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Simpson's type inequalities for strongly (s,m)-convex functions in the second sense and applications

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Abstract: Some new inequalities of Simpson's type for functions whose third derivatives in absolute value at some powers are strongly (s,m)- convex in the second sense are provided. An application to the Simpson's quadrature rule is also provided.

Keywords: Simpson's inequality, strongly (s,m)-convex functions, numerical quadrature rule, Hölder inequality.

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1. Introduction

The inequality below is known in the literature as the Simpson's inequality:

$$\left|\int_a^b f(t)dt - \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \leq \frac{(b-a)^5}{2880}\left\|f^{(4)}\right\|_\infty$$

where $f : [a,b] \to \mathbb{R}$ is a four times continuously differentiable function on (a,b) and $\left\|f^{(4)}\right\|_\infty = \sup_{t\in(a,b)}\left|f^{(4)}(t)\right| < \infty$.

This inequality has been studied and generalized by many authors over the years. For information on recent results about the Simpson's inequality, we refer the reader to [1–11].

Let $I \subset \mathbb{R}$ and I° denote the interior of I. Recall that, a function $f : I \to \mathbb{R}$ is said to be convex, if

$$f(tx+(1-t)y) \leq tf(x) + (1-t)f(y),$$

for all $x, y \in I$ and $t \in [0,1]$. Also, f is said to be quasiconvex, if

$$f(tx+(1-t)y) \leq \max\left\{f(x), f(y)\right\},$$

for all $x, y \in I$ and $t \in [0,1]$.

In [2], Alomari and Hussain presented the following Simpson's type inequalities for functions whose third derivatives in absolute value at certain powers are quasiconvex.

Theorem 1. *Let $f'' : I \subseteq \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function on I° such that $f''' \in L_1([a,b])$, where $a, b \in I$ with $a < b$. If $|f'''|$ is quasiconvex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}&\left|\int_a^b (t)dt - \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\ &\leq \frac{(b-a)^4}{1152}\left[\max\left\{|f'''(a)|, \left|f'''\left(\frac{a+b}{2}\right)\right|\right\} + \max\left\{\left|f'''\left(\frac{a+b}{2}\right)\right|, |f'''(b)|\right\}\right].\end{aligned}$$

Theorem 2. *Let $f'' : I \subseteq \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function on I° such that $f''' \in L_1([a,b])$, where $a,b \in I$ with $a < b$. If $|f'''|^q$ is quasiconvex on $[a,b]$ for some fixed $q > 1$, then the following inequality holds:*

$$\left| \int_a^b (t)dt - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \leq \frac{2^{-1/p}(b-a)^4}{48}\left(\frac{\Gamma(p+1)\Gamma(2p+1)}{\Gamma(3p+2)}\right)^{1/p}$$
$$\times \left[\left(\max\left\{ |f'''(a)|^q, \left| f'''\left(\frac{a+b}{2}\right)\right|^q \right\}\right)^{1/q} + \left(\max\left\{ \left| f'''\left(\frac{a+b}{2}\right)\right|^q, |f'''(b)|^q \right\}\right)^{1/q}\right],$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\Gamma(\cdot)$ is the gamma function (see Definition 8).

Definition 3 ([3])**.** A function $f : [0,b] \to \mathbb{R}, b > 0$, is said to be m-convex, for $m \in [0,1]$, if

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y),$$

for all $x, y \in [0,b]$ and $t \in [0,1]$.

In [4], Özidemir *et al.* presented following Simpson's type inequality for functions whose third derivatives in absolute value at some powers are m-convex.

Theorem 4. *Let $f : I \subset [0,b^*] \to \mathbb{R}$ be a three times differentiable function on I° such that $f''' \in L_1([a,b])$, where $a,b \in I^\circ$ with $a < b, b^* > 0$. If $|f'''|^q$ is m-convex, for $m \in (0,1]$ and $q > 1$, then the following inequality holds:*

$$\left| \int_a^{mb} (t)dt - \frac{mb-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(mb)\right]\right| \leq \frac{2^{-1/p}(mb-a)^4}{96}\left(\frac{\Gamma(p+1)\Gamma(2p+1)}{\Gamma(3p+2)}\right)^{1/p}$$
$$\times \left[\left(\frac{|f'''(a)|^q + 3m|f'''(b)|^q}{4}\right)^{1/q} + \left(\frac{3|f'''(a)|^q + m|f'''(b)|^q}{4}\right)^{1/q}\right]$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\Gamma(\cdot)$ is the gamma function (see Definition 8).

Definition 5 ([12])**.** A function $f : [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense, for $s \in [0,1]$, if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y),$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

In [5], Özidemir *et al.* presented following Simpson's type inequality for functions whose third derivatives in absolute value at some powers are s-convex.

Theorem 6. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a three times differentiable function on I° such that $f''' \in L_1([a,b])$, where $a,b \in I^\circ$ with $a < b$. If $|f'''|^q$ is s-convex in the second sense, for $s \in (0,1]$ and $q > 1$, then the following inequality holds:*

$$\left| \int_a^b (t)dt - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \leq \frac{2^{-1/p}(b-a)^4}{48}\left(\frac{\Gamma(p+1)\Gamma(2p+1)}{\Gamma(3p+2)}\right)^{1/p}$$
$$\times \left[\left(\frac{1}{2^{s+1}(s+1)}|f'''(a)|^q + \frac{2^{s+1}-1}{2^{s+1}(s+1)}|f'''(b)|^q \right) \Big|^q \Big\} \right)^{1/q}$$
$$+ \left(\frac{1}{2^{s+1}(s+1)}|f'''(b)|^q + \frac{2^{s+1}-1}{2^{s+1}(s+1)}|f'''(a)|^q \right) \Big|^q \Big\} \Big)^{1/q} \Big],$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\Gamma(\cdot)$ is the gamma function (see Definition 8).

