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A few comments and some new results on JU-algebras

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Abstract: In this article, we revisit the axioms of JU-algebras previously recognizable as 'pseudo KU-algebras', which we may call as 'weak KU-algebras' and discussed the definitions of some of their substructures. We also associate this class of algebras with the classes of BE-algebras and UP-algebras. In addition, we introduce and analyze some new classes of ideals in this class of algebras.

Keywords: JU-algebras, ideal and filter in JU-algebras, closed ideal, ag-ideal, t-ideal, $(\star)$-ideal and associative ideal.

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1. Introduction

In 1966, Imai and Iseki [1] introduced a notion of BCK-algebras. The concept of BE-algebra as a generalization of BCK-algebra was introduced in 2006 by Kim and Kum in [2]. The concept of KU-algebras was introduced and analyzed in 2009 in [3,4]. KU-algebras are closely related to BE-algebras. Specifically, in the article [5], the authors have shown that KU-algebra is equivalent to a commutative self-distributive BE-algebra. (A BE-algebra A is a self-distributive if $x \cdot (y \cdot z) = (z \cdot y) \cdot (x \cdot z)$ for all $x, y, z \in A$). Additionally, they have shown that every KU-algebra is a BE-algebra [5]. The concept of UP-algebras as a generalization of KU-algebras was introduced by Iampan in [6]. The concept of JU-algebras, as a generalization of KU-algebras, was introduced and analyzed in [7,8].

However, this concept was introduced in [9] by Leerawat and Prabpayak under the name 'pseudo KU-algebra. In doing so, they used the PKU designation for this class of algebra. Since then, this type of generalization of KU-algebra has been in the focus of interest of the academic community (for example see [10,11]).

We are more inclined to refer this concept as 'weak KU-algebra' in the same way as weak BCC-algebra [12]. However, due to the tight connection of this paper to the article [8], we will use the name 'JU-algebra' in what follows.

In this article, we revisit the axioms of JU-algebras and definitions of their substructures. We also link this class of algebras with the classes BE-algebras and UP-algebras. In addition, we introduce and analyze some new classes of ideals in this class of algebra such as closed ideal, ag-ideal, t-ideal, $(\star)$-ideal and associative ideal.

2. Preliminaries

In this section, we take the definitions of JU-algebras, JU-subalgebras, JU-ideals and other important terminologies and some related results from literature [7,8].

2.1. Definition and some comments

Definition 1. [8] An algebra $(A, \cdot, 1)$ of type $(2, 0)$ with a binary operation "$\cdot$" and a fixed element 1 is said to be *JU-algebras* satisfying the following axioms:

(JU-1) $(\forall x, y, z \in A)((y \cdot z) \cdot ((z \cdot x) \cdot (y \cdot x)) = 1)$,
(JU-2) $(\forall x \in A)(1 \cdot x = x)$ and
(JU-3) $(\forall x, y \in A)((x \cdot y \wedge y \cdot x = 1) \implies x = y)$.

We denote this axiom system by [JU].

Lemma 1. *[8] In the axioms system [JU], the following formulae are valid:*

(J_{11}) $(\forall x \in A)(x \cdot x = 1)$,
(J_{12}) $(\forall x, y, z \in A)(z \cdot (y \cdot x) = y \cdot (z \cdot x))$.

Comment 1. *In [3], a KU-algebra is defined as a system* $(A, \cdot, 0)$ *by the following axioms:*

(KU-1) $(\forall x, y, z \in A)((x \cdot y) \cdot ((y \cdot z) \cdot (x \cdot z)) = 0)$,
(KU-2) $(\forall x \in A)(0 \cdot x = x)$,
(KU-3) $(\forall x \in A)(x \cdot 0 = 0)$ *and*
(KU-4) $(\forall x, y \in A)((x \cdot y = 0 \wedge y \cdot x = 0) \implies x = y)$.

We denote this axiom system by [KU]. With [wKU] we denote axiomatic system [KU] without axiom (KU-3). So, [JU] ≡ [wKU] ≡ [PKU].

Recall that in the axiom system [KU], the formula (J_{12}) is a valid formula also [13].

If in the definition of KU-algebras we write 1 instead of 0, then we see that any KU-algebra A is a JU-algebra. Therefore, the concept of JU-algebras is a generalization of the concept of KU-algebras [8].

