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Pseudo-valuations and pseudo-metric on JU-algebras

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Abstract: In this paper we have introduced the concept of pseudo-valuations on JU-algebras and have investigated the relationship between pseudo-valuations and ideals of JU-algebras. Conditions for a real-valued function to be a pseudo-valuation on JU-algebras are given and results based on them have been shown. We have also defined and studied pseudo-metric on JU-algebras and have proved that ϑ being a valuation on a JU-algebras A, the operation $\diamond$ in A is uniformly continuous.

Keywords: JU-algebra, JU-ideal, valuation, pseudo-valuations, pseudo metric.

MSC: 06F35, 03G25, 03C05.

1. Introduction

Pseudo-valuations in residuated lattices was introduced by Busneag [1] where many theorems based on pseudo-valuations in lattice terms and their extension theorem for residuated lattices to pseudo-valuation from valuations are shown using the model of Hilbert algebras [2]. But in fact Pseudo-valuations on a Hilbert algebras was initially introduced by Busneag [3] where it is proved that every pseudo-valuation induces a pseudometric on a Hilbert algebra. Further Busneag [2] proved many results on extensions of pseudo-valuation.

Logical algebras have become the keen interest for researchers in recent years and intensively studied under the influence of different mathematical concepts. Doh and Kang [4] introduced the concept of pseudo-valuation on BCK/BCI algebras and studied several results based on them. Ghorbani [5] defined congruence relations and gave quotient structure of BCI-algebras based on pseudo-valuation. Zhan and Jun [6] studied pseudo valuation on R_0-algebras. Based on the concept of pseudo-valuation in R_0-algebras, Yang and Xin [7] characterized pseudo pre-valuations on EQ-algebras. Mehrshad and Kouhestani studied Pseudo-Valuations on BCK-Algebras [8]. Pseudo-valuations on a BCC-algebra was introduced by Jun *et al.* [9], where they have shown that binary operation in BCC-algebras is uniformly continuous. Recently Moin *et al.* [16] introduced JU-algebras and their p-closure ideals.

UP-algebras were introduced by Iampan [10] as a new branch of logical algebras. Naveed *et al.* [11] introduced the concept of cubic KU-ideals of KU-algebras. Moin and Ali [12] have given the concept of roughness in KU-Algebras recently whereas rough set theory in UP-algebras have been introduced and studied by Moin *et al.* [13]. Next, graph associated to UP-algebras was introduced by Moin *et al.* [14]. Daniel studied pseudo-valuations on UP-algebras in [15].

In this paper, we focus on pseudo-valuation which is applied to JU-algebras and discuss related results. We define pseudo-valuations on JU-algebras using the model of Busneag and introduce a pseudo-metric on JU-algebras. We also prove that the binary operation defined on JU-algebras is uniformly continuous under the induce pseudo-metric.

2. Preliminaries and basic properties of JU-algebras

In this section, we shall introduce JU-algebras, JU-subalgebras, JU-ideals and other important terminologies with examples and some related results.

Definition 1. An algebra $(A, \diamond, 1)$ of type $(2, 0)$ with a single binary operation $\diamond$ is said to be JU-algebras satisfying the following identities: for any $u, v, w \in X$,

(JU_1) $(u \diamond v) \diamond [(v \diamond w) \diamond (u \diamond w)] = 1$,

(JU_2) $1 \diamond u = u$,

(JU_3) $u \diamond v = v \diamond u = 1$ implies $u = v$.

We call the constant 1 of X the fixed element of X. For the sake of convenience, we write X instead of $(X, \diamond, 1)$ to represent a JU-algebra. We define a relation $'' \leq ''$ in X by $v \leq u$ if and only if $u \diamond v = 1$. If we add the condition $u \diamond 1 = 1$ for all $u \in X$ in the definition of JU-algebras, then we get that X is a KU-algebra. Therefore, JU-algebra is a generalization of KU-algebras.

