Article

Characterization of a vector measure: application in the $GL(2;\mathbb{R})$ group

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Abstract: In this paper we characterize a bounded vector measure on Lie compact group $G = GL(2;\mathbb{R})$. It is a question of considering a bounded vector measure $m$ defined from $K(G;E)$ the space of $E$ valued functions with compact support on $G$ and giving its integral form.

Keywords: Vector measure, Haar measure, Lie compact group, absolute continuity.

1. Introduction


In this paper, we treat with a special case by giving a form to a bounded vector measure on $G = GL(2;\mathbb{R})$.

We consider a vector measure which is absolutely continue with respect to Haar measure and then give the integral form [5] to this vector measure. The first essential part of our work is to establish the form of Haar measure on $G = GL(2;\mathbb{R})$. We prove that

$$\mu(f) = \int_{\mathbb{R}^4} \frac{f(x_{11}x_{12};x_{21};x_{22})}{(x_{11}x_{22} - x_{12}x_{21})^2} dx_{11}dx_{12}dx_{21}dx_{22};$$

$\forall f \in K(G;E)$ and $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in GL(2;\mathbb{R})$; is a Haar measure on $G = GL(2;\mathbb{R})$. Once this demonstration achieved we go straight to generalize the form of a vector measure on $K(GL(2;\mathbb{R});\mathbb{R}^4)$.

This paper is organized as follows: in Section 2, we give some definitions related to vector measure and matrices and prove the fundamental theorem which will help us in proving our main result and in Section 3 we present our main result.

2. Preliminaries

In this section, we give basic definitions and concepts concerning with vector measure and Lie groups.

Definition 1. [2] Let $G$ be a locally compact group and $K(G;E)$ be the space of $E$ valued functions with compact support on $G$. A vector measure on $G$ with respect to Banach spaces $E$ and $F$ is a linear map:

$$m : K(G;E) \rightarrow F, \quad f \mapsto m(f)$$

such as $\forall K$ compact in $G$ $\exists k > 0, ||m(f)||_F \leq k ||f||_\infty$. Where $||.||_F$ is the norm on Banach spaces $F$ and $||f||_\infty = \sup\{||f(t)||_E, t \in G\}$ is the norm on $K(G;E)$.

The value $m(f)$ of $m$ in $f \in K(G;E)$ is called integral of $f$ with respect to $m$ and can be written as [5,6]:

$$\int_{G} f(t)dm(t) = m(f).$$
We consider \( GL(2; \mathbb{R}) \) is the set of matrices of order two with real coefficients whose determinant is not equal to zero, i.e.,

\[
GL(2; \mathbb{R}) = \left\{ g = \begin{pmatrix} g_{11} & g_{12} \\
g_{21} & g_{22} \end{pmatrix} ; g_{ij} \in \mathbb{R}; 1 \leq i, j \leq 2 \mid \det g \neq 0 \right\}.
\]

\( G = GL(2; \mathbb{R}) \) is a Lie group. Also \( G \) is a manifold such that, at any point \( g \in G \), there exists an open \( V_g \) of \( G \), an open \( U_g \) of \( \mathbb{R}^4 \) and \( \varphi_g \) a diffeomorphism of \( V_g \) in \( U_g \). So each \( x = \begin{pmatrix} x_{11} & x_{12} \\
x_{21} & x_{22} \end{pmatrix} \in GL(2; \mathbb{R}) \) is assimilated to \((x_{11}; x_{12}; x_{21}; x_{22})\) of \( \mathbb{R}^4 \).

In order to prove our main result, first we prove following fundamental theorem which allow us to get our final result. The following theorem gives us Haar’s measure on \( GL(2; \mathbb{R}) \).

**Theorem 1.** Let \( K(G; E) \) be the space of \( E \) valued functions with compact support on \( G \), where \( G = GL(2; \mathbb{R}) \) and \( E = \mathbb{R}^4 \). Then \( \mu : K(G; E) \to \mathbb{R}^+ \) defined as

\[
\mu(f) = \int_{\mathbb{R}^4} \frac{f(x_{11}; x_{12}; x_{21}; x_{22})}{(x_{11}x_{22} - x_{12}x_{21})^2} dx_{11}dx_{12}dx_{21}dx_{22};
\]

\( \forall f \in K(G; E) \) and \( x = \begin{pmatrix} x_{11} & x_{12} \\
x_{21} & x_{22} \end{pmatrix} \in GL(2; \mathbb{R}) \) is a Haar measure on \( G = GL(2; \mathbb{R}) \).

**Proof.** If \( \mu \) is a Haar measure on \( G = GL(2; \mathbb{R}) \) then \( d\mu(x) = \frac{1}{(\det x)^2} dx \), \( \forall x \in G \). As

\[
d\mu(x) = \frac{1}{(x_{11}x_{22} - x_{12}x_{21})^2} dx_{11}dx_{12}dx_{21}dx_{22}.
\]

Since \( \mu(f) = \int_G f(x)d\mu(x), \forall f \in K(G; E) \), we get:

\[
\mu(f) = \int_{\mathbb{R}^4} \frac{f(x_{11}; x_{12}; x_{21}; x_{22})}{(x_{11}x_{22} - x_{12}x_{21})^2} dx_{11}dx_{12}dx_{21}dx_{22};
\]

Conversely if \( \mu \) is a Haar measure on \( G = GL(2; \mathbb{R}) \) then we have [6]:

\[
\mu(f) = \int_G f(x)|J(L_x)|^{-1} dx \quad \forall f \in K(G; E)
\]