If we followed the formation of the concept of 'weak BCC-algebras' from the 'concept of BCC-algebras', then the name '*weak KU-algebra*' could also be used for a JU-algebra by analogy with the previous one.

If A is a JU-algebra, let us define $\varphi : A \longrightarrow A$ as follows;

$$(\forall x \in A)(\varphi(x) = x \cdot 1)$$

taking the idea from [14]. According to (JU-2), the equality $\varphi(1) = 1$ is valid for mapping φ.

Comment 2. *The concept of UP-algebras was introduced in 2017 in article [6] as a* $(A, \cdot, 0)$ *system that satisfies the following axioms:*

(UP-1) $(\forall x, y, z)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(UP-2) $(\forall x \in A)(0 \cdot x = x)$,
(UP-3) $(\forall x \in A)(x \cdot 0 = 0)$ *and*
(UP-4) $(\forall x, y \in A)((x \cdot y = 0 \wedge y \cdot x = 0) \implies x = y)$.

We denote this axiom system by [UP]. With [wUP] we denote axiomatic system [UP] without axiom (UP-3).

We can transform the formula (JU-1) into the formula (UP-1) using valid equation (J_{12}) [8] and replacing the element 1 by the element 0. However, since formula (J_{12}) does not have to be a valid formula in [UP], we conclude that there is no direct connection between [JU] and [UP]. On the other hand, the system [UP] + (J_{12}) is equivalent to the system [KU] according to theorems in [6], so we conclude that the system [JU] is contained in the system [wUP] + (J_{12}). Therefore, any UP-algebra that additionally satisfies equality (J_{12}) is also a JU-algebra at the same time.

Comment 3. *The concept of BE-algebras is defined in [2] as a system* $(A, \cdot, 1)$ *satisfying the following axioms:*

(BE-1) $(\forall x \in A)(x \cdot x = 1)$,
(BE-2) $(\forall x \in A)(x \cdot 1 = 1)$,
(BE-3) $(\forall x \in A)(1 \cdot x = x)$ *and*
(BE-4) $(\forall x, y, z \in A)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$.

We denote this axiom system by [BE]. The axiomatic system generated by axioms (B-1), (BE-3) and (BE-4) is denoted by [wBE].

It is shown in [5] that every KU-algebra is a BE-algebra. Since any KU-algebra is a JU-algebra, by Comment 1, we get that every BE-algebra is a JU-algebra.

2.2. An order relation

Definition 2. [8] Let A be a JU-algebra. We define a relation "$\leqslant$" in A as follows:

$$\forall x, y \in A)(y \leqslant x \iff x \cdot y = 1.$$

According to claims (J_4), (J_5), (J_6), and claims (J_7), (J_8), the relation "$\leqslant$" is a partial order in A left compatible and right reverse compatible with the internal operation in A [8].

Proposition 1. *Let A be a JU-algebra. Then*

(1) $(\forall x, y \in A)(x \cdot \varphi(y) = y \cdot \varphi(x))$;
(2) $(\forall x, y \in A)(\varphi(x) \cdot \varphi(y) \leqslant y \cdot x)$;
(3) $(\forall x, y \in A)(x \cdot (y \cdot x) \leqslant \varphi(y))$;
(4) $(\forall x, y \in A)(\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y))$;
(5) $(\forall x, y \in A)(x \leqslant y \implies \varphi(y) \leqslant \varphi(x))$.

Proof. Relation (1) is obtained directly from (J_{12}) where, we put $z = 1$.

If we put $x = 1$ and $z = x$ in (JU-1), we get $(y \cdot x) \cdot ((x \cdot 1) \cdot (y \cdot 1)) = 1$. This means $\varphi(x) \cdot \varphi(y) \leqslant y \cdot x$ according to the Definition 2.

If we put $z = 1$ in (JU-1), we get $(y \cdot 1) \cdot ((1 \cdot x) \cdot (y \cdot x)) = 1$. Hence $\varphi(y) \cdot (x \cdot (y \cdot x)) = 1$. So, we have $x \cdot (y \cdot x) \leqslant \varphi(y)$.

Relation (4) is proved in [8] as formula (J_{14}).

Relation (5) is a direct consequence of the right inverse compatibility of order relations with an internal operation in A if we choose $z = 1$. □

Remark 1. The relation (5) of Proposition 1 is a direct consequence of the Proposition 1(4). Indeed, if $x \leqslant y$, then $y \cdot x = 1$. Thus $\varphi(y \cdot x) = 1$. Hence $\varphi(y) \cdot \varphi(x) = 1$ by Proposition 1(4). This means $\varphi(x) \leqslant \varphi(y)$.