Recently, Bracamonte *et al.* [13] introduced the concept of functions that are strongly (s,m)-convex in the second sense as follows.

Definition 7 ([13]). A function $f : [0, \infty) \to \mathbb{R}$ is said to be strongly (s, m)-convex function with modulus $\mu \geq 0$ in the second sense, for $(s, m) \in [0, 1] \times [0, 1]$, if

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t)^s f(y) - \mu t(1-t)(x-y)^2$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$.

Remark 1. If $\mu = 0$ in Definition 7, then we have the definition for (s, m)-convex functions in the second sense. Choosing $\mu = 0$ and $m = 1$ gives the definition of s-convex functions. Also for $\mu = 0$ and $s = 1$, we have m-convex. If $s \in [0, 1]$ and $m = 1$, then we have the class of strongly s-convex functions. Also, if $m \in [0, 1]$ and $s = 1$, then we have the class of strongly m-convex functions.

Motivated by the above results, the goal of this paper is to provide some new Simpson's type inequality for functions whose third derivatives in absolute value at certain powers are strongly (s, m)-convex functions.

We complete this section be recalling the definitions of the gamma, beta and incomplete Beta functions.

Definition 8. The gamma function is given by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-x} dx, \qquad Re(x) > 0,$$

the (complete) beta function is given by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dx, \qquad Re(x) > 0 \text{ and } Re(y) > 0,$$

and the incomplete beta function is given by

$$B_a(x, y) = \int_0^a t^{x-1}(1-t)^{y-1} dx, \qquad Re(x) > 0, Re(y) > 0 \text{ and } 0 < a < 1.$$

Remark 2. The gamma and beta function satisfies the following properties:

1. $\Gamma(x+1) = x\Gamma(x)$,
2. $B(x, y) = B(y, x)$,
3. $B(x, y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

2. Main results

The following lemma by Alomari and Hussain [2] will be very useful in obtaining our main results.

Lemma 9 ([2]). *Let $f : I \to \mathbb{R}$ be a three times differentiable function on I° such that $f''' \in L_1([a, b])$, for $a, b \in I$ and $a < b$. Then the following equality holds;*

$$\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] = (b-a)^4 \int_0^1 p(t) f'''(ta + (1-t)b)dt,$$

where

$$p(t) = \begin{cases} \frac{1}{6}t^2(t - \frac{1}{2}) & 0 \leq t \leq \frac{1}{2}, \\ \frac{1}{6}(t-1)^2(t - \frac{1}{2}) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Theorem 10. *Let $f : [0, \infty) \to \mathbb{R}$ be a three times differentiable function on $(0, \infty)$ such that $f''' \in L_1([a, b])$, for $0 \leq a < b$. If $|f'''|$ is strongly (s, m)-convex with modulus $\mu \geq 0$, for $(s, m) \in (0, 1] \times (0, 1]$, then the following inequality holds:*

$$\left| \int_a^b f(x)dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \right| \leq \frac{(b-a)^4}{6}\left[\frac{2^{-s-4}}{(s+3)(s+4)}\left(\left|f'''(a)\right| + \left|f'''(b)\right|\right)\right.$$
$$\left. + \frac{m2^{-s-4}\left(s^2 + 11s + 2^{s+4}(s-2) + 34\right)}{(s+1)(s+2)(s+3)(s+4)}\left(\left|f'''\left(\frac{a}{m}\right)\right| + \left|f'''\left(\frac{b}{m}\right)\right|\right) \frac{\mu}{960}\left(\left(\frac{b}{m} - a\right)^2 + \left(b - \frac{a}{m}\right)^2\right)\right].$$

