A fixed point theorem for generalized weakly contractive mappings in $b$-metric spaces

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Abstract: In this paper we establish a fixed point theorem for generalized weakly contractive mappings in the setting of $b$-metric spaces and prove the existence and uniqueness of a fixed point for a self-mapping satisfying the established theorem. Our result extends and generalizes the result of Cho [1]. Finally, we provided an example in the support of our main result.

Keywords: Fixed point, generalized weak contractive mapping, $b$-metric space.

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1. Introduction


Rhoades [10] proved that every weakly contractive mapping has a unique fixed point in complete metric spaces. Then, many authors obtained generalizations and extensions of the weakly contractive mappings.

In particular, Choudhury et al. [11] generalized fixed point results for weakly contractive mappings by using altering distance functions. Very recently, Cho [1] introduced the notion of generalized weakly contractive mappings in metric spaces and proved a fixed point theorem for generalized weakly contractive mappings defined on complete metric spaces.

Inspired and motivated by the results of Cho [1] the purpose of this paper is to establish a fixed point result for generalized weakly contractive mappings in the setting of $b$-metric spaces.

2. Preliminaries

In this section, we give basic definitions of concepts concerning a generalized weakly contractive mappings in the setting of $b$-metric spaces.

Definition 1. [2] Let $X$ be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is a $b$-metric if and only if for all $x, y, z \in X$, the following conditions are satisfied:

(a) $d(x, y) = 0$ if and only if $x = y$;
(b) $d(x, y) = d(y, x)$;
(c) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair $(X, d)$ is called a $b$-metric space.

It should be noted that the class of $b$-metric spaces is effectively larger than that of metric spaces, since $b$-metric is metric when $s = 1$. But, in general, the converse is not true.
Example 1. [12] Let $X = R$ and $d : X \times X \rightarrow R^+$ be given by $d(x, y) = (x - y)^2$ for all $x, y \in X$, then $d$ is a $b$-metric on $X$ with $s = 2$ but it is not a metric on $X$, because for $x = 2, y = 4$ and $z = 6$, we have $d(2, 6) \not\leq 2[d(2, 4) + d(4, 6)]$, hence the triangle inequality for a metric does not hold.

Definition 2. A function $f : X \rightarrow R^+$, where $X$ is $b$-metric space is called lower semicontinuous if for all $x \in X$ and $x_n \in X$ with $\lim_{n \rightarrow \infty} x_n = x$, we have

$$f(x) \leq \liminf_{n \rightarrow \infty} f(x_n).$$

Definition 3. [6] Let $X$ be a $b$-metric space and $\{x_n\}$ be a sequence in $X$, we say that

(a) $x_n$ is $b$-converges to $x \in X$ if $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$.
(b) $x_n$ is a $b$-Cauchy sequence if $d(x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$.
(c) $(X, d)$ is $b$-complete if every $b$-Cauchy sequence in $X$ is $b$-convergent.

Definition 4. [1] Let $X$ be a complete metric space with metric $d$, and $T : X \rightarrow X$. Also let $\varphi : X \rightarrow R^+$ be a lower semicontinuous function, then $T$ is called a generalized weakly contractive mapping if it satisfies the following condition:

$$\varphi(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)) \leq \varphi(m(x, y, d, T, \varphi)) - \varphi(l(x, y, d, T, \varphi))$$

where,

$$m(x, y, d, T, \varphi) = \max\{d(x, y) + \varphi(x) + \varphi(y), d(x, Tx) + \varphi(x) + \varphi(Tx), d(y, Ty) + \varphi(y) + \varphi(Ty),$$

$$\frac{1}{2}[d(x, Ty) + \varphi(x) + \varphi(Ty) + d(y, Tx) + \varphi(y) + \varphi(Tx)]\}$$

and $l(x, y, d, T, \varphi) = \max\{d(x, y) + \varphi(x) + \varphi(y), d(y, Ty) + \varphi(y) + \varphi(Ty)\}$, for all $x, y \in X$, where $\varphi : R^+ \rightarrow R^+$ is a continuous with $\varphi(t) = 0$ if and only if $t = 0$ and $\varphi : R^+ \rightarrow R^+$ is a lower semicontinuous function with $\varphi(t) = 0$ if and only if $t = 0$.