3. Some types of JU-ideals

Definition 3. [8] A non-empty subset J of a JU-algebra A is called a JU-*ideal* of A if

(J-1) $1 \in J$ and
(J-2) $(\forall x, y \in A)((x \in J \wedge x \cdot y \in J) \implies y \in J)$.

Lemma 2. *Let J be a JU-*ideal *of a JU-algebra A. Then*

(J-3) $(\forall x, y \in A)((x \leqslant y \wedge y \in J) \implies x \in J)$.

Proof. Let $x, y \in A$ be such that $x \leqslant y$ and $y \in J$. Then $y \cdot x = 1 \in J$ and $y \in J$. Thus $x \in J$ by (J-2). □

Proposition 2. *Let J be a subset of a JU-algebra such that* (J-1) *holds. Then the condition* (J-2) *is equivalent to the condition;*

(J-4) $(\forall x, y, z \in A)((x \cdot (y \cdot z) \in J \wedge y \in J) \implies x \cdot z \in J)$.

Proof. (J-4) $\implies$ (J-2). If we put $x = 1, y = x$ and $z = y$ in (J-4), then we get (J-2) with respect to (JU-2).

(J-2) $\implies$ (J-4). Let $x, y, z \in A$ such that $x \cdots (y \cdot z) \in J$ and $y \in J$. Then $y \cdot (x \cdot z) \in J$ and $y \in J$ by (J_{12}). Thus $x \cdot z \in J$ by (J-2). □

As a consequence of the Proposition 2, we can describe some of the features of the JU-ideals as follows:

Proposition 3. *Let J be a JU-ideal of a JU-algebra A. Then*

(6) $\forall x, y \in A)((\varphi(x) \in J \wedge y \in J) \implies x \cdot y \in J)$ *and*
(7) $(\forall x, y \in A)((x \cdot \varphi(y) \in J \wedge y \in J) \implies \varphi(x))$.

Proof. If we put $z = y$ in *(J-4)*, we get

$$(x \cdot (y \cdot y) \in J \wedge y \in J) \implies x \cdot y \in J.$$

From where, we get *(J-5)* using (J_{11}) in [8]. If we put $z = 1$ in *(J-4)*, we get (7). □

Definition 4. Let J be a JU-ideal of a JU-algebra A, then

(C) J is a *closed ideal* of A if $\{x \in A : \varphi(x) \in J\} = \varphi(J) \subseteq J$ holds.

Theorem 1. *An ideal J of a JU-algebra A is closed if and only if it is a subalgebra of A.*

Proof. Assume that an ideal J is a subalgebra of A and $x \in J$. Then, from $1 \in J$ and $x \in J$ follows $\varphi(x) = x \cdot 1 \in J$ because J is a subalgebra of A. This means $\varphi(J) \subseteq J$.

Conversely, let an ideal J of A is closed and let $x, y \in J$. Then $\varphi(x) \in \varphi(J) \subseteq J$ and $y \in J$. Thus $x \cdot y \in J$ by Proposition 3(6). This means that J is a subalgebra of A. □

As usual, we will write $Ker\varphi = \{x \in A : \varphi(x) = 1\}$. In [8], this set is labeled by B_A.

Corollary 1. *$Ker\varphi$ is a closed JU-ideal of A.*

Proof. Let $x, y \in A$ be such $x \in Ker\varphi$ and $x \cdot y \in Ker\varphi$. Then $\varphi(x) = 1$ and $1 = \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) = 1 \cdot \varphi(y) = \varphi(y)$ by Proposition 1(4) and (JU-2). Thus $y \in Ker\varphi$. So, $Ker\varphi$ is a JU-ideal of A.

Suppose $x \in Ker\varphi$ and $y \in Ker\varphi$. Then, from $\varphi(x) = 1$ and $\varphi(y) = 1$ it follows $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) = 1 \cdot 1 = 1$ by Proposition 1(4). Thus $x \cdot y \in Ker\varphi$. So, $Ker\varphi$ is a subalgebra in A. Therefore $Ker\varphi$ is a closed JU-ideal of A by Theorem 1. □

In what follows, we need the following lemma.