Proof. Using Lemma 9 and the strong (s,m)-convexity of $|f'''|$, we have

$$\left|\int_a^b f(x)dx - \frac{b-a}{6}\Big[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Big]\right| \le (b-a)^4\int_0^1 |p(t)||f'''(ta+(1-t)b)|dt$$
$$= \frac{(b-a)^4}{6}\left[\int_0^{\frac{1}{2}}\left|t^2\Big(t-\frac{1}{2}\Big)\right|\left|f'''(ta+(1-t)b)\right|dt + \int_{\frac{1}{2}}^1\left|(t-1)^2\Big(t-\frac{1}{2}\Big)\right|\left|f'''(ta+(1-t)b)\right|dt\right]$$
$$= \frac{(b-a)^4}{6}\left[\int_0^{\frac{1}{2}} t^2\Big(\frac{1}{2}-t\Big)\left|f'''(ta+(1-t)b)\right|dt + \int_0^{\frac{1}{2}} t^2\Big(\frac{1}{2}-t\Big)\left|f'''(tb+(1-t)a)\right|dt\right]$$
$$\le \frac{(b-a)^4}{6}\left[\int_0^{\frac{1}{2}} t^2\Big(\frac{1}{2}-t\Big)\left(t^s\left|f'''(a)\right| + m(1-t)^s\left|f'''\Big(\frac{b}{m}\Big)\right| - \mu t(1-t)\Big(\frac{b}{m}-a\Big)^2\right)dt\right.$$
$$\left.+\int_0^{\frac{1}{2}} t^2\Big(\frac{1}{2}-t\Big)\left(t^s\left|f'''(b)\right| + m(1-t)^s\left|f'''\Big(\frac{a}{m}\Big)\right| - \mu t(1-t)\Big(b-\frac{a}{m}\Big)^2\right)dt\right]$$
$$= \frac{(b-a)^4}{6}\left[\left|f'''(a)\right|\int_0^{\frac{1}{2}} t^{s+2}\Big(\frac{1}{2}-t\Big)dt\right.$$
$$+m\left|f'''\Big(\frac{b}{m}\Big)\right|\int_0^{\frac{1}{2}} t^2\Big(\frac{1}{2}-t\Big)(1-t)^s dt - \mu\Big(\frac{b}{m}-a\Big)^2\int_0^{\frac{1}{2}} t^3\Big(\frac{1}{2}-t\Big)(1-t)dt$$
$$\left.+\left|f'''(b)\right|\int_0^{\frac{1}{2}} t^{s+2}\Big(\frac{1}{2}-t\Big)dt + m\left|f'''\Big(\frac{a}{m}\Big)\right|\int_0^{\frac{1}{2}} t^2\Big(\frac{1}{2}-t\Big)(1-t)^s dt - \mu\Big(b-\frac{a}{m}\Big)^2\int_0^{\frac{1}{2}} t^3\Big(\frac{1}{2}-t\Big)(1-t)dt\right]$$
$$= \frac{(b-a)^4}{6}\left[\frac{2^{-s-4}}{(s+3)(s+4)}\left(\left|f'''(a)\right|+\left|f'''(b)\right|\right) + \frac{m2^{-s-4}\left(s^2+11s+2^{s+4}(s-2)+34\right)}{(s+1)(s+2)(s+3)(s+4)}\right.$$
$$\left.\times\left(\left|f'''\Big(\frac{a}{m}\Big)\right| + \left|f'''\Big(\frac{b}{m}\Big)\right|\right) - \frac{\mu}{960}\left(\Big(\frac{b}{m}-a\Big)^2 + \Big(b-\frac{a}{m}\Big)^2\right)\right],$$

where

$$\int_0^{\frac{1}{2}} t^{s+2}\Big(\frac{1}{2}-t\Big)dt = \frac{2^{-s-4}}{(s+3)(s+4)}, \tag{1}$$

$$\int_0^{\frac{1}{2}} t^2\Big(\frac{1}{2}-t\Big)(1-t)^s dt = \frac{2^{-s-4}\left(s^2+11s+2^{s+4}(s-2)+34\right)}{(s+1)(s+2)(s+3)(s+4)} \tag{2}$$

and

$$\int_0^{\frac{1}{2}} t^3\Big(\frac{1}{2}-t\Big)(1-t)dt = \frac{1}{960}. \tag{3}$$

□

Theorem 11. *Let $f:[0,\infty)\to\mathbb{R}$ be a three times differentiable function on $(0,\infty)$ such that $f'''\in L_1([a,b])$, for $0\le a<b$. If $|f'''|^q$ is strongly (s,m)-convex with modulus $\mu\ge 0$, for $(s,m)\in(0,1]\times(0,1]$ and $q>1$, then the following inequality holds;*

$$\left|\int_a^b f(x)dx - \frac{b-a}{6}\Big[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Big]\right| \le \frac{(b-a)^4}{48(2^{1/p})}\left(\frac{\Gamma(2p+1)\Gamma(p+1)}{\Gamma(3p+2)}\right)^{\frac{1}{p}}\left[\left(\frac{2^{-s-1}}{s+1}\left|f'''(a)\right|^q\right.\right.$$
$$\left.\left.+\frac{m(1-2^{-s-1})}{s+1}\left|f'''\Big(\frac{b}{m}\Big)\right|^q - \frac{\mu}{12}\Big(\frac{b}{m}-a\Big)^2\right)^{\frac{1}{q}} + \left(\frac{2^{-s-1}}{s+1}\left|f'''(b)\right|^q + \frac{m(1-2^{-s-1})}{s+1}\left|f'''\Big(\frac{a}{m}\Big)\right|^q - \frac{\mu}{12}\Big(b-\frac{a}{m}\Big)^2\right)^{\frac{1}{q}}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 9, the Hölders inequality and the strong (s,m)-convexity of $|f'''|^q$, we have

$$\left|\int_a^b f(x)dx - \frac{b-a}{6}\Big[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Big]\right| \le (b-a)^4\int_0^1 |p(t)||f'''(ta+(1-t)b)|dt$$