Theorem 1. [1] Let $X$ be complete. If $T$ is a generalized weakly contractive mapping, then there exists a unique $z \in X$ such that $z = Tz$ and $\varphi(z) = 0$.

Lemma 1. [12] Suppose $(X, d)$ is a $b$-metric space and $\{x_n\}$ be a sequence in $X$ such that

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) \rightarrow 0.$$

If $\{x_n\}$ is not a $b$-Cauchy sequence, then there exists $\epsilon > 0$ and two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) \geq k$ such that for all positive integer $k$, $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$, $d(x_{m(k)}, x_{n(k-1)}) < \epsilon$ and

(a) $\epsilon \leq \lim_{n \rightarrow \infty} \inf d(x_{m(k)}, x_{n(k)}) \leq \lim_{n \rightarrow \infty} \sup d(x_{m(k)}, x_{n(k)}) \leq s\epsilon$.
(b) $\epsilon \leq \lim_{n \rightarrow \infty} \inf d(x_{m(k)}, x_{n(k)}) \leq \lim_{n \rightarrow \infty} \sup d(x_{m(k+1)}, x_{n(k)}) \leq s^2\epsilon$.
(c) $\epsilon \leq \lim_{n \rightarrow \infty} \inf d(x_{m(k+1)}, x_{n(k+1)}) \leq \lim_{n \rightarrow \infty} \sup d(x_{m(k+1)}, x_{n(k+1)}) \leq s^2\epsilon$.
(d) $\frac{s^2}{2} \leq \lim_{n \rightarrow \infty} \inf d(x_{m(k+1)}, x_{n(k+1)}) \leq \lim_{n \rightarrow \infty} \sup d(x_{m(k+1)}, x_{n(k+1)}) \leq s^3\epsilon$

holds.

3. Results and discussion

In this section, we introduce a generalized weakly contractive mappings in the setting of $b$-metric spaces and prove a fixed point result.

Definition 5. Let $X$ be a $b$-metric space with metric $d$ and parameter $s \geq 1$, $T : X \rightarrow X$, and let $\varphi : X \rightarrow R^+$ be a lower semicontinuous function, then $T$ is called a generalized weakly contractive mapping if satisfies the following condition:

$$\varphi(S^3d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)) \leq \varphi(m(x, y, d, T, \varphi)) - \varphi(l(x, y, d, T, \varphi))$$

(1)
for all \(x, y \in X\), where,
\[
m(x, y, d, T, \varphi) = \max \{d(x, y) + \varphi(x) + \varphi(y), d(x, Tx) + \varphi(x) + \varphi(Tx), d(y, Ty) + \varphi(y) + \varphi(Ty), \frac{1}{2s^2} \{d(x, Ty) + \varphi(x) + \varphi(Ty) + d(y, Tx) + \varphi(y) + \varphi(Tx)\}\}
\]
(2)

and
\[
l(x, y, d, T, \varphi) = \max \{d(x, y) + \varphi(x) + \varphi(y), d(y, Tx) + \varphi(y) + \varphi(Ty)\}.
\]
(3)

for all \(x, y \in X\), where \(\varphi: R^+ \rightarrow R^+\) is a continuous with \(\varphi(t) = 0\) if and only if \(t = 0\) and \(\varphi: R^+ \rightarrow R^+\) is a lower semicontinuous function with \(\varphi(t) = 0\) if and only if \(t = 0\).

**Theorem 2.** Let \(X\) be a complete \(b\)-metric space with metric \(d\) and \(s \geq 1\). If \(T\) is a generalized weakly contractive mapping then \(T\) has a unique fixed point \(u \in X\) such that \(u = Tu\) and \(\varphi(u) = 0\).