Lemma 3. *Let J be a JU-ideal of a JU-algebra A, then*

(8) $(\forall x \in A)(x \in J \implies \varphi^2(x) \in J)$.

Proof. Let us first show that the following holds;

(9) $(\forall x \in A)(\varphi^2(x) \leqslant x)$.

This inequality immediately follows from Proposition 1(2) if we put $y = 1$. Now, from $x \in J$ and $\varphi^2(x) \leqslant x$ it follows $\varphi^2(x) \in J$ according to (J-3). □

The preceding lemma is a motive for introducing the following concept.

Definition 5. Let J be a JU-ideal of JU-algebra A, then

(AG) J is *ag-ideal* of A if $(\forall x \in A)(\varphi^2(x) \in J \implies x \in J)$ holds.

Proposition 4. *Let A be a JU-algebra, then*

(10) $(\forall x \in A)(\varphi^3(x) = \varphi(x))$.

Proof. It has already been shown that $\varphi^2(x) \leqslant x$ is valid. Thus $\varphi(x) \leqslant \varphi^3(x)$ by Proposition 1(5) and $\varphi^3(x) \cdot \varphi(x) = 1$. On the other hand, from $1 = \varphi^2(x) \cdot \varphi^2(x) = \varphi^2(x) \cdot \varphi(\varphi(x))$, it follows $\varphi(x) \cdot \varphi^3(x) = 1$ according to Proposition 1(1). Thus $\varphi^3(x) = \varphi(x)$ by (JU-3). □

Corollary 2. *$Ker\varphi$ is a closed ag-ideal of A.*

Proof. It has already been shown that $Ker\varphi$ is a closed JU-ideal of A. Let $x \in A$ be an arbitrary element such that $\varphi^2(x) \in Ker\varphi$. Then $\varphi(x) = \varphi^3(x) = 1$ by (9) given in proof of Lemma 3. Thus $x \in Ker\varphi$. □

Lemma 4. *If J is a JU-ideal of a JU-algebra A, then $\varphi^2(J) = \{x \in A : \varphi^2(x) \in J\}$ is a JU-ideal of A also. The reverse also applies: If $\varphi^2(J)$ is a JU-ideal of JU-algerba A, then J is a JU-ideal of A also.*

Proof. Since $1 \in J$, then $\varphi^2(1) = 1 \in J$. Thus $1 \in \varphi^2(J)$. Suppose $x \in \varphi^2(J)$ and $x \cdot y \in \varphi^2(J)$. Then $\varphi^2(x) \in J$ and $\varphi^2(x) \cdot \varphi^2(y) = \varphi^2(x \cdot y) \in J$ with respect to Proposition 1(4). Thus $\varphi^2(y) \in J$ and $y \in \varphi^2(J)$. So, $\varphi^2(J)$ is a JU-ideal of JU-algebra A.

It is obvious that $1 = \varphi^2(1) \in J$ holds because $1 \in \varphi^2(J)$ is valid. Let $x, y \in A$ be such $x \in J$ and $x \cdot y \in J$. Then $\varphi^2(x) \in \varphi^2(J)$ and $\varphi^2(x) \cdot \varphi^y = \varphi^2(x \cdot y) \in \varphi^2(J)$. Thus $\varphi^2(y) \in \varphi^2(J)$ by (J-2) because $\varphi^2(J)$ is a JU-ideal of A. Hence $y \in J$. Therefore, J is a JU-ideal of A. □

Theorem 2. *J is a closed a JU-ideal of a JU-algerba A if and only if $\varphi^2(J)$ is a closed JU-ideal of A.*

Proof. Let J be a closed a JU-ideal of a JU-algebra A. Let $x \in A$ be an element such that $x \in \varphi(\varphi^2(J))$. Then $\varphi(x) \in \varphi^2(J)$ and $\varphi^2(\varphi(x)) = \varphi(\varphi^2 x)) \in J$. Thus $\varphi^2(x) \in \varphi(J) \subseteq J$ because J is a closed ideal of A. Hence $x \in \varphi^2(J)$. From this it has shown that the ideal $\varphi^2(J)$ satisfies Definition 4 condition (C).

Let J be a JU-ideal of a JU-algebra A such that $\varphi^2(J)$ is a closed JU-ideal of A. Then, we have

$$\varphi(J) = \varphi^3(J) = \varphi(\varphi^2(J)) \subseteq \varphi^2(J) \subseteq J$$

by Proposition 4(10) and since the inclusion $\varphi^2(J) \subseteq J$ is obviously valid. So, J is a closed JU-ideal of A. □

Before presenting the following theorem, we will introduce the concept of JU-filters in a JU-algebra.