$$
\begin{aligned}
&= \frac{(b-a)^4}{6}\Big[\int_0^{\frac{1}{2}}\Big|t^2\Big(t-\frac{1}{2}\Big)\Big|\Big|f'''(ta+(1-t)b)\Big|dt+\int_{\frac{1}{2}}^1\Big|(t-1)^2\Big(t-\frac{1}{2}\Big)\Big|\Big|f'''(ta+(1-t)b)\Big|dt\Big]\\
&= \frac{(b-a)^4}{6}\Big[\int_0^{\frac{1}{2}}t^2\Big(\frac{1}{2}-t\Big)\Big|f'''(ta+(1-t)b)\Big|dt+\int_0^{\frac{1}{2}}t^2\Big(\frac{1}{2}-t\Big)\Big|f'''(tb+(1-t)a)\Big|dt\Big]\\
&\le \frac{(b-a)^4}{6}\Big[\Big(\int_0^{\frac{1}{2}}\Big[t^2\Big(\frac{1}{2}-t\Big)\Big]^p dt\Big)^{\frac{1}{p}}\Big(\int_0^{\frac{1}{2}}\Big|f'''(ta+(1-t)b)\Big|^q dt\Big)^{\frac{1}{q}}\\
&+\Big(\int_0^{\frac{1}{2}}\Big[t^2\Big(\frac{1}{2}-t\Big)\Big]^p dt\Big)^{\frac{1}{p}}\Big(\int_0^{\frac{1}{2}}\Big|f'''(tb+(1-t)a)\Big|^q dt\Big)^{\frac{1}{q}}\Big]\\
&\le \frac{(b-a)^4}{6}\Big(\int_0^{\frac{1}{2}}\Big[t^2\Big(\frac{1}{2}-t\Big)\Big]^p dt\Big)^{\frac{1}{p}}\Big[\Big(\int_0^{\frac{1}{2}}\Big(t^s\Big|f'''(a)\Big|^q+m(1-t)^s\Big|f'''\Big(\frac{b}{m}\Big)\Big|^q-\mu t(1-t)\Big(\frac{b}{m}-a\Big)^2\Big)dt\Big)^{\frac{1}{q}}\\
&+\Big(\int_{\frac{1}{2}}^1\Big(t^s\Big|f'''(b)\Big|^q+m(1-t)^s\Big|f'''\Big(\frac{a}{m}\Big)\Big|^q-\mu t(1-t)\Big(b-\frac{a}{m}\Big)^2\Big)dt\Big)^{\frac{1}{q}}\Big]\\
&= \frac{(b-a)^4}{6}\Big(\int_0^{\frac{1}{2}}\Big[t^2\Big(\frac{1}{2}-t\Big)\Big]^p dt\Big)^{\frac{1}{p}}\Big[\Big(\Big|f'''(a)\Big|^q\int_0^{\frac{1}{2}}t^s dt+m\Big|f'''\Big(\frac{b}{m}\Big)\Big|^q\int_0^{\frac{1}{2}}(1-t)^s dt-\mu\Big(\frac{b}{m}-a\Big)^2\\
&\times\int_0^{\frac{1}{2}}t(1-t)dt\Big)^{\frac{1}{q}}+\Big(\Big|f'''(b)\Big|^q\int_0^{\frac{1}{2}}t^s dt+m\Big|f'''\Big(\frac{a}{m}\Big)\Big|^q\int_0^{\frac{1}{2}}(1-t)^s dt-\mu\Big(b-\frac{a}{m}\Big)^2\int_0^{\frac{1}{2}}t(1-t)dt\Big)^{\frac{1}{q}}\Big]\\
&= \frac{(b-a)^4}{48(2^{1/p})}\Big(\frac{\Gamma(2p+1)\Gamma(p+1)}{\Gamma(3p+2)}\Big)^{\frac{1}{p}}\Big[\Big(\frac{2^{-s-1}}{s+1}\Big|f'''(a)\Big|^q+\frac{m(1-2^{-s-1})}{s+1}\Big|f'''\Big(\frac{b}{m}\Big)\Big|^q\\
&-\frac{\mu}{12}\Big(\frac{b}{m}-a\Big)^2\Big)^{\frac{1}{q}}+\Big(\frac{2^{-s-1}}{s+1}\Big|f'''(b)\Big|^q+\frac{m(1-2^{-s-1})}{s+1}\Big|f'''\Big(\frac{a}{m}\Big)\Big|^q-\frac{\mu}{12}\Big(b-\frac{a}{m}\Big)^2\Big)^{\frac{1}{q}}\Big],
\end{aligned}
$$

where

$$\int_0^{\frac{1}{2}}\Big[t^2\Big(\frac{1}{2}-t\Big)\Big]^p dt=\frac{1}{2^{3p+1}}B(2p+1,p+1)=\frac{\Gamma(2p+1)\Gamma(p+1)}{2^{3p+1}\Gamma(3p+2)},$$

$$\int_0^{\frac{1}{2}}t^s dt=\frac{2^{-s-1}}{s+1}$$

and

$$\int_0^{\frac{1}{2}}(1-t)^s dt=\frac{1-2^{-s-1}}{s+1}.$$

This completes the proof.