**Proof.** Let \(x_0 \in X\) be fixed and define a sequence \(\{x_n\}\) by \(x_1 = Tx_0, x_2 = Tx_1, \ldots, x_{n+1} = Tx_n\) for all \(n = 0, 1, 2, \ldots\). If \(x_n = x_{n+1}\) for some \(n, x_n = x_{n+1} = Tx_n\) is a fixed point of \(T\).

Assume \(x_n \neq x_{n+1}\) for all \(n = 0, 1, 2, \ldots\). From (2) by using \(x = x_n\) and \(y = x_{n+1}\), we have
\[
m(x_{n-1}, x_n, d, T, \varphi) = \max \{d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n), d(x_{n-1}, Tx_n) + \varphi(x_{n-1}) + \varphi(Tx_n), \frac{1}{2s^2} \{d(x_{n-1}, Tx_n) + \varphi(x_{n-1}) + \varphi(Tx_n) + d(x_n, Tx_n) + \varphi(x_n) + \varphi(Tx_n)\}\} = \max \{d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n), d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_{n+1}), \frac{1}{2s^2} \{d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n) + \varphi(x_{n+1})\}\}.
\]

Since
\[
\frac{1}{2s^2} \{d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n) + \varphi(x_{n+1})\} \leq \frac{1}{2s} \{sd(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n) + \varphi(x_{n+1})\} \leq \frac{1}{2s} \{\varphi(x_{n-1}) + \varphi(x_n) + d(x_{n-1}, x_n) + \varphi(x_{n+1})\} \leq \frac{1}{2s} \{d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n) + d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n) + d(x_n, x_{n+1}) + \varphi(x_{n+1})\} \leq \frac{1}{2s} \{d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n) + d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n) + d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n) + \varphi(x_{n+1})\}\}.
\]
So, we obtain
\[
m(x_{n-1}, x_n, d, T, \varphi) = \max \{d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n) + \varphi(x_{n+1})\}.
\]
(4)

Similarly from (3)
\[
l(x_{n-1}, x_n, d, T, \varphi) = \max \{d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n), d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n) + \varphi(x_{n+1})\}.
\]
(5)

Then (1) becomes
\[
\psi(s^3d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) \leq \psi(m(x_{n-1}, x_n, d, T, \varphi)) - \psi(l(x_{n-1}, x_n, d, T, \varphi)).
\]
(6)

Now, if \(d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n) \leq d(x_{n-1}, x_n) + \varphi(x_{n}) + \varphi(x_{n+1})\), for some positive integer \(n\) then (6) becomes
\[
\psi(s^3d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) \leq \psi(d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n) + \varphi(x_{n+1})) - \psi(d(x_{n-1}, x_n) + \varphi(x_n) + \varphi(x_{n+1})).
\]

It follows \(\psi(d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n) + \varphi(x_{n+1})) \leq \psi(s^3d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) \leq \psi(d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n) + \varphi(x_{n+1}))\), which is a contradiction. Thus,
\[
d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n) > d(x_{n-1}, x_n) + \varphi(x_{n+1}) + \varphi(x_{n+1}).
\]
(7)

From (4), (5) and (7), we obtain
\[
m(x_{n-1}, x_n, d, T, \varphi) = l(x_{n-1}, x_n, d, T, \varphi) = d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n),
\]
(8)

So (6) becomes:
\[
\psi(s^3d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) \leq \psi(m(x_{n-1}, x_n, d, T, \varphi)) - \psi(l(x_{n-1}, x_n, d, T, \varphi)).
\]
(9)
From (7), the sequence \( (d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})) \) is decreasing and bounded below. Hence \( d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) \to r \) as \( n \to \infty \) for some \( r \geq 0 \). Assume \( r > 0 \) and letting \( n \to \infty \) in (9) and using the continuity of \( \psi \) and the lower semicontinuity of \( \varphi \), we have

\[
\psi(s^3 r) \leq \psi(r) - \lim_{n \to \infty} \inf \varphi(d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)) \\
\leq \psi(r) - \lim_{n \to \infty} \varphi(d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)) \\
= \psi(r) - \varphi(r).
\]