Definition 6. A subset F of a JU-algerba A is a JU-*filter* of A if the following hold:

(F-1) $1 \in F$ and
(F-2) $(\forall x, y \in A)((x \cdot y \in F \wedge y \in F) \implies x \in F)$.

Lemma 5. *If F is a JU-filter of a JU-algerba A, then*

(F-3) $(\forall x, y \in A)(x \in F \wedge x \leqslant y) \implies y \in F)$.

Proof. Let $x, y \in A$ be such that $x \in F$ and $x \leqslant y$. Then $x \in F$ and $y \cdot x = 1 \in F$. Thus $y \in F$ by (F-3). □

Lemma 6. *If F is a JU-filter of a JU-algerba A, then*

(11) $(\forall x \in A)(\varphi(x) \in F \implies x \in F)$.

Proof. If we put $y = 1$ in (F-2), we immediately get (11). □

Remark 2. In [8], the concept of strong JU-ideal of a JU-algebra was introduced in the way we introduced the concept of JU-filters.

Theorem 3. *For a JU-ideal J of a JU-algerba A the following are equivalent:*

(a) *J is an ag-ideal of A;*
(b) *J is a JU-filter of A;*
(c) $(\forall x, y, z \in A)(((x \cdot z) \cdot (y \cdot z) \in J \wedge y \in J) \implies x \in J)$*; and*
(d) $(\forall x, y \in A)(\varphi(x \cdot y) \in J \wedge y \in J) \implies x \in J)$.

Proof. **(a)$\Longrightarrow$ (b).** Let J be an ag-ideal of A and $x \leqslant y \wedge y \in J$. Then $y \cdot x = 1 \in J$ and $y \in J$. Thus $\varphi^2(y) \cdot \varphi^2(x) = \varphi^2(y \cdot x) = 1 \in J$ and $\varphi^2(y) \in \varphi^2(J) \subseteq J$. Hence $\varphi^2(x)J$ because J is an ideal of A. From here, it is follows $x \in J$ since J is an ag-ideal of A. Therefore, subset J is a JU-filter of A.
(b)$\Longrightarrow$ (c). From (JU-1), written in the form $(y \cdot x) \cdot ((x \cdot z) \cdot (y \cdot z)) = 1 \in J$, and from $(x \cdot z) \cdot (y \cdot z) \in J$ we get $y \cdot x \in J$ according to (F-2). Now, from $y \cdot x \in J$ and $y \in J$ it follows $x \in J$ according to (J-2).
(c)$\Longrightarrow$ (d). If we put $z = 1$ in (c), we get $\varphi(x) \cdot \varphi(y) \in J y \in J$. hence (d) with respect to (4).
(d)$\Longrightarrow$ (a). If we put $y = 1$ in (d), we get (a) with respect to (J-1).
□

Let's introduce the following type of JU-ideals.

Definition 7. A subset J of a JU-algebra A is a t-*ideal* of A if the following hold:

(J-1) $1 \in J$ and
(t) $(\forall x, y, z \in A)(((x \cdot y) \cdot z \in J \wedge y \in J) \implies x \cdot z \in J)$.

Lemma 7. *A t-ideal of a JU-algebra A is a JU-ideal of A.*

Proof. If we put $x = 1, y = x$ and $z = y$ in Dentition 7(t), we get condition (J-2). □

In the following proposition we give some of the characteristics of this type of JU-ideals.

Theorem 4. *Let J be a t-ideal of a JU-algerba A, then*

(e) $(\forall x, z \in A)(\varphi(x) \cdot z \in J \implies x \cdot z \in J)$;
(f) $(\forall x, z \in A)(\varphi(x) \cdot z \in J \implies \varphi^2(x) \cdot z \in J)$; *and*
(g) $(\forall x \in A)(x \cdot \varphi(x) \in J)$.

Proof. If we put $y = 1$ in Definition 7(t), we get (e) with respect to (J-1).

Let $x, z \in A$ be such that $\varphi(x) \cdot z \in J$. Then

$$\varphi(\varphi^2(x)) \cdot z = \varphi^3(x) \cdot z = \varphi(x) \cdot z \in J \implies \varphi^2(x) \cdot z \in J$$

by (e).

