Open Journal of Mathematical Sciences

PSR Press

Article

Fractional integrals inequalities for exponentially m-convex functions

Sajid Mehmood[1,*] and Ghulam Farid[1]

[1] Department of Mathematics, COMSATS University Islamabad, Attock Campus, Pakistan.; ghlmfarid@cuiatk.edu.pk(G.F)

* Correspondence: smjg227@gmail.com

Received: 18 October 2019; Accepted: 9 February 2020; Published: 24 March 2020.

Abstract: Fractional integral operators are very useful in mathematical analysis. This article investigates bounds of generalized fractional integral operators by exponentially m-convex functions. Furthermore, a Hadamard type inequality have been analyzed and, special cases of established results have been discussed.

Keywords: Convex function, exponentially m-convex functions, fractional integrals operators, bounds.

MSC: 26A51, 26A33, 26D15.

1. Introduction

Fractional integral operators play a vital role in the advancement of mathematical inequalities and many integral inequalities have been established in literature. In [1], Farid established the bounds of the Riemann-Liouville fractional integral operators for convex function. For more information related to fractional integral inequalities, the readers are referred to [2–10].

Definition 1. Let $\psi \in L_1[a,b]$ with $0 \leq a < b$. Then the left-sided and right-sided Riemann-Liouville fractional integral operators of a function ψ of order $\sigma > 0$ are defined as follows:

$$\zeta^{\sigma}_{a^+}\psi(u) = \frac{1}{\Gamma(\sigma)}\int_a^u (u-\xi)^{\sigma-1}\psi(\xi)d\xi, \quad u > a \tag{1}$$

and

$$\zeta^{\sigma}_{b^-}\psi(u) = \frac{1}{\Gamma(\sigma)}\int_u^b (\xi-u)^{\sigma-1}\psi(\xi)d\xi, \quad u < b, \tag{2}$$

where $\Gamma(\sigma)$ is the Gamma function defined as $\Gamma(\sigma) = \int_0^{\infty} t^{\sigma-1}e^{-t}dt$.

In [9], Mubeen *et al.* defined the following Riemann-Liouville k-fractional integral operators;

Definition 2. Let $\psi \in L_1[a,b]$ with $0 \leq a < b$. Then the left-sided and right-sided Riemann-Liouville k-fractional integral operators of a function ψ of order $\sigma, k > 0$ are defined as follows;

$$\zeta^{\sigma,k}_{a^+}\psi(u) = \frac{1}{k\Gamma_k(\sigma)}\int_a^u (u-\xi)^{\frac{\sigma}{k}-1}\psi(\xi)d\xi, \quad u > a \tag{3}$$

and

$$\zeta^{\sigma,k}_{b^-}\psi(u) = \frac{1}{k\Gamma_k(\sigma)}\int_u^b (\xi-u)^{\frac{\sigma}{k}-1}\psi(\xi)d\xi, \quad u < b, \tag{4}$$

where $\Gamma_k(\sigma)$ is the k-Gamma function defined as $\Gamma_k(\sigma) = \int_0^{\infty} t^{\sigma-1}e^{-\frac{t^k}{k}}dt$.

In [11], generalized Riemann-Liouville fractional integral operators are given as follows;

Definition 3. Let $\psi \in L_1[a,b]$ with $0 \leq a < b$ and ϕ be an increasing and positive function on $(a,b]$ having a continuous derivative ϕ' on (a,b). Then the left-sided and right-sided generalized Riemann-Liouville

fractional integral operators of a function ψ with respect to another function ϕ on $[a,b]$ of order $\sigma > 0$ are defined as follows;

$$\zeta^{\sigma}_{\phi,a^+}\psi(u) = \frac{1}{\Gamma(\sigma)}\int_a^u (\phi(u)-\phi(\xi))^{\sigma-1}\phi'(\xi)\psi(\xi)d\xi,\ u > a \tag{5}$$

and

$$\zeta^{\sigma}_{\phi,b^-}\psi(u) = \frac{1}{\Gamma(\sigma)}\int_u^b (\phi(\xi)-\phi(u))^{\sigma-1}\phi'(\xi)\psi(\xi)d\xi,\ u < b. \tag{6}$$

In [7], Kwun *et al.* defined the generalized Riemann-Liouville k-fractional integral operators as follows;