It follows that \( \psi(r) \leq \psi(s^3 r) \leq \psi(r) - \varphi(r) < \psi(r) \), which is a contradiction, hence we have \( r = 0 \) and consequently, \( \lim_{n \to \infty} [d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})] = 0. \) Implies

\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \quad (10)
\]

\[
\lim_{n \to \infty} \varphi(x_n) = \lim_{n \to \infty} \varphi(x_{n+1}) = 0. \quad (11)
\]

Now, we prove that the sequence \( \{x_n\} \) is a \( b \)-Cauchy sequence. If \( \{x_n\} \) is not a \( b \)-Cauchy sequence, then by Lemma 1 there exists \( \varepsilon > 0 \) and sequences of positive integers \( m(k) \) and \( n(k) \) such that for all positive integer \( k \), \( n(k) > m(k) \geq k \), \( d(x_{m(k)}, x_{n(k)}) \geq \varepsilon \) and \( d(x_{m(k)}, x_{n(k)-1}) < \varepsilon \) and conditions from (a)-(d) of 1 hold.

From (2) and by setting \( x = x_{m(k)} \) and \( y = x_{n(k)} \), we have:

\[
m(x_{m(k)}, x_{n(k)}, d, T, \varphi) = \max \left\{ \left( d(x_{m(k)}, x_{n(k)}) + \varphi(x_{m(k)}) + \varphi(x_{n(k)}) \right), \left( d(x_{m(k)}, x_{n(k)}), d(x_{m(k)}, T_{x_{m(k)}}) + \varphi(x_{m(k)}) \right) \right\} + \varphi(T_{x_{m(k)}}, d(x_{m(k)}, d(T_{x_{m(k)}}) + \varphi(x_{m(k)})) + \varphi(x_{m(k)})) + d(x_{n(k)}, x_{m(k)+1}) + \varphi(x_{m(k)}) + \varphi(x_{n(k)+1}) \right\} = \max \left\{ d(x_{m(k)}, x_{n(k)}) + \varphi(x_{m(k)}) + \varphi(x_{n(k)}) \right\}.
\]

Taking the limit as \( k \to \infty \) and using (10), (11) and Lemma 1, we have

\[
\lim_{k \to \infty} m(x_{m(k)}, x_{n(k)}, d, T, \varphi) = \max \left\{ d(x_{m(k)}, x_{n(k)}) + \varphi(x_{m(k)}) + \varphi(x_{n(k)}) \right\} \leq \max \left\{ \varepsilon, 0, \frac{1}{2s^2} (s^2 \varepsilon + s^2 \varepsilon) \right\} = \varepsilon.
\]

Similarly from (3), we have

\[
\lim_{k \to \infty} l(x_{m(k)}, x_{n(k)}, d, T, \varphi) = \max \left\{ d(x_{m(k)}, x_{n(k)}) + \varphi(x_{m(k)}) + \varphi(x_{n(k)}) \right\} \leq \max \left\{ \varepsilon, 0 \right\} = \varepsilon.
\]

Now from (1), we have

\[
\psi(s^3 d(T_{x}, Ty) + \varphi(Tx) + \varphi(Ty)) = \psi(s^3 d(x_{m(k)+1}, x_{n(k)+1}) + \varphi(x_{m(k)+1}) + \varphi(x_{n(k)+1}) \leq \psi(m(x_{m(k)+1}, x_{n(k)+1}, d, T, \varphi)) - \psi(l(x_{m(k)+1}, x_{n(k)+1}, d, T, \varphi)).
\]

Letting \( k \to \infty \), using (11), (12), (13), applying the continuity of \( \psi \) and lower semicontinuity of \( \varphi \), we have,

\[
\lim_{k \to \infty} \psi(s^3 d(x_{m(k)+1}, x_{n(k)+1})) \leq \psi(\varepsilon) - \psi(\varepsilon).
\]

