frac{\partial \phi}{\partial n}|_{\Sigma} \) arbitrary, we have
\[
c_0^2 \text{div} (\rho_0 \tilde{u})|_{\Sigma} + \hat{T}u|_{\Sigma} + \rho_0 g n|_{\Sigma} u_n|_{\Sigma} = 0,
\]
i.e the dynamic condition (30). Since
\[
\Phi_0 = \rho_0 \nabla \Psi, \quad \text{with} \quad \Psi = \phi - \frac{c_0^2}{\rho_0} \text{div} \left( \rho_0 \nabla \phi \right),
\]
the Equations (33) and (34) give
\[
\text{div} \left( \rho_0 \nabla \Psi \right) = 0 \quad \text{in} \quad \Omega; \quad \frac{\partial \Psi}{\partial n}|_{\Sigma} = 0. \tag{35}
\]

This Weumann problem has for solution only \( \Psi = \text{constant} \) and consequently
\[
\ddot{\tilde{u}} - \nabla \left( \frac{c_0^2}{\rho_0} \text{div} \left( \rho_0 \tilde{u} \right) \right) = 0.
\]

6. The problem is a classical vibration problem

Step 1.

We precise the field of the admissible displacements by introducing the space \( V \):
\[
V = \left\{ \vec{u} \in \mathcal{L}^2(\Omega) \stackrel{\text{def}}{=} \left[ L^2(\Omega) \right]^3 ; \quad \vec{u} = \nabla \phi; \quad \phi \in \tilde{H}^1(\Omega); \quad \text{div} (\rho_0 \vec{u}) \in L^2(\Omega); \right\}
\]
equipped with the hilbertian norm defined by
\[
\|\vec{u}\|_V^2 = \int_{\Omega} \rho_0 |\vec{u}|^2 \, d\Omega + \int_{\Omega} |\text{div} (\rho_0 \vec{u})|^2 \, d\Omega + \|u_n|_{\Sigma}|_{H^{1/2}(\Sigma)},
\]
and the space \( H \) completion of \( V \) for the norm associated to the scalar product
\[
(\vec{u}, \vec{v})_H = \int_{\Omega} \rho_0 \vec{u} \cdot \vec{v} \, d\Omega.
\]

Setting
\[
a(\vec{u}, \vec{v}) = \int_{\Omega} \frac{c_0^2}{\rho_0} \text{div} \left( \rho_0 \vec{u} \right) \text{div} \left( \rho_0 \vec{v} \right) \, d\Omega + \int_{\Sigma} \left( \hat{T}u|_{\Sigma} + \rho_0 g n|_{\Sigma} u_n|_{\Sigma} \right) \vec{u} \cdot \vec{n} \, d\Sigma,
\]
we obtain the precise variational formulation of the problem. To find \( \vec{u}(\cdot) \in V \) such that
\[
(\ddot{\vec{u}}, \vec{u})_H + a(\vec{u}, \vec{u}) = 0 \quad \forall \vec{u} \in V.
\]

**Step 2.**

Let us study the hermitian sesquilinear form
\[
\mathcal{C}(u_n|\Sigma, \tilde{u}_n|\Sigma) \overset{\text{def}}{=} \int_{\Sigma} (\hat{T}u_n|\Sigma + \rho_0|\Sigma \gamma_n|\Sigma) \tilde{u}_n|\Sigma \, d\Sigma
\]
\( \mathcal{C} \) is continuous on \( H^{1/2}(\Sigma) \times H^{1/2}(\Sigma) \) and we have:
\[
\mathcal{C}(u_n|\Sigma, u_n|\Sigma) \geq \left(C^{-2} \max_{\Sigma} \rho_0|\Sigma \gamma \right) \left\| u_n|\Sigma \right\|_{H^{1/2}(\Sigma)}^2.
\]

In the following, we suppose that \( \mathcal{C} \) is coercive, i.e
\[
C^{-2} \max_{\Sigma} \rho_0|\Sigma \gamma > 0
\]
(for example, if \( \max_{\Sigma} \rho_0|\Sigma \gamma \) is sufficiently small).

Then, \( \left[\mathcal{C}(u_n|\Sigma, u_n|\Sigma)\right]^{1/2} \) defines on \( H^{1/2}(\Sigma) \) a norm that is equivalent to the classical norm of \( H^{1/2}(\Sigma) \).

**Step 3.**

In order to prove that the problem is a classical vibration problem, we use the method that is introduced in [7]. We must prove that

a) \( [a(\vec{u}, \vec{u})]^{1/2} \) defines on \( V \) a norm equivalent to \( \| \vec{u} \|_V \).

b) The imbedding \( V \subset H \), obviously dense and continuous, hence compact. We omit the proof that is strictly identical to the proof in [7], p66-68. Therefore there exists a denumerable infinity of positive real eigenvalues \( \omega^2_p \):
\[
0 < \omega^2_1 \leq \omega^2_2 \leq \cdots \leq \omega^2_p \leq \cdots ; \quad \omega^2_p \to +\infty \quad \text{when} \quad p \to +\infty.
\]

The eigenelements \( \vec{u}_p \) form an orthonormal basis in \( H \) and an orthogonal basis in \( V \) equipped with the scalar product \( (\vec{u}, \tilde{u})_V \).

To each eigenmotion \( \vec{u}_p \) of the gas corresponds an eigenmotion \( \{\vec{u}_p\} \) of the elastic body verifying
\[
\left\| \vec{u}_p \right\|_{E(\gamma')} \leq 2c \left\| u_n|\Sigma \right\|_{H^{1/2}(\Sigma)}.
\]

\[\blacksquare\]

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**References**