Lemma 2. *If X is a JU-algebra, then $(X, \leq)$ is a partial ordered set i.e.,*

(J_4) $u \leq u$,

(J_5) $u \leq v, v \leq u$, *implies* $u = v$,

(J_6) $u \leq w, w \leq v$, *implies* $u \leq v$.

Proof. Putting $v = w = 1$ in (JU_1) we get $u \diamond u = 1$, i.e. $u \leq u$ which proves (J_4). (J_5) directly follows from (JU_3). For (J_6) take $u \leq w$ and $w \leq v$ implies that $w \diamond u = 1$ and $v \diamond w = 1$. By (JU_1), we have $v \diamond u = 1$ implies that $u \leq v$. □

Further we have the following Lemma for a JU-algebra X.

Lemma 3. *If A is a JU-algebra, then following inequalities holds for any $u, v, w \in A$:*

(J_7) $u \leq v$ *implies* $v \diamond w \leq u \diamond w$,

(J_8) $u \leq v$ *implies* $w \diamond u \leq w \diamond v$,

(J_9) $(w \diamond u) \diamond (v \diamond u) \leq v \diamond w$,

(J_{10}) $(v \diamond u) \diamond u \leq v$.

Proof. (J_7), (J_8) and (J_9) follows from (JU_1) by adequate replacement of elements. (J_{10}) follows from (JU_1) and Definition 1. □

Next, we have the following Lemmas.

Lemma 4. *Any JU-algebra X satisfies following conditions for any $u, v, w \in A$,*

(J_{11}) $u \diamond u = 1$,

(J_{12}) $w \diamond (v \diamond u) = v \diamond (w \diamond u)$,

(J_{13}) *If* $(u \diamond v) \diamond v = 1$, *then A is a KU-algebra*,

(J_{14}) $(v \diamond u) \diamond 1 = (v \diamond 1) \diamond (u \diamond 1)$.

Proof. Putting $v = w = 1$ in JU_1, we get; $u \diamond u = 1$ which proves (J_{11}). For (J_{12}), we have $(w \diamond u) \diamond u \leq w$. By putting $v = 1$ in (JU_1) and using (J_7), we get

$$w \diamond (v \diamond u) \leq ((w \diamond u) \diamond u) \diamond (v \diamond u). \tag{1}$$

Replace w by $w \diamond u$ in (JU_1), we get $[v \diamond (w \diamond u)] \diamond [((w \diamond u) \diamond u) \diamond (v \diamond u)] = 1$, which implies

$$((w \diamond u) \diamond u) \diamond (v \diamond u) \leq v \diamond (w \diamond u). \tag{2}$$

From (1), (2) and Lemma 2(J_6) we get,

$$w \diamond (v \diamond u) \leq v \diamond (w \diamond u). \tag{3}$$

Further by replacing v with w and w with v in (3), we get

$$v \diamond (w \diamond u) \leq w \diamond (v \diamond u). \tag{4}$$

Now (3), (4) and (J_5) yields, $w \diamond (v \diamond u) = v \diamond (w \diamond u)$.

In order to prove (J_{13}), we just needs to show that $u \diamond 1 = 1, \ \forall \ u \in A$. Replacing $v \to 1, u \to 1, w \to u$ in (JU_1), we obtained, $(1 \diamond u) \diamond [(u \diamond 1) \diamond (1 \diamond 1)] = 1 \Rightarrow u \diamond [(u \diamond 1) \diamond 1] = 1 \Rightarrow u \diamond 1 = 1$ (by using $v = 1$ in the given condition of (J_{13})).