This implies that

\[
\psi(\varepsilon) = \psi(s^3 \frac{\varepsilon}{s^2}) \leq \psi(s^3 \lim_{k \to \infty} d(x_{m(k)+1}, x_{n(k)+1})) \leq \psi(\varepsilon) - \psi(\varepsilon) < \psi(\varepsilon),
\]

which is a contradiction. Therefore \( \{x_n\} \) is a \( b \)-Cauchy sequence. Now since \( \{x_n\} \) is a \( b \)-Cauchy and \( X \) is \( b \)-complete we have,

\[
\lim_{n \to \infty} x_n = u \in X.
\]
Since $\phi$ is lower semicontinuous,

$$\varphi(u) \leq \lim_{n \to \infty} \inf \varphi(x_n) \leq \lim_{n \to \infty} \varphi(x_n) = 0,$$

which implies

$$\varphi(u) = 0. \quad (14)$$

Now from (2) by putting $x = x_n$ and $y = u$, we have

$$m(x_n, u, d, T, \varphi) = \max \{d(x_n, u) + \varphi(x_n) + \varphi(u), d(x_n, Tx_n) + \varphi(x_n) + \varphi(Tx_n), d(u, Tu) + \varphi(u) + \varphi(Tu), \frac{1}{2}\{d(x_n, Tu) + \varphi(x_n) + \varphi(Tu) + d(u, Tu) + \varphi(u) + \varphi(Tu), \frac{1}{2n}\{d(x_n, Tu) + \varphi(x_n) + \varphi(Tu) + d(u, x_{n+1}) + \varphi(u) + \varphi(x_{n+1})\}\}.$$

Applying the limit as $n \to \infty$ and using (10), (11) and (14) we have

$$\lim_{n \to \infty} m(x_n, u, d, T, \varphi) = d(u, Tu) + \varphi(Tu). \quad (15)$$

Similarly

$$\lim_{n \to \infty} l(x_n, u, d, T, \varphi) = d(u, Tu) + \varphi(Tu). \quad (16)$$

Then using (1), we have

$$\psi(s^3d(Tx_n, Tu) + \varphi(Tx_n) + \varphi(Tu)) = \psi(s^3d(x_{n+1}, Tu) + \varphi(x_{n+1}) + \varphi(Tu)) \leq \psi(m(x_n, u, d, T, \varphi)) - \varphi(l(x_n, u, d, T, \varphi)).$$

Letting $n \to \infty$, using (14), (15), (16) and by using the continuity of $\psi$ and lower semicontinuity of $\varphi$, we have

$$\psi(s^3d(x_{n+1}, Tu) + \varphi(x_{n+1}) + \varphi(Tu)) = \psi(s^3d(u, Tu) + \varphi(Tu)) \leq \psi(m(x_n, u, d, T, \varphi)) - \varphi(l(x_n, u, d, T, \varphi)) = \psi(d(u, Tu) + \varphi(Tu)) - \varphi(d(u, Tu) + \varphi(Tu)).$$

This implies

$$\psi(d(u, Tu) + \varphi(Tu)) \leq \psi(s^3d(u, Tu) + \varphi(Tu)) \leq \psi(d(u, Tu) + \varphi(Tu)) - \varphi(d(u, Tu) + \varphi(Tu)).$$

This holds if and only if, $\psi(d(u, Tu) + \varphi(Tu)) = 0$ and then from the property of $\phi$ we have,

$$d(u, Tu) + \varphi(Tu) = 0.$$

Hence, $d(u, Tu) = 0$ so that $u = Tu$ and $\varphi(Tu) = 0$. Since $u = Tu$ this implies $\varphi(u) = 0$.

Therefore $u$ is fixed point of $T$.

**Uniqueness**

Suppose $v$ is another fixed point of $T$. Then $Tv = v$ and $\varphi(v) = 0$.