To prove (g), we note that $(x \cdot 1) \cdot (x \cdot 1) = 1 \in J$ and $1 \in J$ follows $x \cdot (x \cdot 1) \in J$ by Definition 7(t). So we have $x \cdot \varphi(x) \in J$. □

Proposition 5. *The condition (e) of Theorem 4 is equivalent to the condition (t) of Definition 7.*

Proof. Theorem 4 already proves that (t) $\implies$ (e), the inverse implication (e) $\implies$ (t) remains to be proved.

We can write (JU-1) in the form $(y \cdot z) \cdot (y \cdot x) \leqslant z \cdot x$ according to (J_{12}). If we put $z = 1$, $x = y$ and $y = x$ in the previous inequality, we get $\varphi(x) \cdot (x \cdot y) \leqslant y$. From here, it follows $\varphi(x) \cdot (x \cdot y) \in J$ by (J-3) and the hypothesis $y \in J$. On the other hand, if we put $z = x \cdot y$, $x = z$ and $y = \varphi(x)$ in (JU-1) written in the form $(y \cdot z) \cdot (y \cdot x) \leqslant z \cdot x$, then we get $(\varphi(x) \cdot (x \cdot y)) \cdot (\varphi(x) \cdot z) \leqslant (x \cdot y) \cdot z$. Thus $(\varphi(x) \cdot (x \cdot y)) \cdot (\varphi(x) \cdot z) \in J$ by (J-3) and the hypothesis $(x \cdot y) \cdot z \in J$.

Now, from $(\varphi(x) \cdot (x \cdot y)) \cdot (\varphi(x) \cdot z) \in J$ and $\varphi(x) \cdot (x \cdot y) \in J$ it follows $\varphi(x) \cdot z \in J$. Thus $x \cdot z \in J$ by (e). □

Proposition 6. *The condition (f) of Theorem 4 is equivalent to the condition (e) of Theorem 4.*

Proof. Let $x, z \in A$ be elements such that $\varphi(x) \cdot z \in J$. Then $\varphi^2(x) \cdot z \in J$ by (f). On the other hand, it follows $x \cdot z \leqslant \varphi^2(x) \cdot z$ from (9) with respect to right reverse compatibility of the operation in A with the order in A. Now, from $x \cdot z \leqslant \varphi^2(x) \cdot z$ and $\varphi^2(x) \cdot z \in J$ it follows $x \cdot z \in J$ by (J-3). This has shown that formula (e) is a consequence of formula (f).

As Theorem 4 already shows that (e) $\implies$ (f), we conclude that the condition (e) is equivalent to the condition (f). □

Proposition 7. *The condition (g) of Theorem 4 is equivalent to the condition (e) of Theorem 4.*

Proof. Assume (g). Let $x, y, z \in A$ be elements such that $\varphi(x) \cdot z \in J$. If we put $y = x$, $x = z$ and $z = \varphi(x)$ in (JU-1), written in the form $(z \cdot x) \cdot (t \cdot x) \leqslant y \cdot z$, we get $(\varphi(x) \cdot z) \cdot (x \cdot z) \leqslant x \cdot \varphi(x)$. From here, it follows $(\varphi(x) \cdot z) \cdot (x \cdot z) \in J$ by (J-3) and the hypothesis (g): $\varphi(x) \cdot x \in J$. Now, from $(\varphi(x) \cdot z) \cdot (x \cdot z) \in J$ it follows $x \cdot z \in J$ by (J-2) and hypothesis $\varphi(x) \cdot z \in J$. We have shown by this that (g) $\implies$ (e).

In Theorem 4 it is shown that (t) $\implies$ (g). Since (t) $\iff$ (g) is a valid formula, by Proposition 5, we conclude that the equivalence of (e) $\iff$ (g) is a valid formula, too. □

The concept of p-ideals of a JU-algebra was introduced and analyzed in [8].

Definition 8. [8] A subset J of a JU-algebra A is called a p-*ideal* of A if

(J-1) $1 \in J$ and
(p) $(\forall x, y, z \in A)((z \cdot x) \cdot (z \cdot y) \in J \wedge y \in J) \implies x \in J)$.