Using (J_{12}) for any $u, v \in A$ in the followings we see that, $(v \diamond 1) \diamond (u \diamond 1) = (v \diamond 1) \diamond u \diamond [(v \diamond u) \diamond (v \diamond u)] = (v \diamond 1) \diamond [(v \diamond u) \diamond (u \diamond (v \diamond u))] = (v \diamond u) \diamond [(v \diamond 1) \diamond (v \diamond (u \diamond u))] = (v \diamond u) \diamond [(v \diamond 1) \diamond (v \diamond 1)] = (v \diamond u) \diamond 1$, which shows that (J_{14}) holds. □

Definition 5. A non-empty subset I of a JU-algebra A is called a JU-ideal of A if it satisfies the following conditions:

(1) $1 \in I$,
(2) $u \diamond (v \diamond w) \in I, v \in I$ implies $u \diamond w \in I$, for all $u, v, w \in I$.

3. Pseudo-valuations and pseudo-metric

Definition 6. A real-valued function ϑ on a JU-algebra A is called a pseudo-valuation on A if it satisfies the following two conditions:

(1) $\vartheta(1) = 0$,
(2) $\vartheta(u \diamond w) \leq \vartheta(u \diamond (v \diamond w)) + \vartheta(v)$ for all $u, v, w \in A$.

A pseudo-valuation ϑ on a JU-algebra A satisfying the following condition:
$\vartheta(u) = 0 \Rightarrow u = 1$ for all $u \in A$ is called a valuation on A.

Example 1. Let $A = \{1, 2, 3, 4\}$ be a set with operation $\diamond$. A Cayley table for A is defined as

$\diamond$	1	2	3	4
1	1	2	3	4
2	1	1	1	4
3	1	2	1	4
4	1	2	1	1

Here A is a JU-algebra. We find that a real valued function defined on A by $\vartheta(1) = 0$, $\vartheta(2) = \vartheta(3) = 1$, and $\vartheta(4) = 3$, is a pseudo-valuation on A.

Proposition 7. *Let ϑ be a pseudo-valuation on a JU-algebras A. Then we have*

(1) $u \leq v \Rightarrow \vartheta(v) \leq \vartheta(u)$.
(2) $\vartheta((u \diamond (v \diamond w)) \diamond w) \leq \vartheta(u) + \vartheta(v)$ for all $u, v, w \in A$.

Proof. **(1)** Let $u, v \in A$ be such that $u \leq v$. Replacing $u = 1, v = u, w = v$ in Definition 6 and Definition 1, we get $\vartheta(v) = \vartheta(1 \diamond v) \leq \vartheta(1 \diamond (u \diamond v)) + \vartheta(u) = \vartheta(1 \diamond 1) + \vartheta(u) = \vartheta(1) + \vartheta(u) = \vartheta(u)$.

(2) If we replace u by $u \diamond (v \diamond w)$ in Definition 6(2), then we get

$$\vartheta((u \diamond (v \diamond w)) \diamond w) \leq \vartheta((u \diamond (v \diamond w)) \diamond (v \diamond w)) + \vartheta(v),$$

again applying Definition 6 (2) by choosing $u = u \diamond (v \diamond w)$ and $w = v \diamond w$, we get

$$\vartheta((u \diamond (v \diamond w)) \diamond w) \leq \vartheta[(u \diamond (v \diamond w)) \diamond (u \diamond (v \diamond w))] + \vartheta(u) + \vartheta(v) = \vartheta(1) + \vartheta(u) + \vartheta(v)$$

$$\Rightarrow \vartheta((u \diamond (v \diamond w)) \diamond w) \leq \vartheta(u) + \vartheta(v).$$

□

Corollary 8. *A pseudo-valuation ϑ on a JU-algebra A satisfies the inequality $\vartheta(u) \geq 0$ for all $u \in A$.*

Proposition 9. *If ϑ is a pseudo-valuation on a JU-algebra A, then we have $\vartheta((u \diamond v) \diamond v) \leq \vartheta(u)$ for all $u, v \in A$.*

Proof. It is easy to see that the required inequality holds by considering $v = 1$ and $w = v$ in Proposition 7 (2) and using Definition 1. □

Following results are devoted to find conditions for a real valued function on a JU-algebra A to be a pseudo-valuation.