By (1) with $x = u$ and $y = v$

$$\psi(s^3d(Tu, Tv) + \varphi(Tu) + \varphi(Tv)) = \psi(s^3d(u, v)) \leq \psi(m(Tu, Tv, d, T, \varphi)) - \varphi(l(Tu, Tv, d, T, \varphi)).$$

From (2) we have

$$m(Tu, Tv, d, T, \varphi) = \max \{d(Tu, Tv) + \varphi(Tu) + \varphi(Tv), d(Tu, T^2u) + \varphi(Tu) + \varphi(T^2u), d(Tv, T^2v) + \varphi(Tv) + \varphi(T^2v), \frac{1}{2}\{d(Tu, T^2v) + \varphi(Tu) + \varphi(T^2v) + d(Tv, T^2v) + \varphi(Tv) + \varphi(T^2v)\}\} = \max \{d(u, v) + \varphi(u) + \varphi(v) + \varphi(Tu) + \varphi(Tv), \frac{1}{2n}\{d(u, v) + \varphi(u) + \varphi(v) + d(v, u) + \varphi(v) + \varphi(u)\}\} = d(u, v).$$

Similarly from (3), we have
(Tu, Tv, d, T, \varphi) = \max \{d(Tu, Tv) + \varphi(Tu) + \varphi(Tv), d(Tv, T^2v) + \varphi(Tv) + \varphi(T^2v) \}
= d(u, v) + \varphi(u) + \varphi(v), d(v, v) + \varphi(v) + \varphi(v) \}= d(u, v).

Then using (1) and the continuity of \( \varphi \), we have
\[ \varphi(d(u, v)) \leq \varphi(s^2d(u, v)) \leq \varphi(d(u, v)) - \varphi(d(u, v)). \]

This holds if \( \varphi(d(u, v)) = 0 \) and then we have \( d(u, v) = 0 \). Hence \( u = v \). Therefore, \( T \) has a unique fixed point.

**Example 2.** Let \( X = [0, 1] \) and \( d(x, y) = (x - y)^2 \). Then \( (X, d) \) is \( b \)-metric space with \( s = 2 \). Define \( T : X \to X \),
\[ \varphi : X \to R^+ \text{ and } \psi, \phi : R^+ \to R^+ \text{ by } T(x) = \begin{cases} 0 & \text{if } x \leq \frac{1}{2}, \\ \frac{1}{s} \text{ if } x \in (\frac{1}{2}, 1], \end{cases} \phi(t) = \begin{cases} t & \text{if } t \leq 1, \\ \frac{1}{s} & \text{if } t > 1. \end{cases} \]

And
\[ \varphi(t) = \begin{cases} 2t & \text{if } t > 1, \\ t & \text{if } 0 \leq t \leq 1. \end{cases} \]

**Case I:** If \( x, y \in [0, \frac{1}{2}] \) and \( x \geq y \), then
\[ \varphi [s^2d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)] = \varphi [2^2(Tx - Ty)^2 + \varphi(Tx) + \varphi(Ty)] = \varphi [8(0) + \varphi(0) + \varphi(0)] = 0. \]

Also \( d(x, y) + \varphi(x) + \varphi(y) = (x - y)^2 + \varphi(x) + \varphi(y) = (x - y)^2 + x + y, \)
\[ d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) = (x - Ty)^2 + \varphi(Tx) = x^2 + x, \]
\[ \frac{1}{s} d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) = \frac{1}{s} (y - Ty)^2 + \varphi(Tx) = y^2 + y, \]
\[ \frac{1}{s} d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) = \frac{1}{s} (y - Ty)^2 + \varphi(Tx) = \frac{1}{s} (x^2 + x + y^2 + y). \]

And
\[ m(x, y, d, T, \varphi) = \max \{(x - y)^2 + x + y, x^2 + x, y^2 + y, \frac{1}{s} (x^2 + x + y^2 + y) \}, \]
\[ \frac{1}{s} (x^2 + x + y^2 + y) \leq \max \{x^2 + x, y^2 + y\} = x^2 + x, \]
\[ m(x, y, d, T, \varphi) = \max \{x - y^2 + x + y, x^2 + x\} \]
and
\[ l(x, y, d, T, \varphi) = \max \{(x - y)^2 + x + y, y^2 + y\}. \]

But
\[ (x - y)^2 + x + y \geq x + y \geq y + y \geq y^2 + y, \]
so, \( l(x, y, d, T, \varphi) = (x - y)^2 + x + y \) and \( \varphi[l(x, y, d, T, \varphi)] = \frac{1}{s} (x - y)^2 + x + y \).