Lemma 8. *Any p-ideal J of a JU-algebra A is a JU-ideal of A.*

Proof. If we put $z = 1$ in Definition 8(p), we get (JU-2). □

Theorem 5. *Let J be a p-ideal of a JU-algerba A. Then*

(h) $(\forall y, z \in A)((z \cdot x) \cdot \varphi(z) \in J \implies x \in J)$;
(k) $(\forall x, z \in A)(z \cdot \varphi(z \cdot x) \in J \implies x \in J)$;
(m) $(\forall x, z \in A)((\varphi(z \cdot x) \in J \wedge z \in J) \implies x \in J)$;

(n) $(\forall x \in A)(\varphi(x) \in J \Longrightarrow x \in J)$.

Proof. If we put $y = 1$ in Definition 8(p), we get (h).
The condition (k) is obtained from condition (h) by applying equality (1) of Proposition 1.
If we put $y = z$ in in Definition 8(p), we get (m).
If we put $z = 1$ in (m), we get (n). □

In what follows, we introduce and analyze a new type of JU-ideal in JU-algebras.

Definition 9. Let J be a JU-ideal of a A. J is a $(\star)$-*ideal* of A if

($\star$) $(\forall x, y \in A)(\neg(x \in J) \wedge y \in J) \Longrightarrow x \cdot y \in J)$.

Proposition 8. *Let J be a $(\star)$-ideal of a JU-algebra A. Then*

(q) $(\forall x \in A)(\neg(x \in J) \Longrightarrow \varphi(x) \in J)$; *and*
(r) $(\forall x, y \in A)((\neg(x \cdot y \in J) \wedge y \in J) \Longrightarrow x \in J)$.

Proof. The Condition (q) is obtained by putting $y = 1$ in Definition 9($\star$).
The Condition (q) can be obtained from the contraposition of Definition 9($\star$). □

Theorem 6. *An ideal J of a JU-algebra A is a closed $(\star)$-ideal if and only if $\varphi(A) = \{\varphi(x) : x \in A\} \subseteq J$.*

Proof. Let J be an JU-ideal of A. If $\varphi(A) \subseteq J$, then obviously $\varphi(J) \subseteq J$ holds, i.e., the ideal J is closed.

Let $x, y \in A$ be arbitrary elements. Then $y \cdot (x \cdot y) \leqslant \varphi(x) \in J$ by Proposition 1(3) and hypothesis $\varphi(A) \subseteq J$. Thus $y \cdot (x \cdot y) \in J$ by Lemma 2. Hence $x \cdot y \in J$ by (J-2). So, J is a $(\star)$-ideal of A.

Let J be a closed $(\star)$-ideal of A. For $x \in A$, we have $x \in A \vee \neg(\in J)$. If $x \in J$, then $\varphi(x) \in J$ because J is a closed ideal of A.

If $\neg(x \in J)$, then from $\neg(x \in J) \wedge 1 \in J$ it follows $\varphi(x) = x \cdot 1 \in J$ because J is a $(\star)$-ideal of A. □

Before we finish this section, let us introduce another type of JU-ideal.

Definition 10. Let J be a JU-ideal of a JU-algebra A. J is an *associative* JU-ideal of A is

(A) $(\forall x, z \in A)(\varphi(z) \cdot x \in J \Longrightarrow x \cdot z \in J)$.

Theorem 7. *An associative ideal J of a JU-algebra A is closed and ag-ideal of A.*

Proof. If we put $x = 1$ in Definition 10(A), we get $\varphi(z) \cdot 1 \in J \Longrightarrow 1 \cdot z \in J$, i.e. we get $\varphi^2(z) \in J \Longrightarrow z \in J$. This means that J is an ag-ideal of A.

If we put $z = 1$ in Definition 10(A), we get $\varphi(1) \cdot x \in J \Longrightarrow x \cdot 1 \in J$, i.e. we get $x \in J \Longrightarrow \varphi(x) \in J$. So, J is a closed JU-ideal of A. □

4. Conclusion

In [7,8], the concept of JU-algebras is introduced and analyzed. However, this concept was introduced earlier in [9] under the name 'pseudo KU-algebra'. This author is more inclined to use the term 'weak KU-algebra' for this generalization of KU-algebras. In this paper, we have introduced and analyzed the concepts of a few new types of JU-ideals of a JU-algebra such as closed ideal, ag-ideal, t-ideal, $(\star)$-ideal and associative ideal. This article opens the possibility of introducing and analyzing several different types of JU filters (Definition 6) in these algebras.

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