Definition 4. Let $\psi \in L_1[a,b]$ with $0 \le a < b$ and ϕ be an increasing and positive function on $(a,b]$ having a continuous derivative ϕ' on (a,b). Then the left-sided and right-sided generalized Riemann-Liouville k-fractional integral operators of a function ψ with respect to another function ϕ on $[a,b]$ of order $\sigma, k > 0$ are defined as follows:

$$\zeta^{\sigma,k}_{\phi,a^+}\psi(u) = \frac{1}{k\Gamma_k(\sigma)}\int_a^u (\phi(u)-\phi(\xi))^{\frac{\sigma}{k}-1}\phi'(\xi)\psi(\xi)d\xi,\ u > a \tag{7}$$

and

$$\zeta^{\sigma,k}_{\phi,b^-}\psi(u) = \frac{1}{k\Gamma_k(\sigma)}\int_u^b (\phi(\xi)-\phi(u))^{\frac{\sigma}{k}-1}\phi'(\xi)\psi(\xi)d\xi,\ u < b. \tag{8}$$

For suitable settings of ϕ and k, some interesting consequences can be achieved which are given in following remark;

Remark 1.
1. For $k = 1$, (7) and (8) fractional integrals coincide with (5) and (6) fractional integrals.
2. By taking ϕ as identity function, (7) and (8) fractional integrals coincide with (1) and (2) fractional integrals.
3. For $k = 1$, along with ϕ as identity function, (7) and (8) fractional integrals coincide with (3) and (4) fractional integrals.
4. For $\phi(u) = \frac{u^{1+s}}{1+s}$, (7) and (8) produce conformable fractional integrals given in [12].
5. For $k = 1$, along with $\phi(u) = \frac{u^{\rho}}{\rho}$, $\rho > 0$, (7) and (8) produce Katugampola fractional integrals given in [2].
6. For $k = 1$, along with $\phi(u) = \frac{u^{\tau+s}}{\tau+s}$, (7) and (8) produce conformable fractional integrals given in [13].
7. For $\phi(u) = \frac{(u-a)^s}{s}$, $s > 0$ in (7) and $\phi(u) = -\frac{(b-u)^s}{s}$, $s > 0$ in (8) then conformable (k,s)-fractional integrals.
8. For $\phi(u) = \frac{(u-a)^s}{s}$, $s > 0$ in (7) and $\phi(u) = -\frac{(b-u)^s}{s}$, $s > 0$ in (8), along with $k = 1$, then conformable fractional integrals are achieved given in [14] are achieved.

Next, we give the definition of exponentially convex function.

Definition 5. [15,16] A function $\psi : [a,b] \to \mathbb{R}$ is said to be exponentially convex if for all $x, y \in [a,b]$ and $u \in [0,1]$, the following inequality holds;

$$e^{\psi(ux+(1-u)y)} \le ue^{\psi(x)} + (1-u)e^{\psi(y)}. \tag{9}$$

The concept of exponentially m-convex functions was introduced by Rashid *et al.* in [17]. It is defined as follows;

Definition 6. A function $\psi : [a,b] \to \mathbb{R}$ is said to be exponentially m-convex, where $m \in (0,1]$, if for all $x, y \in [a,b]$ and $u \in [0,1]$, the following inequality holds;

$$e^{\psi(ux+m(1-u)y)} \le ue^{\psi(x)} + m(1-u)e^{\psi(y)}. \tag{10}$$

Remark 2. For $m = 1$ in (10), (9) is achieved.

The aim of this research is to establish the bounds of the fractional integral operators defined in Definition 4. To establish these bounds exponentially m-convexity has been utilized. The established results provide all the possible outcomes of fractional integral operators given in remark (1). The breakup of this paper is:

in Section 2, the first result provides the bounds of the generalized Riemann-Liouville k-fractional integral operators defined in Equations (7) and (8) for exponentially m-convex functions. The last result of Section 2 provides the fractional Hadamard type inequality. Furthermore, special cases of established results are also discussed. In Section 3, we give some applications of presented results.

2. Main result

First, we give the following bounds of the sum of the left-sided and right-sided fractional integral operators.