The translation on the left \( L_x : y \mapsto xy \quad x; y \in G = GL(2; \mathbb{R}) \)

\[
L_x(y) = \begin{pmatrix} x_{11} & x_{12} \\
x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\
y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} x_{11}y_{11} + x_{12}y_{21} & x_{11}y_{12} + x_{12}y_{22} \\
x_{21}y_{11} + x_{22}y_{21} & x_{21}y_{12} + x_{22}y_{22} \end{pmatrix}
\]

\[
L_x(y) = (x_{11}y_{11} + x_{12}y_{21}; x_{11}y_{12} + x_{12}y_{22}; x_{21}y_{11} + x_{22}y_{21}; x_{21}y_{12} + x_{22}y_{22})
\]

\[
J(L_x) = \frac{dL}{dy} = \begin{pmatrix} x_{11} & 0 & x_{12} & 0 \\
0 & x_{11} & 0 & x_{12} \\
x_{21} & 0 & x_{22} & 0 \\
0 & x_{21} & 0 & x_{22} \end{pmatrix}
\]

The Jacobian gives:

\[
|J(L_x)| = \left| \begin{array}{cccc} x_{11} & 0 & x_{12} & 0 \\
0 & x_{22} & 0 & x_{12} \\
x_{21} & 0 & x_{22} & 0 \\
0 & x_{21} & 0 & x_{22} \end{array} \right| + x_{12} 
\]
form. Using Theorem 2, we get

$$m(x) = |J(L_\mu)| = \left| \frac{dL}{dy} \right| = (x_{11}x_{22})^2 - x_{11}x_{12}x_{21}x_{22} - x_{11}x_{12}x_{21}x_{22} + \left( x_{12}x_{21} \right)^2$$

$$= (x_{11}x_{22})^2 - 2x_{11}x_{12}x_{21}x_{22} + \left( x_{12}x_{21} \right)^2$$

$$= (x_{11}x_{22} - x_{21}x_{12})^2$$

$$= (\det x)^2$$

According to the relationship (3) we have:

$$\mu(f) = \int_G f(x)|J(L_\mu)|^{-1} dx \quad \forall f \in K(G; E) \quad x \in G$$

$$= \int_G f(x) \left( (\det x)^2 \right)^{-1} dx$$

$$= \int_G f(x) (\det x)^2 dx$$

$$= \int_{\mathbb{R}^4} f(x_{11}; x_{12}; x_{21}; x_{22}) \frac{x_{11}x_{22} - x_{21}x_{12}}{(x_{11}x_{22})^2} dx_{11} dx_{12} dx_{21} dx_{22};$$

The last form of $\mu$ is a 4 linear, alternating, positive, finite, left-invariant form so it is a Haar measure on $GL(2; \mathbb{R})$.

The following theorem is of a capital importance.

**Theorem 2.** Let $m$ be a bounded vector measure on compact Lie group $G$ and $E$ and $F$ two Banach spaces. If $m$ is a continuous alternating linear form in $L^p(G; E)$ then $m$ is absolutely continuous.

The following theorems are important, because once we establish the form of the Haar measure on $G$, it will be easy to establish a vector measure on $K(G; E)$.

**Theorem 3.** [3] Let $G$ be a compact group, $m$ be a bounded vector measure on $G$ and $p$ and $q$ are two conjugates numbers with $p \geq 1$. Then the following two assertions are equivalent:

1. $\forall h \in L^p(G; E), m \ast h \in C(G; E)$,
2. $\exists f \in L^q(G; E)$ such as $m = f \mu$.

**Theorem 4.** [3] Let $\mu$ be a Haar measure on compact Lie group $G$, $p \in [1, \infty]$ and $q$ conjugate of $p$. If $\phi$ is a linear continuous form on $L^p(G; E)$ then there exists a map $f \in L^q(G, E)$ such as for any $g \in L^p$, we have $\phi(g) = \int_G g f d\mu$.

We use the duality theorem for $p = 1$ and $q = \infty$.

3. **Main result**

In this section, we give our main result.

**Theorem 5.** Let $K(G; E)$ be the space of $E$ valued functions with compact support on $G$, where $G = GL(2; \mathbb{R})$ and $E = \mathbb{R}^4$. If $m$ is a vector measure on $K(G; E)$, then

$$m(f) = \int_{\mathbb{R}^4} g(x_{11}; x_{12}; x_{21}; x_{22}) f(x_{11}; x_{12}; x_{21}; x_{22}) \frac{x_{11}x_{22} - x_{21}x_{12}}{(x_{11}x_{22})^2} dx_{11} dx_{12} dx_{21} dx_{22};$$

(4)

$$\forall \ f \in K(GL(2; \mathbb{R}); \mathbb{R}^4), \ g \in L^\infty(GL(2; \mathbb{R}); \mathbb{R}^4), \ x \in GL(2; \mathbb{R})$$

**Proof.** Since $m$ being a vector measure on $K(GL(2; \mathbb{R}); \mathbb{R}^4)$, so $m$ is continuous, alternating and bounded linear form. Using Theorem 2, we get $m \ll \mu$. 
The Theorems 3 and 4 allow us to write \( dm = g d\mu; \quad g \in L^\infty(GL(2; \mathbb{R})) \), which implies

\[
dm(x) = \frac{g(x)}{(\det x)^2} dx;
\]

\[
m(f) = \int_G \frac{g(x)f(x)}{(\det x)^2} dx \quad f \in K(GL(2; \mathbb{R}); \mathbb{R}^4)
\]

\[
= \int_{\mathbb{R}^4} \frac{g(x_{11}; x_{12}; x_{21}; x_{22})f(x_{11}; x_{12}; x_{21}; x_{22})}{(x_{11}x_{22} - x_{12}x_{21})^2} dx_{11}dx_{12}dx_{21}dx_{22}.
\]

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