Theorem 10. *Let ϑ be a real valued function on a JU-algebra A satisfying the following conditions:*

(a) If $\vartheta(a) \leq \vartheta(u)$ for all $u \in A$, then $\vartheta(a) = 0$,

(b) $\vartheta(u \diamond v) \leq \vartheta(v)$ for all $u, v \in A$,

(c) $\vartheta((u \diamond (v \diamond w)) \diamond w) \leq \vartheta(u) + \vartheta(v)$,

(d) $\vartheta(v \diamond (u \diamond w)) \leq \vartheta(u \diamond (v \diamond w))$.

Then ϑ is a pseudo-valuation on A.

Proof. From Lemma 4 and given condition (b), we have $\vartheta(1) = \vartheta(u \diamond u) \leq \vartheta(u)$ for all $u \in A$ and hence $\vartheta(1) = 0$, using given condition (a). Now, from Definition 1, Lemma 4 and given condition (c), we get $\vartheta(v) = \vartheta(1 \diamond v) = \vartheta(((u \diamond v) \diamond (u \diamond v)) \diamond v) \leq \vartheta(u \diamond v) + \vartheta(u)$ for all $u, v \in A$. It follows from given condition (d) that $\vartheta(u \diamond w) \leq \vartheta(v \diamond (u \diamond w)) + \vartheta(v) \leq \vartheta(u \diamond (v \diamond w)) + \vartheta(v)$ for all $u, v, w \in A$. Therefore ϑ is a pseudo-valuation on A. □

Corollary 11. *Let ϑ be a real-valued function on a JU-algebra A satisfying the following conditions:*

(a) $\vartheta(1) = 0$,

(b) $\vartheta(u \diamond v) \leq \vartheta(v)$, for all $u, v \in A$,

(c) $\vartheta((u \diamond (v \diamond w) \diamond w)) \leq \vartheta(u) + \vartheta(v)$ for all $u, v, w \in A$,

(d) $\vartheta(v \diamond (u \diamond w)) \leq \vartheta(u \diamond (v \diamond w))$.

Then ϑ is a pseudo-valuation on A.

Theorem 12. *If ϑ is a pseudo-valuation on a JU-algebra A, then $\vartheta(v) \leq \vartheta(u \diamond v) + \vartheta(u)$, for all $u, v \in A$.*

Proof. Let $m = (u \diamond v) \diamond v$ for any $u, v \in A$, and $n = u \diamond v$. Then $v = 1 \diamond v = (((u \diamond v) \diamond v) \diamond ((u \diamond v) \diamond v)) \diamond v = (m \diamond (n \diamond v)) \diamond v$. It follows from Proposition 7(2) and Proposition 9 that $\vartheta(v) = \vartheta((m \diamond (n \diamond v)) \diamond v) \leq \vartheta(m) + \vartheta(n) = \vartheta((u \diamond v) \diamond v) + \vartheta(u \diamond v) \leq \vartheta(u) + \vartheta(u \diamond v)$. This completes the proof. □

Theorem 13. *Let ϑ be a real-valued function on a JU-algebra A satisfying the following conditions.*

(1) $\vartheta(1) = 0$,

(2) $\vartheta(v) \leq \vartheta(u \diamond v) + \vartheta(u)$,

(3) $\vartheta(v \diamond (u \diamond w)) \leq \vartheta(u \diamond (v \diamond w))$ for all $u, v, w \in A$.

Then ϑ is a pseudo-valuation on A.