Theorem 1. *Let $\psi, \phi : [a,b] \longrightarrow \mathbb{R}$ be two functions such that ϕ be differentiable and $\psi \in L[a,b]$ with $a < b$. Also let ψ be exponentially m-convex, and ϕ be strictly increasing on $[a,b]$ with $\phi' \in L_1[a,b]$. Then for $u \in [a,b]$ and $\sigma, \tau \geq k$, the following inequality holds;*

$$k\left(\Gamma_k(\sigma)\zeta^{\sigma,k}_{\phi,a^+}e^{\psi(u)} + \Gamma_k(\tau)\zeta^{\tau,k}_{\phi,b^-}e^{\psi(u)}\right) \leq \frac{(\phi(u)-\phi(a))^{\frac{\sigma}{k}-1}}{u-a}\left[(u-a)\left(me^{\psi\left(\frac{u}{m}\right)}\phi(u) - e^{\psi(a)}\phi(a)\right)\right.$$
$$\left. - \left(me^{\psi\left(\frac{u}{m}\right)} - e^{\psi(a)}\right)\int_a^u \phi(\xi)d\xi\right] + \frac{(\phi(b)-\phi(u))^{\frac{\tau}{k}-1}}{b-u}\left[(b-u)\left(e^{\psi(b)}\phi(b) - me^{\psi\left(\frac{u}{m}\right)}\phi(u)\right)\right.$$
$$\left. - \left(e^{\psi(b)} - me^{\psi\left(\frac{u}{m}\right)}\right)\int_u^b \phi(\xi)d\xi\right]. \tag{11}$$

Proof. From exponentially m-convexity of ψ, we have

$$e^{\psi(\xi)} \leq \frac{u-\xi}{u-a}e^{\psi(a)} + m\frac{\xi-a}{u-a}e^{\psi\left(\frac{u}{m}\right)}. \tag{12}$$

Under given assumptions for the function ϕ: for $\sigma \geq k$, the following inequality holds;

$$\phi'(\xi)(\phi(u)-\phi(\xi))^{\frac{\sigma}{k}-1} \leq \phi'(\xi)(\phi(u)-\phi(a))^{\frac{\sigma}{k}-1}, \quad \xi \in [a,u] \quad and \quad u \in [a,b]. \tag{13}$$

Multiplying (12) with (13) and integrating over $[a,u]$, we have

$$\int_a^u (\phi(u)-\phi(\xi))^{\frac{\sigma}{k}-1}\phi'(\xi)e^{\psi(\xi)}d\xi$$
$$\leq \frac{(\phi(u)-\phi(a))^{\frac{\sigma}{k}-1}}{u-a}\left[e^{\psi(a)}\int_a^u (u-\xi)\phi'(\xi)d\xi + me^{\psi\left(\frac{u}{m}\right)}\int_a^u (\xi-a)\phi'(\xi)d\xi\right].$$

By using (7), we get the following estimation;

$$k\Gamma_k(\sigma)\zeta^{\sigma,k}_{\phi,a^+}e^{\psi(u)} \leq \frac{(\phi(u)-\phi(a))^{\frac{\sigma}{k}-1}}{u-a}\left[(u-a)\left(me^{\psi\left(\frac{u}{m}\right)}\phi(u) - e^{\psi(a)}\phi(a)\right) - \left(me^{\psi\left(\frac{u}{m}\right)} - e^{\psi(a)}\right)\int_a^u \phi(\xi)d\xi\right]. \tag{14}$$

Again, from exponentially m-convexity of ψ we have

$$e^{\psi(\xi)} \leq \frac{\xi-u}{b-u}e^{\psi(b)} + m\frac{b-\xi}{b-u}e^{\psi\left(\frac{u}{m}\right)}. \tag{15}$$

Now, for $\tau \geq k$, the following inequality holds;

$$\phi'(\xi)(\phi(\xi)-\phi(u))^{\frac{\tau}{k}-1} \leq \phi'(\xi)(\phi(b)-\phi(u))^{\frac{\tau}{k}-1}, \quad \xi \in [u,b] \quad and \quad u \in [a,b]. \tag{16}$$

Multiplying (15) with (16) and integrating over $[u,b]$, we have

$$\int_u^b (\phi(\xi)-\phi(u))^{\frac{\tau}{k}-1}\phi'(\xi)e^{\psi(\xi)}d\xi \leq \frac{(\phi(b)-\phi(u))^{\frac{\tau}{k}-1}}{b-u}\left[e^{\psi(b)}\int_u^b (\xi-u)\phi'(\xi)d\xi + me^{\psi\left(\frac{u}{m}\right)}\int_u^b (b-\xi)\phi'(\xi)d\xi\right].$$

By using (8), we get the following estimation;

$$k\Gamma_k(\tau)\zeta^{\tau,k}_{\phi,b^-}e^{\psi(u)} \leq \frac{(\phi(b)-\phi(u))^{\frac{\tau}{k}-1}}{b-u}\left[(b-u)\left(e^{\psi(b)}\phi(b)-me^{\psi\left(\frac{u}{m}\right)}\phi(u)\right)-\left(e^{\psi(b)}-me^{\psi\left(\frac{u}{m}\right)}\right)\int_u^b\phi(\xi)d\xi\right] \quad (17)$$