□

Theorem 12. *Let $f:[0,\infty)\to\mathbb{R}$ be a three times differentiable function on $(0,\infty)$ such that $f'''\in L_1([a,b])$, for $0\le a<b$. If $|f'''|^q$ is strongly (s,m)-convex with modulus $\mu\ge 0$, for $(s,m)\in(0,1]\times(0,1]$ and $q>1$, then the following inequality holds;*

$$
\begin{aligned}
&\Big|\int_a^b f(x)dx-\frac{b-a}{6}\Big[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Big]\Big|\le\frac{(b-a)^4}{6}\Big(\frac{1}{192}\Big)^{\frac{1}{p}}\Big[\Big(\frac{2^{-s-4}}{(s+3)(s+4)}\Big|f'''(a)\Big|^q\\
&+\frac{m2^{-s-4}\Big(s^2+11s+2^{s+4}(s-2)+34\Big)}{(s+1)(s+2)(s+3)(s+4)}\Big|f'''\Big(\frac{b}{m}\Big)\Big|^q-\frac{\mu}{960}\Big(\frac{b}{m}-a\Big)^2\Big)^{\frac{1}{q}}\\
&+\Big(\frac{2^{-s-4}}{(s+3)(s+4)}\Big|f'''(b)\Big|^q+\frac{m2^{-s-4}\Big(s^2+11s+2^{s+4}(s-2)+34\Big)}{(s+1)(s+2)(s+3)(s+4)}\Big|f'''\Big(\frac{a}{m}\Big)\Big|^q-\frac{\mu}{960}\Big(b-\frac{a}{m}\Big)^2\Big)^{\frac{1}{q}}\Big],
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 9, the Hölders inequality and the strong (s,m)-convexity of $|f'''|^q$, we have

$$
\begin{aligned}
&\left|\int_a^b f(x)dx - \frac{b-a}{6}\Big[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Big]\right| \le (b-a)^4\int_0^1 |p(t)||f'''(ta+(1-t)b)|dt \\
= &\ \frac{(b-a)^4}{6}\Big[\int_0^{\frac{1}{2}}\Big|t^2\Big(t-\frac{1}{2}\Big)\Big|\Big|f'''(ta+(1-t)b)\Big|dt + \int_{\frac{1}{2}}^1\Big|(t-1)^2\Big(t-\frac{1}{2}\Big)\Big|\Big|f'''(ta+(1-t)b)\Big|dt\Big] \\
= &\ \frac{(b-a)^4}{6}\Big[\int_0^{\frac{1}{2}} t^2\Big(\frac{1}{2}-t\Big)\Big|f'''(ta+(1-t)b)\Big|dt + \int_0^{\frac{1}{2}} t^2\Big(\frac{1}{2}-t\Big)\Big|f'''(tb+(1-t)a)\Big|dt\Big] \\
\le &\ \frac{(b-a)^4}{6}\Big(\int_0^{\frac{1}{2}} t^2\Big(\frac{1}{2}-t\Big)dt\Big)^{\frac{1}{p}}\Big[\Big(\int_0^{\frac{1}{2}} t^2\Big(\frac{1}{2}-t\Big)\Big|f'''(ta+(1-t)b)\Big|^q dt\Big)^{\frac{1}{q}} \\
&+\Big(\int_0^{\frac{1}{2}} t^2\Big(\frac{1}{2}-t\Big)\Big|f'''(tb+(1-t)a)\Big|^q dt\Big)^{\frac{1}{q}}\Big] \\
\le &\ \frac{(b-a)^4}{6}\Big(\int_0^{\frac{1}{2}} t^2\Big(\frac{1}{2}-t\Big)dt\Big)^{\frac{1}{p}}\Big[\Big(\int_0^{\frac{1}{2}} t^2\Big(\frac{1}{2}-t\Big)\Big(t^s\Big|f'''(a)\Big|^q + m(1-t)^s\Big|f'''\Big(\frac{b}{m}\Big)\Big|^q - \mu t(1-t) \\
&\Big(\frac{b}{m}-a\Big)^2\Big)dt\Big)^{\frac{1}{q}} + \Big(\int_0^{\frac{1}{2}} t^2\Big(\frac{1}{2}-t\Big)\Big(t^s\Big|f'''(b)\Big|^q + m(1-t)^s\Big|f'''\Big(\frac{a}{m}\Big)\Big|^q - \mu t(1-t)\Big(b-\frac{a}{m}\Big)^2\Big)dt\Big)^{\frac{1}{q}}\Big] \\
= &\ \frac{(b-a)^4}{6}\Big(\int_0^{\frac{1}{2}} t^2\Big(\frac{1}{2}-t\Big)dt\Big)^{\frac{1}{p}}\Big[\Big(\Big|f'''(a)\Big|^q\int_0^{\frac{1}{2}} t^{s+2}\Big(\frac{1}{2}-t\Big)dt + m\Big|f'''\Big(\frac{b}{m}\Big)\Big|^q\int_0^{\frac{1}{2}} t^2\Big(\frac{1}{2}-t\Big)(1-t)^s dt \\
&-\mu\Big(\frac{b}{m}-a\Big)^2\int_0^{\frac{1}{2}} t^3(1-t)\Big(\frac{1}{2}-t\Big)dt\Big)^{\frac{1}{q}} + \Big(\Big|f'''(b)\Big|^q\int_0^{\frac{1}{2}} t^{s+2}\Big(\frac{1}{2}-t\Big)dt \\
&+m\Big|f'''\Big(\frac{a}{m}\Big)\Big|^q\int_0^{\frac{1}{2}} t^2\Big(\frac{1}{2}-t\Big)(1-t)^s dt - \mu\Big(b-\frac{a}{m}\Big)^2\int_0^{\frac{1}{2}} t^3(1-t)\Big(\frac{1}{2}-t\Big)dt\Big)^{\frac{1}{q}}\Big]. \qquad (4)
\end{aligned}
$$