If \( m(x, y, d, T, \varphi) = (x - y)^2 + x + y \) then (1) becomes
\[ 0 \leq \frac{5}{8} [(x - y)^2 + x + y] - \frac{1}{5} [(x - y)^2 + x + y] = \frac{1}{5} [(x - y)^2 + x + y], \]
and
\[ m(x, y, d, T, \varphi) = \max \{(x - y)^2 + x + y, x^2 + x, y^2 + y, \frac{1}{s} (x^2 + x + y^2 + y) \}, \]
\[ \frac{1}{s} (x^2 + x + y^2 + y) \leq \max \{x^2 + x, y^2 + y\} = y^2 + y, \]
implies
\[ m(x, y, d, T, \varphi) = \max \{(x - y)^2 + x + y, y^2 + y\}. \]

Similarly
\[ l(x, y, d, T, \varphi) = \max \{(x - y)^2 + x + y, y^2 + y\}. \]
Now, if \( m(x, y, d, T, \varphi) = (x - y)^2 + x + y \), \( l(x, y, d, T, \varphi) \), (1) becomes
\[ 0 \leq \frac{5}{8} [(x - y)^2 + x + y]. \]
If \( m(x, y, d, T, \varphi) = y^2 + y = l(x, y, d, T, \varphi) \), (1) becomes
\[ 0 \leq \frac{5}{8} [(x - y)^2 + x + y]. \]

**Case II:** If \( x \in [0, \frac{1}{2}] \) and \( y \in (\frac{1}{2}, 1] \). This implies \( x < y \). Then
\[ \varphi [s^2d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)] = \varphi [2^2(Tx - Ty)^2 + \varphi(Tx) + \varphi(Ty)] = \varphi [8(0) - \frac{1}{10} 2^2 + \varphi(0) + \varphi(\frac{1}{16})] = \frac{5}{4} (\frac{1}{16} + \frac{1}{16}) = \frac{15}{128}. \]

Also \( d(x, y) + \varphi(x) + \varphi(y) = (x - y)^2 + \varphi(x) + \varphi(y) = (x - y)^2 + x + y, \)
\[ d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) = (x - Ty)^2 + \varphi(Tx) = x^2 + x, \]
\[ d(Ty, Tx) + \varphi(Ty) + \varphi(Tx) = (y - Tx)^2 + \varphi(Tx) = y^2 + y, \]
\[ \frac{1}{s} d(Ty, Tx) + \varphi(Ty) + \varphi(Tx) = \frac{1}{s} (x^2 + x + y^2 + y) = \frac{1}{s} (x^2 + x + y^2 + y) \]
\[ \frac{1}{s} d(Ty, Tx) + \varphi(Ty) + \varphi(Tx) = \frac{1}{s} (x - Ty)^2 + \varphi(Tx) = \frac{1}{s} (x^2 + x + y^2 + y) \]
\[ \frac{1}{s} d(Ty, Tx) + \varphi(Ty) + \varphi(Tx) = \frac{1}{s} (x - Ty)^2 + \varphi(Tx) = \frac{1}{s} (x^2 + x + y^2 + y) \]
Case III: If $x, y \in (\frac{1}{8}, 1]$ and $x \geq y$. Then $\psi[s^3d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)] = \psi[8(\frac{1}{16} - \frac{1}{32})^2 + \frac{1}{16} + \frac{1}{16}] = \psi[\frac{1}{8}] = \frac{5}{32}$.