From (14) and (17) we achieve (11). □

Corollary 1. *If we put $\sigma=\tau$ in (11), then following inequality holds;*

$$\zeta^{\sigma,k}_{\phi,a^+}e^{\psi(u)}+\zeta^{\sigma,k}_{\phi,b^-}e^{\psi(u)} \leq \frac{(\phi(u)-\phi(a))^{\frac{\sigma}{k}-1}}{k\Gamma_k(\sigma)(u-a)}\left[(u-a)\left(me^{\psi\left(\frac{u}{m}\right)}\phi(u)-e^{\psi(a)}\phi(a)\right)-\left(me^{\psi\left(\frac{u}{m}\right)}-e^{\psi(a)}\right)\int_a^u\phi(\xi)d\xi\right]$$
$$+\frac{(\phi(b)-\phi(u))^{\frac{\sigma}{k}-1}}{k\Gamma_k(\sigma)(b-u)}\left[(b-u)\left(e^{\psi(b)}\phi(b)-me^{\psi\left(\frac{u}{m}\right)}\phi(u)\right)-\left(e^{\psi(b)}-me^{\psi\left(\frac{u}{m}\right)}\right)\int_u^b\phi(\xi)d\xi\right]. \quad (18)$$

Corollary 2. *Under the supposition of Theorem 1, let $m=1$. Then the following inequality for exponentially convex function holds:*

$$k\left(\Gamma_k(\sigma)\zeta^{\sigma,k}_{\phi,a^+}e^{\psi(u)}+\Gamma_k(\tau)\zeta^{\tau,k}_{\phi,b^-}e^{\psi(u)}\right)$$
$$\leq \frac{(\phi(u)-\phi(a))^{\frac{\sigma}{k}-1}}{u-a}\left[(u-a)\left(e^{\psi(u)}\phi(u)-e^{\psi(a)}\phi(a)\right)-\left(e^{\psi(u)}-e^{\psi(a)}\right)\int_a^u\phi(\xi)d\xi\right]$$
$$+\frac{(\phi(b)-\phi(u))^{\frac{\tau}{k}-1}}{b-u}\left[(b-u)\left(e^{\psi(b)}\phi(b)-e^{\psi(u)}\phi(u)\right)-\left(e^{\psi(b)}-e^{\psi(u)}\right)\int_u^b\phi(\xi)d\xi\right]. \quad (19)$$

Corollary 3. *Suppose $k=1$, then under under the supposition of Theorem 1, the following inequality for generalized Riemann-Liouville fractional integral operators holds;*

$$\Gamma(\sigma)\zeta^{\sigma}_{\phi,a^+}e^{\psi(u)}+\Gamma(\tau)\zeta^{\tau}_{\phi,b^-}e^{\psi(u)} \leq \frac{(\phi(u)-\phi(a))^{\sigma-1}}{u-a}\left[(u-a)\left(me^{\psi\left(\frac{u}{m}\right)}\phi(u)-e^{\psi(a)}\phi(a)\right)\right.$$
$$\left.-\left(me^{\psi\left(\frac{u}{m}\right)}-e^{\psi(a)}\right)\int_a^u\phi(\xi)d\xi\right]+\frac{(\phi(b)-\phi(u))^{\tau-1}}{b-u}\left[(b-u)\left(e^{\psi(b)}\phi(b)-me^{\psi\left(\frac{u}{m}\right)}\phi(u)\right)\right.$$
$$\left.-\left(e^{\psi(b)}-me^{\psi\left(\frac{u}{m}\right)}\right)\int_u^b\phi(\xi)d\xi\right]. \quad (20)$$

Corollary 4. *Suppose $\phi(u)=u$, then under the supposition of Theorem 1, the following inequality for Riemann-Liouville k-fractional integral operators holds;*

$$2k\left(\Gamma_k(\sigma)\zeta^{\sigma,k}_{a^+}e^{\psi(u)}+\Gamma_k(\tau)\zeta^{\tau,k}_{b^-}e^{\psi(u)}\right) \leq (u-a)^{\frac{\sigma}{k}}\left[e^{\psi(a)}+me^{\psi\left(\frac{u}{m}\right)}\right]+(b-u)^{\frac{\tau}{k}}\left[e^{\psi(b)}+me^{\psi\left(\frac{u}{m}\right)}\right].$$

Corollary 5. *Suppose $\phi(u)=u$ and $k=1$, then under the supposition of Theorem 1, the following inequality for Riemann-Liouville fractional integral operators holds;*

$$2\left(\Gamma(\sigma)\zeta^{\sigma}_{a^+}e^{\psi(u)}+\Gamma(\tau)\zeta^{\tau}_{b^-}e^{\psi(u)}\right) \leq (u-a)^{\sigma}\left[e^{\psi(a)}+me^{\psi\left(\frac{u}{m}\right)}\right]+(b-u)^{\tau}\left[e^{\psi(b)}+me^{\psi\left(\frac{u}{m}\right)}\right].$$

We need following lemma in the proof of next result.