The desired inequality follows from (4) by using (1), (2), (3) and

$$\int_0^{\frac{1}{2}} t^2\Big(\frac{1}{2}-t\Big)dt = \frac{1}{192}.$$

□

2.1. Other Simpson's type integral inequalities

Theorem 13. *Let $f : [0,\infty) \to \mathbb{R}$ be a three times differentiable function on $(0,\infty)$ such that $f''' \in L_1([a,b])$, for $0 \le a < b$. If $|f'''|^q$ is strongly (s,m)-convex with modulus $\mu \ge 0$, for $(s,m) \in (0,1] \times (0,1]$ and $q > 1$, then the following inequality holds;*

$$
\begin{aligned}
&\left|\int_a^b f(x)dx - \frac{b-a}{6}\Big[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Big]\right| \le \frac{(b-a)^4}{24}\Big(\frac{1}{8(2p+1)(p+1)}\Big)^{\frac{1}{p}} \\
&\times\Big[\Big(\frac{1}{2^{s+2}(s+1)(s+2)}\Big|f'''(a)\Big|^q + \frac{m(2s+2^{-s})}{4(s+1)(s+2)}\Big|f'''\Big(\frac{b}{m}\Big)\Big|^q - \frac{\mu}{64}\Big(\frac{b}{m}-a\Big)^2\Big)^{\frac{1}{q}} \\
&+\Big(\frac{1}{2^{s+2}(s+1)(s+2)}\Big|f'''(b)\Big|^q + \frac{m(2s+2^{-s})}{4(s+1)(s+2)}\Big|f'''\Big(\frac{a}{m}\Big)\Big|^q - \frac{\mu}{64}\Big(b-\frac{a}{m}\Big)^2\Big)^{\frac{1}{q}}\Big],
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 9, the Hölders inequality and the strong (s,m)-convexity of $|f'''|^q$, we have

$$
\begin{aligned}
&\left|\int_a^b f(x)dx - \frac{b-a}{6}\Big[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Big]\right| \le (b-a)^4\int_0^1 |p(t)||f'''(ta+(1-t)b)|dt \\
&= \frac{(b-a)^4}{6}\Big[\int_0^{\frac{1}{2}}\Big|t^2\Big(t-\frac{1}{2}\Big)\Big|\Big|f'''(ta+(1-t)b)\Big|dt + \int_{\frac{1}{2}}^1\Big|(t-1)^2\Big(t-\frac{1}{2}\Big)\Big|\Big|f'''(ta+(1-t)b)\Big|dt\Big]
\end{aligned}
$$

$$= \frac{(b-a)^4}{6}\Big[\int_0^{\frac{1}{2}} t^2\Big(\frac{1}{2}-t\Big)\Big|f'''(ta+(1-t)b)\Big|dt + \int_0^{\frac{1}{2}} t^2\Big(\frac{1}{2}-t\Big)\Big|f'''(tb+(1-t)a)\Big|dt\Big]$$
$$\leq \frac{(b-a)^4}{6}\Big(\int_0^{\frac{1}{2}} t^{2p}\Big(\frac{1}{2}-t\Big)dt\Big)^{\frac{1}{p}}\Big[\Big(\int_0^{\frac{1}{2}}\Big(\frac{1}{2}-t\Big)\Big|f'''(ta+(1-t)b)\Big|^q dt\Big)^{\frac{1}{q}}$$
$$+\Big(\int_0^{\frac{1}{2}}\Big(\frac{1}{2}-t\Big)\Big|f'''(tb+(1-t)a)\Big|^q dt\Big)^{\frac{1}{q}}\Big]$$
$$\leq \frac{(b-a)^4}{6}\Big(\int_0^{\frac{1}{2}} t^{2p}\Big(\frac{1}{2}-t\Big)dt\Big)^{\frac{1}{p}}\Big[\Big(\int_0^{\frac{1}{2}}\Big(\frac{1}{2}-t\Big)\Big(t^s\Big|f'''(a)\Big|^q + m(1-t)^s\Big|f'''\Big(\frac{b}{m}\Big)\Big|^q - \mu t(1-t)\Big(\frac{b}{m}-a\Big)^2\Big)dt\Big)^{\frac{1}{q}}$$
$$+\Big(\int_0^{\frac{1}{2}}\Big(\frac{1}{2}-t\Big)\Big(t^s\Big|f'''(b)\Big|^q + m(1-t)^s\Big|f'''\Big(\frac{a}{m}\Big)\Big|^q - \mu t(1-t)\Big(b-\frac{a}{m}\Big)^2\Big)dt\Big)^{\frac{1}{q}}\Big]$$
$$= \frac{(b-a)^4}{6}\Big(\int_0^{\frac{1}{2}} t^{2p}\Big(\frac{1}{2}-t\Big)dt\Big)^{\frac{1}{p}}\Big[\Big(\Big|f'''(a)\Big|^q\int_0^{\frac{1}{2}} t^s\Big(\frac{1}{2}-t\Big)dt$$
$$+m\Big|f'''\Big(\frac{b}{m}\Big)\Big|^q\int_0^{\frac{1}{2}}\Big(\frac{1}{2}-t\Big)(1-t)^s dt - \mu\Big(\frac{b}{m}-a\Big)^2\int_0^{\frac{1}{2}} t(1-t)\Big(\frac{1}{2}-t\Big)dt\Big)^{\frac{1}{q}}$$
$$+\Big(\Big|f'''(b)\Big|^q\int_0^{\frac{1}{2}} t^s\Big(\frac{1}{2}-t\Big)dt + m\Big|f'''\Big(\frac{a}{m}\Big)\Big|^q\int_0^{\frac{1}{2}}\Big(\frac{1}{2}-t\Big)(1-t)^s dt - \mu\Big(b-\frac{a}{m}\Big)^2\int_0^{\frac{1}{2}} t(1-t)\Big(\frac{1}{2}-t\Big)dt\Big)^{\frac{1}{q}}\Big].$$