Proof. For any $u, v, a, b \in A$, and using 4 with given condition (2) and (3) we get, $\vartheta(u \diamond v) \leq \vartheta(v \diamond (u \diamond v)) + \vartheta(v) \leq \vartheta(u \diamond (v \diamond v)) + \vartheta(v) = \vartheta(v \diamond (1)) + \vartheta(v) = \vartheta(1) + \vartheta(v) = \vartheta(v)$. Also,

$$\begin{aligned} \vartheta[(b \diamond (a \diamond u)) \diamond u] &\leq \vartheta[a \diamond ((b \diamond (a \diamond u)) \diamond u)] + \vartheta(a) \\ &\leq \vartheta[(b \diamond (a \diamond u)) \diamond (a \diamond u)] + \vartheta(a) \\ &\leq \vartheta[b \diamond [(b \diamond (a \diamond u)) \diamond (a \diamond u)]] + \vartheta(a) + \vartheta(b) \\ &\leq \vartheta[(b \diamond (a \diamond u)) \diamond (b \diamond (a \diamond u))] + \vartheta(a) + \vartheta(b) \\ &= \vartheta(1) + \vartheta(a) + \vartheta(b) \\ &= \vartheta(a) + \vartheta(b). \end{aligned}$$

By Corollary 11, we get that ϑ is a pseudo-valuation on A. □

Proposition 14. *If ϑ is a pseudo-valuation on a JU-algebra A, then*

$$a \leq b \diamond u \Rightarrow \vartheta(u) \leq \vartheta(a) + \vartheta(b) \text{ for all } a, b, u \in A. \tag{5}$$

Proof. Suppose that $a, b, u \in A$ such that $a \leq b \diamond u$. Then by Proposition 7 (2) and Theorem 12, we have $\vartheta(u) \leq \vartheta((a \diamond (b \diamond u)) \diamond u) + \vartheta(a \diamond (b \diamond u)) = \vartheta((a \diamond (b \diamond u)) \diamond u) + \vartheta(1) = \vartheta((a \diamond (b \diamond u)) \diamond u)$
$\leq \vartheta(a) + \vartheta(b)$. □

Proposition 15. *Suppose that A is JU-algebra. Then every pseudo-valuation ϑ on A satisfies the following inequality:*
$\vartheta(u \diamond w) \leq \vartheta(u \diamond v) + \vartheta(v \diamond w)$, *for all $u, v, w \in A$.*

Proof. It follows from JU_1 and Proposition 14. □

Theorem 16. *If ϑ is a pseudo-valuation on a JU-algebra A, then the set $I := \{u \in A | \vartheta(u) = 0\}$ is an ideal of A.*

Proof. We have $\vartheta(1) = 0$ and hence $1 \in I$. Next, $u, v, w \in A$ be such that $v \in I$ and $u \diamond (v \diamond w) \in I$. Then $\vartheta(v) = 0$ and $\vartheta(u \diamond (v \diamond w)) = 0$. By Definition 6(2), we get $\vartheta(u \diamond w) \leq \vartheta(u \diamond (v \diamond w)) + \vartheta(v) = 0$ so that $\vartheta(u \diamond w) = 0$. Hence $u \diamond w \in I$, therefore I is an ideal of A. □

Example 2. [16] Let $A = \{1, 2, 3, 4, 5\}$ in which $\diamond$ is defined by the following table

$\diamond$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	3	4	5
3	1	2	1	4	4
4	1	1	3	1	3
5	1	1	1	1	1

It is easy to see that A is a JU-algebra. Now, define a real-valued function ϑ on A by $\vartheta(1) = \vartheta(2) = \vartheta(3) = 0$, $\vartheta(4) = 3$, and $\vartheta(5) = 1$. Then $I := \{u \in A \mid \vartheta(u) = 0\} = \{1, 2, 3\}$ is the ideal of A. But ϑ is not a pseudo-valuation as $\vartheta(3 \diamond 5) \nleq \vartheta(3 \diamond (5 \diamond 5)) + \vartheta(5)$.

For a real-valued function ϑ on a JU-algebra A, define a mapping $d_\vartheta : X \times X \to \mathbb{R}$ by $d_\vartheta(u, v) = \vartheta(u \diamond v) + \vartheta(v \diamond u)$ for all $(u, v) \in A \times A$. We have following result.