Lemma 1. *Let $\psi:[a,b]\to\mathbb{R}$ be an exponentially m-convex function. If $e^{\psi(\xi)}=e^{\psi\left(\frac{a+b-\xi}{m}\right)}$ then for $\xi\in[a,b]$ and $m\in(0,1]$, the following inequality holds;*

$$e^{\psi\left(\frac{a+b}{2}\right)} \leq \frac{1}{2}(m+1)e^{\psi(\xi)}. \quad (21)$$

Proof. We have

$$\frac{a+b}{2}=\frac{1}{2}\left(\frac{\xi-a}{b-a}b+\frac{b-\xi}{b-a}a\right)+m\frac{1}{2}\left(\frac{\frac{\xi-a}{b-a}a+\frac{b-\xi}{b-a}b}{m}\right). \quad (22)$$

Since ψ is exponentially m-convex, we have

$$e^{\psi\left(\frac{a+b}{2}\right)} \leq \frac{1}{2}\left[e^{\psi\left(\frac{\xi-a}{b-a}b+\frac{b-\xi}{b-a}a\right)} + me^{\psi\left(\frac{\frac{\xi-a}{b-a}a+\frac{b-\xi}{b-a}b}{m}\right)}\right] = \frac{1}{2}\left[e^{\psi(\xi)} + me^{\psi\left(\frac{a+b-\xi}{m}\right)}\right] \tag{23}$$

Using given condition $e^{\psi(\xi)} = e^{\psi\left(\frac{a+b-\xi}{m}\right)}$ in (23), the inequality (21) can be achieved. □

Theorem 2. *Let $\psi, \phi : [a,b] \subset [0,\infty) \longrightarrow \mathbb{R}$, be two functions such that ϕ be differentiable and $\psi \in L_1[a,b]$ with $a < b$. Also let ψ be exponentially m-convex, $e^{\psi(\xi)} = e^{\psi\left(\frac{a+b-\xi}{m}\right)}$ and ϕ be strictly increasing on $[a,b]$ with $\phi' \in L_1[a,b]$. Then for $\xi \in [a,b]$ and $\sigma, \nu, k > 0$, the following inequalities hold;*

$$\frac{2k}{(m+1)}e^{\psi\left(\frac{a+b}{2}\right)}\left[\frac{(\phi(b)-\phi(a))^{\frac{\sigma}{k}+1}}{\sigma+k} + \frac{(\phi(b)-\phi(a))^{\frac{\tau}{k}+1}}{\tau+k}\right] \leq k\left(\Gamma_k(\sigma+k)\zeta^{\sigma+k,k}_{\phi,b^-}e^{\psi(a)} + \Gamma_k(\tau+k)\zeta^{\tau+k,k}_{\phi,a^+}e^{\psi(b)}\right)$$
$$\leq \frac{(\phi(b)-\phi(a))^{\frac{\sigma}{k}} + (\phi(b)-\phi(a))^{\frac{\tau}{k}}}{(b-a)}\left[(b-a)\left(e^{\psi(b)}\phi(b) - me^{\psi\left(\frac{a}{m}\right)}\phi(a)\right) - \left(e^{\psi(b)} - me^{\phi\left(\frac{a}{m}\right)}\right)\int_a^b \phi(\xi)d\xi\right] \tag{24}$$

Proof. Multiplying both sides of (21) with $(\phi(\xi)-\phi(a))^{\frac{\sigma}{k}}\phi'(\xi)$ and integrating over $[a,b]$, we have

$$e^{\psi\left(\frac{a+b}{2}\right)}\int_a^b (\phi(\xi)-\phi(a))^{\frac{\sigma}{k}}\phi'(\xi)d\xi \leq \frac{1}{2}(m+1)\int_a^b (\phi(\xi)-\phi(a))^{\frac{\sigma}{k}}\phi'(\xi)e^{\psi(\xi)}d\xi. \tag{25}$$