The desired inequality follows from (5) by using the fact that

$$\int_0^{\frac{1}{2}} t^{2p}\Big(\frac{1}{2}-t\Big)dt = \frac{1}{2^{2p+2}(2p+1)(2p+2)},$$

$$\int_0^{\frac{1}{2}} t^s\Big(\frac{1}{2}-t\Big)dt = \frac{1}{2^{s+2}(s+1)(s+2)},$$

$$\int_0^{\frac{1}{2}}\Big(\frac{1}{2}-t\Big)(1-t)^s dt = \frac{2s+2^{-s}}{4(s+1)(s+2)},$$

and

$$\int_0^{\frac{1}{2}} t(1-t)\Big(\frac{1}{2}-t\Big)dt = \frac{1}{64}.$$

□

Theorem 14. *Let $f:[0,\infty)\to\mathbb{R}$ be a three times differentiable function on $(0,\infty)$ such that $f'''\in L_1([a,b])$, for $0\leq a<b$. If $|f'''|^q$ is strongly (s,m)-convex with modulus $\mu\geq 0$, for $(s,m)\in(0,1]\times(0,1]$ and $q>1$, then the following inequality holds;*

$$\Big|\int_a^b f(x)dx - \frac{b-a}{6}\Big[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Big]\Big| \leq \frac{(b-a)^4}{12}\Big(\frac{1}{8}B(3,p+1)\Big)^{\frac{1}{p}}\Big[\Big(\frac{1}{2^{s+3}(s+3)}\Big|f'''(a)\Big|^q$$
$$+mB_{\frac{1}{2}}(3,s+1)\Big|f'''\Big(\frac{b}{m}\Big)\Big|^q - \frac{3\mu}{320}\Big(\frac{b}{m}-a\Big)^2\Big)^{\frac{1}{q}} + \Big(\frac{1}{2^{s+3}(s+3)}\Big|f'''(b)\Big|^q + mB_{\frac{1}{2}}(3,s+1)\Big|f'''\Big(\frac{a}{m}\Big)\Big|^q$$
$$-\frac{3\mu}{320}\Big(b-\frac{a}{m}\Big)^2\Big)^{\frac{1}{q}}\Big], \tag{5}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 9, the Hölders inequality and the strong (s,m)-convexity of $|f'''|^q$, we have

$$\Big|\int_a^b f(x)dx - \frac{b-a}{6}\Big[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Big]\Big| \leq (b-a)^4\int_0^1 |p(t)||f'''(ta+(1-t)b)|dt$$
$$= \frac{(b-a)^4}{6}\Big[\int_0^{\frac{1}{2}}\Big|t^2\Big(t-\frac{1}{2}\Big)\Big|\Big|f'''(ta+(1-t)b)\Big|dt + \int_{\frac{1}{2}}^1\Big|(t-1)^2\Big(t-\frac{1}{2}\Big)\Big|\Big|f'''(ta+(1-t)b)\Big|dt\Big]$$

$$
\begin{aligned}
&= \frac{(b-a)^4}{6}\left[\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)\left|f'''(ta+(1-t)b)\right|dt+\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)\left|f'''(tb+(1-t)a)\right|dt\right]\\
&\leq \frac{(b-a)^4}{6}\left(\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)^p dt\right)^{\frac{1}{p}}\left[\left(\int_0^{\frac{1}{2}} t^2\left|f'''(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}} t^2\left|f'''(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right]\\
&\leq \frac{(b-a)^4}{6}\left(\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)^p dt\right)^{\frac{1}{p}}\left[\left(\int_0^{\frac{1}{2}} t^2\left(t^s\left|f'''(a)\right|^q+m(1-t)^s\left|f'''\left(\frac{b}{m}\right)\right|^q\right.\right.\right.\\
&\left.\left.-\mu t(1-t)\left(\frac{b}{m}-a\right)^2\right)dt\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}} t^2\left(t^s\left|f'''(b)\right|^q+m(1-t)^s\left|f'''\left(\frac{a}{m}\right)\right|^q-\mu t(1-t)\left(b-\frac{a}{m}\right)^2\right)dt\right)^{\frac{1}{q}}\Bigg]\\
&= \frac{(b-a)^4}{6}\left(\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)^p dt\right)^{\frac{1}{p}}\left[\left(\left|f'''(a)\right|^q\int_0^{\frac{1}{2}} t^{s+2}dt+m\left|f'''\left(\frac{b}{m}\right)\right|^q\int_0^{\frac{1}{2}} t^2(1-t)^s dt-\mu\left(\frac{b}{m}-a\right)^2\right.\right.\\
&\left.\times\int_0^{\frac{1}{2}} t^3(1-t)dt\right)^{\frac{1}{q}}+\left(\left|f'''(b)\right|^q\int_0^{\frac{1}{2}} t^{s+2}dt+m\left|f'''\left(\frac{a}{m}\right)\right|^q\int_0^{\frac{1}{2}} t^2(1-t)^s dt-\mu\left(b-\frac{a}{m}\right)^2\int_0^{\frac{1}{2}} t^3(1-t)dt\right)^{\frac{1}{q}}\Bigg].
\end{aligned}
\tag{6}
$$