Theorem 17. *Let A is a JU-algebra. If a real-valued function ϑ on A is a pseudo-valuation on A, then d_ϑ is a pseudo-metric on A, and so (X, d_ϑ) is a pseudo-metric space. (The d_ϑ is called pseudo-metric induced by pseudo-valuation ϑ.)*

Proof. Clearly, $d_\vartheta\ (u, v) \geq 1$, $m_\vartheta\ (u, u) = 1$ and $m_\vartheta\ (u, v) = m_\vartheta(v, u)$ for all $u, v \in A$. For any $u, v, w \in A$ from Proposition 15, we get $d_\vartheta(u, v) + d_\vartheta(v, w) = [\vartheta(u \diamond v) + \vartheta(v \diamond u)] + [\vartheta(v \diamond w) + \vartheta(w \diamond v)] = [\vartheta(u \diamond v) + \vartheta(v \diamond w)] + [\vartheta(w \diamond v) + \vartheta(v \diamond u)] \geq \vartheta(u \diamond w) + \vartheta(w \diamond u) = d_\vartheta(u, w)$. Hence (X, d_ϑ) is a pseudo-metric space. □

Proposition 18. *Let A is a JU-algebra. Then every pseudo-metric d_ϑ induced by pseudo-valuation ϑ satisfies the following inequalities:*
(1) $d_\vartheta(u, v) \geq d_\vartheta(x \diamond u, x \diamond v)$,
(2) $d_\vartheta(u \diamond v, x \diamond y) \leq d_\vartheta(u \diamond v, x \diamond v) + d_\vartheta(x \diamond v, x \diamond y)$ for all $u, v, x, y \in A$.

Proof. (1) Let $u, v, a \in A$. By JU_1 $u \diamond v \leq (x \diamond v) \diamond (x \diamond u)$ and $v \diamond u \leq (x \diamond u) \diamond (x \diamond v)$. It follows from Proposition 7(1) that $\vartheta(u \diamond v) \geq \vartheta((x \diamond v) \diamond (x \diamond u))$ and $\vartheta(v \diamond u) \geq \vartheta((x \diamond u) \diamond (x \diamond v))$. So $d_\vartheta(u, v) = \vartheta(u \diamond v) + \vartheta(v \diamond u) \geq \vartheta((x \diamond u) \diamond (x \diamond u)) + \vartheta((x \diamond u) \diamond (x \diamond v)) = d_\vartheta(x \diamond u, x \diamond v)$.
(2) Followed by definition of pseudo-metric. □

Theorem 19. *Let ϑ be a real-valued function on a JU-algebra A, if d_ϑ is a pseudo-metric on A, then $(X \times X, d_\vartheta^\diamond)$ is a pseudo-metric space, where*

$$d_\vartheta^\diamond((u, v), (a, b)) = \max\{d_\vartheta(u, a), d_\vartheta(v, b)\} \text{ for all } (u, v), (a, b) \in A \times A.$$

Proof. Suppose d_ϑ is a pseudo-metric on A. For any $(u, v), (a, b) \in A \times A$, we have $d_\vartheta^\diamond((u, v), (u, v)) = \max\{d_\vartheta(u, u), d_\vartheta(v, v)\} = 0$ and

$$d_\vartheta^\diamond((u, v), (a, b)) = \max\{d_\vartheta(u, a), d_\vartheta(v, b)\} = \max\{d_\vartheta(a, u), d_\vartheta(b, v)\} = d^\diamond((a, b), (u, v)).$$

Now let $(u,v),(a,b),(u,v) \in A \times A$. Then we have

$$\begin{aligned} d^{\diamond}_{\vartheta}((u,v),(u,v)) + d^{\diamond}_{\vartheta}((u,v),(a,b)) &= \max\{d_{\vartheta}(u,u), d_{\vartheta}(v,v)\} + \max\{d_{\vartheta}(u,a), d_{\vartheta}(v,b)\} \\ &\geq \max\{d_{\vartheta}(u,u) + d_{\vartheta}(u,a), d_{\vartheta}(v,v) + d_{\vartheta}(v,b)\} \\ &\geq \max\{d_{\vartheta}(u,a), d_{\vartheta}(v,b)\} = d^{\diamond}_{\vartheta}((u,v),(a,b)). \end{aligned}$$