By using (8), we get

$$\frac{2k(\phi(b)-\phi(a))^{\frac{\sigma}{k}+1}}{(\sigma+k)(m+1)}e^{\psi\left(\frac{a+b}{2}\right)} \leq k\Gamma_k(\sigma+k)\zeta^{\sigma+k,k}_{\phi,b^-}e^{\psi(a)}. \tag{26}$$

Similarly, multiplying both sides of (21) with $(\phi(b)-\phi(\xi))^{\frac{\tau}{k}}\phi'(\xi)$ and integrating over $[a,b]$, one can get the following inequality:

$$\frac{2k(\phi(b)-\phi(a))^{\frac{\tau}{k}+1}}{(\tau+k)(m+1)}e^{\psi\left(\frac{a+b}{2}\right)} \leq k\Gamma_k(\tau+k)\zeta^{\tau+k,k}_{\phi,a^+}e^{\psi(b)}. \tag{27}$$

On the other hand from exponentially m-convexity of ψ, we have

$$e^{\psi(\xi)} \leq \frac{\xi-a}{b-a}e^{\psi(b)} + m\frac{b-\xi}{b-a}e^{\psi\left(\frac{a}{m}\right)}. \tag{28}$$

Under given assumptions for the function ϕ for $\sigma, k > 0$, the following inequality holds;

$$\phi'(\xi)(\phi(\xi)-\phi(a))^{\frac{\sigma}{k}} \leq \phi'(\xi)(\phi(b)-\phi(a))^{\frac{\sigma}{k}}, \qquad \xi \in [a,b]. \tag{29}$$

Multiplying (28) with (29) and integrating over $[a,b]$, we have

$$\int_a^b (\phi(\xi)-\phi(a))^{\frac{\sigma}{k}}\phi'(\xi)e^{\psi(\xi)}d\xi \leq \frac{(\phi(b)-\phi(a))^{\frac{\sigma}{k}}}{b-a}\left[e^{\psi(b)}\int_a^b(\xi-a)\phi'(\xi)d\xi + me^{\psi\left(\frac{a}{m}\right)}\int_a^b(b-\xi)\phi'(\xi)d\xi\right].$$

By using (8), we get

$$k\Gamma_k(\sigma+k)\zeta^{\sigma+k,k}_{\phi,b^-}e^{\psi(a)} \leq \frac{(\phi(b)-\phi(a))^{\frac{\sigma}{k}}}{b-a}\left[(b-a)\left(e^{\psi(b)}\phi(b) - me^{\psi\left(\frac{a}{m}\right)}\phi(a)\right) - \left(e^{\psi(b)} - me^{\phi\left(\frac{a}{m}\right)}\right)\int_a^b \phi(\xi)d\xi\right] \tag{30}$$

Now for $\xi \in [a,b]$ and $\tau, k > 0$, the following inequality holds true;

$$\phi'(\xi)(\phi(b)-\phi(\xi))^{\frac{\tau}{k}} \leq \phi'(\xi)(\phi(b)-\phi(a))^{\frac{\tau}{k}}. \tag{31}$$

Multiplying (28) with (31) and integrating over $[a,b]$, we have

$$\int_a^b (\phi(b)-\phi(\xi))^{\frac{\tau}{k}}\phi'(\xi)e^{\psi(\xi)}d\xi \leq \frac{(\phi(b)-\phi(a))^{\frac{\tau}{k}}}{b-a}\left[e^{\psi(b)}\int_a^b(\xi-a)\phi'(\xi)d\xi + me^{\psi\left(\frac{a}{m}\right)}\int_a^b(b-\xi)\phi'(\xi)d\xi\right].$$

By using (8), we get

$$k\Gamma_k(\tau+k)\zeta^{\tau+k,k}_{\phi,a^+}e^{\psi(b)} \leq \frac{(\phi(b)-\phi(a))^{\frac{\tau}{k}}}{b-a}\left[(b-a)\left(e^{\psi(b)}\phi(b)-me^{\psi\left(\frac{a}{m}\right)}\phi(a)\right)-\left(e^{\psi(b)}-me^{\phi\left(\frac{a}{m}\right)}\right)\int_a^b\phi(\xi)d\xi\right] \quad (32)$$