The desired inequality follows from (6) and using the fact that

$$\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)^p dt=\frac{1}{2^{p+3}}B(3,p+1),$$

$$\int_0^{\frac{1}{2}} t^{s+2}dt=\frac{1}{2^{s+3}(s+3)},$$

$$\int_0^{\frac{1}{2}} t^2\left(1-t\right)^s dt=B_{\frac{1}{2}}(3,s+1),$$

and

$$\int_0^{\frac{1}{2}} t^3\left(1-t\right)dt=\frac{3}{320}.$$

This completes the proof of the theorem. □

3. Applications to Simpson's Formula

Let $\mathcal{P}$ be a division of the interval $[a,b]$, i.e, $\mathcal{P}: a=x_0<x_1<\cdots<x_{n-1}<x_n=b, h_i=\frac{x_{i+1}-x_i}{2}$ and consider the Simpson's formula

$$S(f,\mathcal{P})=\sum_{i=0}^{n-1}\frac{f(x_i)+4f(x_i+h_i)+f(x_{i+1})}{3}h_i.$$

For any function $f:[a,b]\to\mathbb{R}$, we let $E_S(f,\mathcal{P})$ denote the approximation error of the integral $\int_a^b f(x)dx$ with respect to the Simpson's formula. That is,

$$E_S(f,\mathcal{P})=\int_a^b f(x)dx-S(f,\mathcal{P}).$$

We have the following estimate for $E_S(f,\mathcal{P})$ under the condition that $|f'''|$ is strongly (s,m)-convex.

Proposition 15. *Let $f:[0,\infty)\to\mathbb{R}$ be a three times differentiable function on $(0,\infty)$ such that $f'''\in L_1([a,b])$, for $0\leq a<b$. If $|f'''|$ is strongly (s,m)-convex with modulus $\mu\geq 0$, for $(s,m)\in(0,1]\times(0,1]$, then for any division $\mathcal{P}$ of $[a,b]$, the following inequality holds:*

$$\left|E_S(f,\mathcal{P})\right| \leq \frac{1}{6}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^4\left[\frac{2^{-s-4}}{(s+3)(s+4)}\left(\left|f'''(x_i)\right|+\left|f'''(x_{i+1})\right|\right)\right.$$
$$\left.+\frac{m2^{-s-4}\left(s^2+11s+2^{s+4}(s-2)+34\right)}{(s+1)(s+2)(s+3)(s+4)}\left(\left|f'''\left(\frac{x_i}{m}\right)\right|+\left|f'''\left(\frac{x_{i+1}}{m}\right)\right|\right)-\frac{\mu}{960}\left(\left(\frac{x_{i+1}}{m}-x_i\right)^2+\left(x_{i+1}-\frac{x_i}{m}\right)^2\right)\right].$$

Proof. First we observe that

$$E_S(f,\mathcal{P})=\sum_{i=0}^{n-1}\left[\int_{x_i}^{x_{i+1}}f(x)dx-\frac{f(x_i)+4f(x_i+h_i)+f(x_{i+1})}{3}h_i\right].$$

It follows that

$$\left|E_S(f,\mathcal{P})\right| \leq \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}}f(x)dx-\frac{f(x_i)+4f(x_i+h_i)+f(x_{i+1})}{3}h_i\right|. \tag{7}$$

Now, by applying Theorem 10 to the subinterval $[x_i,x_{i+1}],(i=0,1,\cdots,n-1)$, we have

$$\begin{aligned}&\left|\int_{x_i}^{x_{i+1}}f(x)dx-\frac{x_{i+1}-x_i}{6}\left[f(x_i)+4f\left(\frac{x_i+x_{i+1}}{2}\right)+f(x_{i+1})\right]\right|\\ &\leq \frac{(x_{i+1}-x_i)^4}{6}\left[\frac{2^{-s-4}}{(s+3)(s+4)}\left(\left|f'''(x_i)\right|+\left|f'''(x_{i+1})\right|\right)\right.\\ &+\frac{m2^{-s-4}\left(s^2+11s+2^{s+4}(s-2)+34\right)}{(s+1)(s+2)(s+3)(s+4)}\left(\left|f'''\left(\frac{x_i}{m}\right)\right|+\left|f'''\left(\frac{x_{i+1}}{m}\right)\right|\right)\\ &\left.-\frac{\mu}{960}\left(\left(\frac{x_{i+1}}{m}-x_i\right)^2+\left(x_{i+1}-\frac{x_i}{m}\right)^2\right)\right].\end{aligned} \tag{8}$$

The desired inequality follows from (7) and using (8). This completes the proof. □

4. Conclusion

We have introduced five main results related to the Simpson's type inequalities for strongly (s,m)-convex functions in the second sense. Our results generalizes some results in the literature. Several other interesting inequalities could be derived from our results by considering different values of the parameters s,m and μ. Some application to the Simpson's quadrature formula has also been provided.

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