Hence $(X \times X, d^{\diamond}_{\vartheta})$ is a pseudo-metric space. □

Corollary 20. *If $\vartheta : X \to \mathbb{R}$ is a pseudo-valuation on a JU-algebra A, then $(X \times X, d^{\diamond}_{\vartheta})$ is a pseudo-metric space.*

Theorem 21. *Let A is a JU-algebra. If $\vartheta : X \to \mathbb{R}$ is a valuation on A, then (X, d_{ϑ}) is a metric space.*

Proof. Suppose ϑ is a valuation on A, then (X, d_{ϑ}) is a pseudo-metric space by Theorem 19. Further consider $u, v \in A$ be such that $d_{\vartheta}(u,v) = 0$, then $0 = d_{\vartheta}(u,v) = \vartheta(u \diamond v) + \vartheta(v \diamond u)$, and hence $\vartheta(u \diamond v) = 0$ and $\vartheta(v \diamond u) = 0$ since $\vartheta(u) \geq 0$ for all $u \in A$ and, since ϑ is a valuation on A, it follows that $u \diamond v = 1$ and $v \diamond u = 1$ so from (condition in the given theorem) that $u = v$. Hence (X, d_{ϑ}) is a metric space. □

Theorem 22. *Let A is a JU-algebra. If $\vartheta : X \to \mathbb{R}$ is a valuation on A, then $(X \times X, d^{\diamond}_{\vartheta})$ is a metric space.*

Proof. From Corollary 20, we know that $(X \times X, d^{\diamond}_{\vartheta})$ is a pseudo-metric space. Suppose $(u,v),(a,b) \in A \times A$ be such that $d^{\diamond}_{\vartheta}((u,v),(a,b)) = 0$, then $0 = d^{\diamond}_{\vartheta}((u,v),(a,b)) = \max\{d_{\vartheta}(u,a), d_{\vartheta}(v,b)\}$, and so $d_{\vartheta}(u,a) = 0 = d_{\vartheta}(v,b)$. Since $d_{\vartheta}(u,v) \geq 0$ for all $(u,v) \in A \times A$. Hence $0 = d_{\vartheta}(u,a) = \vartheta(u \diamond a) + \vartheta(a \diamond u)$ and $0 = d_{\vartheta}(v,b) = \vartheta(v \diamond b) + \vartheta(b \diamond v)$. It follows that $\vartheta(u \diamond a) = 0 = \vartheta(a \diamond u)$ and $\vartheta(v \diamond b) = 0 = \vartheta(b \diamond v)$ so that $u \diamond a = 1 = a \diamond u$ and $v \diamond b = 0 = b \diamond v$. Now we have $a = u$ and $b = v$, and so $(u,v) = (a,b)$, therefore $(X \times X, d^{\diamond}_{\vartheta})$ is a metric space. □

Theorem 23. *Let A is a JU-algebra. If ϑ is a valuation on A, then the operation $\diamond$ in A is uniformly continuous.*

Proof. Consider for any $\epsilon > 0$, if $d^{\diamond}_{\vartheta}((u,v),(a,b)) < \frac{\epsilon}{2}$ then $d_{\vartheta}(u,a) < \frac{\epsilon}{2}$ and $d_{\vartheta}(v,b) < \frac{\epsilon}{2}$. This implies that $d_{\vartheta}(u \diamond v, a \diamond b) \leq d_{\vartheta}(u \diamond v, a \diamond v) + d_{\vartheta}(a \diamond v, a \diamond b) \leq d_{\vartheta}(u,a) + d_{\vartheta}(v,b) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ (from Proposition 18). Therefore the operation $\diamond : X \times X \to A$ is uniformly continuous. □

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