By adding inequalities (30) and (32), second inequality (24) is achieved. □

Corollary 6. *If we put $\sigma=\tau$ in (24), we get following inequalities;*

$$\begin{aligned}&\frac{4k(\phi(b)-\phi(a))^{\frac{\sigma}{k}+1}}{(m+1)(\sigma+k)}e^{\psi\left(\frac{a+b}{2}\right)} \leq k\Gamma_k(\sigma+k)\left(\zeta^{\sigma+k,k}_{\phi,b^-}e^{\psi(a)}+\zeta^{\sigma+k,k}_{\phi,a^+}e^{\psi(b)}\right)\\ &\leq \frac{2(\phi(b)-\phi(a))^{\frac{\sigma}{k}}}{(b-a)}\left[(b-a)\left(e^{\psi(b)}\phi(b)-me^{\psi\left(\frac{a}{m}\right)}\phi(a)\right)-\left(e^{\psi(b)}-me^{\phi\left(\frac{a}{m}\right)}\right)\int_a^b\phi(\xi)d\xi\right].\end{aligned}$$

Corollary 7. *Suppose $m=1$, then under the supposition of Theorem 2, the following inequalities for exponentially convex function hold;*

$$\begin{aligned}&ke^{\psi\left(\frac{a+b}{2}\right)}\left[\frac{(\phi(b)-\phi(a))^{\frac{\sigma}{k}+1}}{\sigma+k}+\frac{(\phi(b)-\phi(a))^{\frac{\tau}{k}+1}}{\tau+k}\right] \leq k\left(\Gamma_k(\sigma+k)\zeta^{\sigma+k,k}_{\phi,b^-}e^{\psi(a)}+\Gamma_k(\tau+k)\zeta^{\tau+k,k}_{\phi,a^+}e^{\psi(b)}\right)\\ &\leq \frac{(\phi(b)-\phi(a))^{\frac{\sigma}{k}}+(\phi(b)-\phi(a))^{\frac{\tau}{k}}}{(b-a)}\left[(b-a)\left(e^{\psi(b)}\phi(b)-e^{\psi(a)}\phi(a)\right)-\left(e^{\psi(b)}-e^{\phi(a)}\right)\int_a^b\phi(\xi)d\xi\right].\end{aligned}$$

Corollary 8. *Suppose $k=1$, then under the supposition of Theorem 2, the following inequalities for generalized Riemann-Liouville fractional integral operators hold;*

$$\begin{aligned}&\frac{2}{(m+1)}e^{\psi\left(\frac{a+b}{2}\right)}\left[\frac{(\phi(b)-\phi(a))^{\sigma+1}}{\sigma+1}+\frac{(\phi(b)-\phi(a))^{\tau+1}}{\tau+1}\right] \leq \Gamma(\sigma+1)\zeta^{\sigma+1}_{\phi,b^-}e^{\psi(a)}+\Gamma(\tau+1)\zeta^{\tau+1}_{\phi,a^+}e^{\psi(b)}\\ &\leq \frac{(\phi(b)-\phi(a))^{\sigma}+(\phi(b)-\phi(a))^{\tau}}{(b-a)}\left[(b-a)\left(e^{\psi(b)}\phi(b)-me^{\psi\left(\frac{a}{m}\right)}\phi(a)\right)-\left(e^{\psi(b)}-me^{\phi\left(\frac{a}{m}\right)}\right)\int_a^b\phi(\xi)d\xi\right].\end{aligned} \quad (33)$$

Corollary 9. *Suppose $\phi(u)=u$, then under the supposition of Theorem 2, the following inequalities for Riemann-Liouville k-fractional integral operators hold;*

$$\begin{aligned}&\frac{2k}{(m+1)}e^{\psi\left(\frac{a+b}{2}\right)}\left[\frac{(b-a)^{\frac{\sigma}{k}+1}}{\sigma+k}+\frac{(b-a)^{\frac{\tau}{k}+1}}{\tau+k}\right] \leq k\left(\Gamma_k(\sigma+k)\zeta^{\sigma+k,k}_{b^-}e^{\psi(a)}+\Gamma_k(\tau+k)\zeta^{\tau+k,k}_{a^+}e^{\psi(b)}\right)\\ &\leq \frac{(b-a)^{\frac{\sigma}{k}+1}+(b-a)^{\frac{\tau}{k}+1}}{2}\left[e^{\psi(b)}+me^{\psi\left(\frac{a}{m}\right)}\right]\end{aligned} \quad (34)$$

Corollary 10. *Suppose $\phi(u)=u$ and $k=1$, then under the supposition of Theorem 2, the following inequalities for Riemann-Liouville fractional integral operators hold;*

$$\begin{aligned}&\frac{2}{(m+1)}e^{\psi\left(\frac{a+b}{2}\right)}\left[\frac{(b-a)^{\sigma+1}}{\sigma+1}+\frac{(b-a)^{\tau+1}}{\tau+1}\right] \leq \Gamma(\sigma+1)\zeta^{\sigma+1}_{b^-}e^{\psi(a)}+\Gamma(\tau+1)\zeta^{\tau+1}_{a^+}e^{\psi(b)}\\ &\leq \frac{(b-a)^{\sigma+1}+(b-a)^{\tau+1}}{2}\left[e^{\psi(b)}+me^{\psi\left(\frac{a}{m}\right)}\right]\end{aligned} \quad (35)$$

3. Applications

In this section, we give the applications of the results proved in previous section.