Also $d(x, y) + \varphi(x) + \varphi(y) = (x - y)^2 + x + y, d(Tx, Ty) + \varphi(Tx) = (x - \frac{1}{16})^2 + x + \frac{1}{16}$.

$\frac{1}{2}d(x, Ty) + \varphi(x) + \varphi(Ty) = \frac{1}{8}(x - \frac{1}{16})^2 + x + \frac{1}{16}$.

This implies that $m(x, y, d, T, \varphi) = \max\{\varphi(y) + \varphi(x) + \varphi(Ty) + d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)\} = \frac{1}{8}(x - \frac{1}{16})^2 + x + \frac{1}{16} + (y - \frac{1}{16})^2 + y + \frac{1}{16}$

Now if, $m(x, y, d, T, \varphi) = (x - y)^2 + x + y$, then (1) becomes $\frac{5}{32} \leq \frac{5}{8}[(x - y)^2 + x + y] - \frac{1}{8}[(x - y)^2 + x + y] = \frac{5}{16}[(x - y)^2 + x + y]$.

If $m(x, y, d, T, \varphi) = (x - y)^2 + x + \frac{1}{16}$, then we have, $\frac{5}{8}[(x - \frac{1}{16})^2 + x + \frac{1}{16}] = \frac{5}{16}[(x - \frac{1}{16})^2 + x + \frac{1}{16}] > \frac{5}{32}.

Let $x, y \in (\frac{1}{16}, 1]$ and $x < y$. Then $m(x, y, d, T, \varphi) = \max\{\varphi(x)^2 + x + y, (y - \frac{1}{16})^2 + y + \frac{1}{16}\} = l(x, y, d, T, \varphi)$.

Now if $m(x, y, d, T, \varphi) = l(x, y, d, T, \varphi)$, then (1) becomes $\frac{5}{32} \leq \frac{5}{8}[(y - \frac{1}{16})^2 + y + \frac{1}{16}] - \frac{1}{8}[(y - \frac{1}{16})^2 + y + \frac{1}{16}] = \frac{5}{16}[(y - \frac{1}{16})^2 + y + \frac{1}{16}]$.

Case IV: If $x \in (\frac{1}{4}, 1), y \in [0, \frac{1}{4})$. This implies $x > y$. Then $\psi[s^3d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)] = \psi[2^3(Tx - Ty)^2 + \varphi(Tx) + \varphi(Ty)] = \frac{5}{8}[\left(\frac{1}{16} - 0\right)^2 + \frac{7}{8} + \frac{1}{16}] = \frac{15}{128}$.

Also $d(x, y) + \varphi(x) + \varphi(y) = (x - y)^2 + x + y$.

$d(x, Ty) + \varphi(x) + \varphi(Ty) = x^2 + \frac{7}{8} + \frac{17}{256}$.

This implies that $m(x, y, d, T, \varphi) = \max\{\varphi(y) + \varphi(x) + \varphi(Ty) + d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)\} = \frac{1}{8}[x^2 + x + y^2 + \frac{7}{8} + \frac{17}{256}]$.

And $m(x, y, d, T, \varphi) = \max\{(x - y)^2 + x^2 + \frac{7}{8} + \frac{17}{256}, d(x, Tx) + \varphi(x) + \varphi(Tx) + \varphi(y) + \varphi(y) + \varphi(Ty)\} = \frac{1}{8}[x^2 + x + y^2 + \frac{7}{8} + \frac{17}{256}]$.

$m(x, y, d, T, \varphi) = \max\{(x - y)^2 + x^2 + \frac{7}{8} + \frac{17}{256}, d(x, Ty) + \varphi(x) + \varphi(Ty) + d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)\} = \frac{1}{8}[x^2 + x + y^2 + \frac{7}{8} + \frac{17}{256}]$.

Thus all the condition of Theorem (2) are satisfied and $0$ is the unique fixed point of $T$.

Remark 1. If we take $s = 1$ in Theorem (2) we get the result of Cho [1]. Hence Our result generalizes the result of Cho [1] and related results in the literature.

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References


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