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Characterization of a vector measure: application in the $GL(2;\mathbb{R})$ group

Abalo Douhadji[1,*] and Yaovi Awussi[2]

1 Department of Mathematics, University of Lomé, PObox 1515, Lomé, Togo.
2 Department of Mathematics, Mathematics and Applications Laboratory, University of Lomé, PObox 1515, Lomé, Togo.; jawussi@yahoo.fr
* Correspondence: douhadjiabalo@gmail.com

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Abstract: In this paper we characterize a bounded vector measure on Lie compact group $G = GL(2;\mathbb{R})$. It is a question of considering a bounded vector measure m defined from $K(G;E)$ the space of E valued functions with compact support on G and giving its integral form.

Keywords: Vector measure, Haar measure, Lie compact group, absolute continuity.

1. Introduction

In 1989, Assiamoua [1] worked on the properties of vector measure, introduced by Diestel in [2]. In 2013, Awussi [3] proved that any bounded vector measure is absolutely continue with respect to Haar measure. In 2013, Mensah [4] worked on Fourier-Stieljes transform of vector measures on compact groups.

In this paper, we treat with a special case by giving a form to a bounded vector measure on $G = GL(2;\mathbb{R})$. We consider a vector measure which is absolutely continue with respect to Haar measure and then give the integral form [5] to this vector measure. The first essential part of our work is to establish the form of Haar measure on $G = GL(2;\mathbb{R})$. We prove that

$$\mu(f) = \int_{\mathbb{R}^4} \frac{f(x_{11};x_{12};x_{21};x_{22})}{(x_{11}x_{22} - x_{12}x_{21})^2} dx_{11}dx_{12}dx_{21}dx_{22};$$

$\forall f \in K(G;E)$ and $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in GL(2;\mathbb{R})$; is a Haar measure on $G = GL(2;\mathbb{R})$. Once this demonstration achieved we go straight to generalize the form of a vector measure on $K(GL(2;\mathbb{R});\mathbb{R}^4)$.

This paper is organized as follows: in Section 2, we give some definitions related to vector measure and matrices and prove the fundamental theorem which will help us in proving our main result and in Section 3 we present our main result.

2. Preliminaries

In this section, we give basic definitions and concepts concerning with vector measure and Lie groups.

Definition 1. [2] Let G be a locally compact group and $K(G;E)$ be the space of E valued functions with compact support on G. A vector measure on G with respect to Banach spaces E and F is a linear map:

$$\begin{aligned} m: \quad K(G;E) &\to F \\ f &\mapsto m(f) \end{aligned}$$

such as $\forall$ K compact in G $\exists\, a_K > 0, \|m(f)\|_F \leq a_K \|f\|_\infty$. Where $\|.\|_F$ is the norm on Banach spaces F and $\|f\|_\infty = sup\{\|f(t)\|_E, t \in G\}$ is the norm on $K(G;E)$.

The value $m(f)$ of m in $f \in K(G;E)$ is called integral of f with respect to m and can be written as [5,6]:

$$\int_G f(t)dm(t) = m(f).$$

We consider $GL(2;\mathbb{R})$ is the set of matrices of order two with real coefficients whose determinant is not equal to zero, *i.e.*,

$$GL(2;\mathbb{R}) = \left\{ g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}; g_{ij} \in \mathbb{R}; 1 \leqslant i,j \leqslant 2 | detg \neq 0 \right\}.$$

$G = GL(2;\mathbb{R})$ is a Lie group. Also G is a manifold such that, at any point $g \in G$, there exists an open V_g of G, an open U_g of $\mathbb{R}^4$ and φ_g a diffeomorphism of V_g in U_g. So each $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in GL(2;\mathbb{R})$ is assimilated to $(x_{11}; x_{12}; x_{21}; x_{22})$ of $\mathbb{R}^4$.

In order to prove our main result, first we prove following fundamental theorem which allow us to get our final result. The following theorem gives us Haar's measure on $GL(2;\mathbb{R})$.

Theorem 1. *Let $K(G;E)$ be the space of E valued functions with compact support on G, where $G = GL(2;\mathbb{R})$ and $E = \mathbb{R}^4$. Then $\mu : K(G;E) \to \mathbb{R}^+$ defined as*

$$\mu(f) = \int_{\mathbb{R}^4} \frac{f(x_{11}; x_{12}; x_{21}; x_{22})}{(x_{11}x_{22} - x_{12}x_{21})^2} dx_{11}dx_{12}dx_{21}dx_{22}; \tag{1}$$

$\forall f \in K(G;E)$ and $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in GL(2;\mathbb{R})$ is a Haar measure on $G = GL(2;\mathbb{R})$.

Proof. If μ is a Haar measure on $G = GL(2;\mathbb{R})$ then $d\mu(x) = \frac{1}{(detx)^2} dx, \quad \forall x \in G$. As

$$d\mu(x) = \frac{1}{(x_{11}x_{22} - x_{12}x_{21})^2} dx_{11}dx_{12}dx_{21}dx_{22}.$$

Since $\mu(f) = \int_G f(x) d\mu(x)$, $\forall f \in K(G;E)$, we get:

$$\mu(f) = \int_{\mathbb{R}^4} \frac{f(x_{11}; x_{12}; x_{21}; x_{22})}{(x_{11}x_{22} - x_{12}x_{21})^2} dx_{11}dx_{12}dx_{21}dx_{22}; \tag{2}$$