Theorem 3. *Under the assumptions of Theorem 1, we have*

$$k\left(\Gamma_k(\sigma)\zeta^{\sigma,k}_{\phi,a^+}e^{\psi(b)}+\Gamma_k(\tau)\zeta^{\tau,k}_{\phi,b^-}e^{\psi(a)}\right)\leq\frac{(\phi(b)-\phi(a))^{\frac{\sigma}{k}-1}}{b-a}\left[(b-a)\left(me^{\psi\left(\frac{b}{m}\right)}\phi(b)-e^{\psi(a)}\phi(a)\right)\right.$$
$$\left.-\left(me^{\psi\left(\frac{b}{m}\right)}-e^{\psi(a)}\right)\int_a^b\phi(\xi)d\xi\right]+\frac{(\phi(b)-\phi(a))^{\frac{\tau}{k}-1}}{b-a}\left[(b-a)\left(e^{\psi(b)}\phi(b)-me^{\psi\left(\frac{a}{m}\right)}\phi(a)\right)\right.$$
$$\left.-\left(e^{\psi(b)}-me^{\psi\left(\frac{a}{m}\right)}\right)\int_a^b\phi(\xi)d\xi\right]. \tag{36}$$

Proof. If we put $u=a$ in (11), we get

$$k\Gamma_k(\tau)\zeta^{\tau,k}_{\phi,b^-}e^{\psi(a)}\leq\frac{(\phi(b)-\phi(a))^{\frac{\tau}{k}-1}}{b-a}\left[(b-a)\left(e^{\psi(b)}\phi(b)-me^{\psi\left(\frac{a}{m}\right)}\phi(a)\right)-\left(e^{\psi(b)}-me^{\psi\left(\frac{a}{m}\right)}\right)\int_a^b\phi(\xi)d\xi\right]. \tag{37}$$

If we put $u=b$ in (11), we get

$$k\Gamma_k(\sigma)\zeta^{\sigma,k}_{\phi,a^+}e^{\psi(b)}\leq\frac{(\phi(b)-\phi(a))^{\frac{\sigma}{k}-1}}{b-a}\left[(b-a)\left(me^{\psi\left(\frac{b}{m}\right)}\phi(b)-e^{\psi(a)}\phi(a)\right)-\left(me^{\psi\left(\frac{b}{m}\right)}-e^{\psi(a)}\right)\int_a^b\phi(\xi)d\xi\right]. \tag{38}$$

By adding inequalities (37) and (38), inequality (36) can be achieved. □

Corollary 11. *If we put $\sigma=\tau$ in (36), then the following inequality holds;*

$$\zeta^{\sigma,k}_{\phi,a^+}e^{\psi(b)}+\zeta^{\sigma,k}_{\phi,b^-}e^{\psi(a)}\leq\frac{(\phi(b)-\phi(a))^{\frac{\sigma}{k}-1}}{k\Gamma_k(\sigma)(b-a)}\left[(b-a)\left(\left(me^{\psi\left(\frac{b}{m}\right)}\phi(b)-e^{\psi(a)}\phi(a)\right)\right.\right.$$
$$\left.\left.+\left(e^{\psi(b)}\phi(b)-me^{\psi\left(\frac{a}{m}\right)}\phi(a)\right)\right)-\left(\left(me^{\psi\left(\frac{b}{m}\right)}-e^{\psi(a)}\right)+\left(e^{\psi(b)}-me^{\psi\left(\frac{a}{m}\right)}\right)\right)\int_a^b\phi(\xi)d\xi\right] \tag{39}$$

Corollary 12. *If we put $\sigma=k=m=1$ and $\phi(u)=u$ in (39), then the following inequality holds;*

$$\frac{1}{b-a}\int_a^b e^{\psi(\xi)}d\xi\leq\frac{e^{f(a)}+e^{f(b)}}{2}. \tag{40}$$

Remark 3. Similar relation can be achieved by applying Theorem 2, so we leave it for readers.

Author Contributions: Both authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: "The authors declare no conflict of interest."

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