Conversely if μ is a Haar measure on $G = GL(2;\mathbb{R})$ then we have [6]:

$$\mu(f) = \int_G f(x) |J(L_x)|^{-1} dx \quad \forall f \in K(G;E) \tag{3}$$

The translation on the left $L_x : y \mapsto xy \quad x; y \in G = GL(2;\mathbb{R})$

$$L_x(y) = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} x_{11}y_{11} + x_{12}y_{21} & x_{11}y_{12} + x_{12}y_{22} \\ x_{21}y_{11} + x_{22}y_{21} & x_{21}y_{12} + x_{22}y_{22} \end{pmatrix}$$

$$L_x(y) = (x_{11}y_{11} + x_{12}y_{21}; x_{11}y_{12} + x_{12}y_{22}; x_{21}y_{11} + x_{22}y_{21}; x_{21}y_{12} + x_{22}y_{22})$$

$$J(L_x) = \frac{dL}{dy} = \begin{pmatrix} x_{11} & 0 & x_{12} & 0 \\ 0 & x_{11} & 0 & x_{12} \\ x_{21} & 0 & x_{22} & 0 \\ 0 & x_{21} & 0 & x_{22} \end{pmatrix}$$

The Jacobian gives:

$$|J(L_x)| = \left| \frac{dL}{dy} \right| = x_{11} \begin{vmatrix} x_{11} & 0 & x_{12} \\ 0 & x_{22} & 0 \\ x_{21} & 0 & x_{22} \end{vmatrix} + x_{12} \begin{vmatrix} 0 & x_{11} & x_{12} \\ x_{21} & 0 & 0 \\ 0 & x_{21} & x_{22} \end{vmatrix}$$

$$
\begin{aligned}
|J(L_x)| = \left|\frac{dL}{dy}\right| &= (x_{11}x_{22})^2 - x_{11}x_{12}x_{21}x_{22} - x_{11}x_{12}x_{21}x_{22} + (x_{12}x_{21})^2 \\
&= (x_{11}x_{22})^2 - 2x_{11}x_{12}x_{21}x_{22} + (x_{12}x_{21})^2 \\
&= (x_{11}x_{22} - x_{21}x_{12})^2 \\
&= (det\, x)^2
\end{aligned}
$$

According to the relationship (3) we have:

$$
\begin{aligned}
\mu(f) &= \int_G f(x)|J(L_x)|^{-1}dx \quad \forall f \in K(G;E) \quad x \in G \\
&= \int_G f(x) \left((det\, x)^2\right)^{-1} dx \\
&= \int_G \frac{f(x)}{(det\, x)^2} dx \\
&= \int_{\mathbb{R}^4} \frac{f(x_{11};x_{12};x_{21};x_{22})}{(x_{11}x_{22} - x_{12}x_{21})^2} dx_{11}dx_{12}dx_{21}dx_{22};
\end{aligned}
$$

The last form of μ is a 4 linear, alternating, positive, finite, left-invariant form so it is a Haar measure on $GL(2;\mathbb{R})$.

□

The following theorem is of a capital importance.

Theorem 2. *Let m be a bounded vector measure on compact Lie group G and E and F two Banach spaces. If m is a continuous alternating linear form in $L^p(G;E)$ then m is absolutely continuous.*

The following theorems are important, because once we establish the form of the Haar measure on G, it will be easy to establish a vector measure on $K(G;E)$.

Theorem 3. *[3] Let G be a compact group, m be a bounded vector measure on G and p and q are two conjugates numbers with $p \geq 1$. Then the following two assertions are equivalent:*

1. *$\forall h \in L^p(G;E), m * h \in C(G;E)$,*
2. *$\exists f \in L^q(G;E)$ such as $m = f\mu$.*

Theorem 4. *[3] Let μ be a Haar measure on compact Lie group G, $p \in [1, \infty[$ and q conjugate of p. If ϕ is a linear continuous form on $L^p(G;E)$ then there exists a map $f \in L^q(G,E)$ such as for any $g \in L^p$, we have $\phi(g) = \int_G gfd\mu$.*

We use the duality theorem for $p = 1$ and $q = \infty$.

3. Main result

In this section, we give our main result.

Theorem 5. *Let $K(G;E)$ be the space of E valued functions with compact support on G, where $G = GL(2;\mathbb{R})$ and $E = \mathbb{R}^4$. If m is a vector measure on $K(G;E)$, then*

$$
m(f) = \int_{\mathbb{R}^4} \frac{g(x_{11};x_{12};x_{21};x_{22})f(x_{11};x_{12};x_{21};x_{22})}{(x_{11}x_{22} - x_{12}x_{21})^2} dx_{11}dx_{12}dx_{21}dx_{22}; \tag{4}
$$

$$
\forall \quad f \in K(GL(2;\mathbb{R});\mathbb{R}^4), \quad g \in L^\infty(GL(2;\mathbb{R});\mathbb{R}^4), \quad x \in GL(2;\mathbb{R})
$$

Proof. Since m being a vector measure on $K(GL(2;\mathbb{R});\mathbb{R}^4)$, so m is continuous, alternating and bounded linear form. Using Theorem 2, we get $m \ll \mu$.

The Theorems 3 and 4 allow us to write $dm = gd\mu; \quad g \in L^{\infty}(GL(2;\mathbb{R}))$, which implies

$$\begin{aligned} dm(x) &= \frac{g(x)}{(detx)^2}dx; \\ m(f) &= \int_G \frac{g(x)f(x)}{(detx)^2}dx \quad f \in K(GL(2;\mathbb{R});\mathbb{R}^4) \\ &= \int_{\mathbb{R}^4} \frac{g(x_{11};x_{12};x_{21};x_{22})f(x_{11};x_{12};x_{21};x_{22})}{(x_{11}x_{22}-x_{12}x_{21})^2}dx_{11}dx_{12}dx_{21}dx_{22}. \end{aligned}$$

□

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