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Positive solutions for boundary value problem of sixth-order elastic beam equation

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Abstract: In this paper, we study the existence of positive solutions for boundary value problem of sixth-order elastic beam equation of the form $-u^{(6)}(t) = q(t)f(t, u(t), u'(t), u''(t), u'''(t), u^{(4)}(t), u^{(5)}(t)),\ \ 0 < t < 1$, with conditions $u(0) = u'(1) = u''(0) = u'''(1) = u^{(4)}(0) = u^{(5)}(1) = 0$, where $f \in C([0,1] \times [0,\infty) \times [0,\infty) \times (-\infty,0] \times (-\infty,0] \times [0,\infty) \times [0,\infty) \to [0,\infty))$. The boundary conditions describe the deformation of an elastic beam simply supported at left and clamped at right by sliding clamps. We give sufficient conditions that allow us to obtain the existence of positive solution. The main tool used in the proof is the Leray-Schauder nonlinear alternative and Leray-Schauder fixed point theorem. As an application, we also give example to illustrate the results obtained.

Keywords: Green's function, positive solution, Leary-Schauder nonlinear alternative, fixed point theorem, boundary value problem.

MSC: 34B15, 34B18.

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1. Introduction

The study differential equations arise in variety of different areas of applied mathematics, physics and many applications of engineering and sciences. For example, the deformations of an elastic beam are described by a differential equation, often referred as the beam equation, see [1–3] and references therein for more details. Wuest [4] derived a model for beams and pipes that leads to a sixth-order differential equation.

Many authors studied the existence of positive solutions for sixth-order boundary value problem using different methods, for example, minimization theorem, global bifurcation theorem, operator spectral theorem and fixed point theorem in cone, see [5–10] and the references therein.

Also, in papers [11–15] the authors proved the existence of solutions for higher-order (2m-th-order) m-point boundary value problem

$$u^{(2m)}(t) = f(t, u(t), u'(t), ..., u^{(2m-2)}(t), u^{(2m-1)}(t)), \quad 0 \le t \le 1,$$
$$u^{(2i)}(0) = u^{(2i)}(1) = 0. \qquad 0 \le i \le m-1,$$

where $(-1)^n f : (0,1) \times \mathbb{R}^n \to [0,\infty)$.

Recently in 2016, Mirzei [16] studied the existence and nonexistence of positive solution for sixth-order boundary value problems (SBVP):

$$-u^{(6)}(t) = \lambda f(t, u(t)), \quad 0 < t < 1.$$
$$u(0) = u'(0) = u''(0) = 0, \quad u(1) = u'(1) = u''(1) = 0,$$

where λ is a parameter, $f : [0,1] \times [0,\infty) \to [0,\infty)$. The method used is the fixed point theorem in cones.

The aim of this paper is to establish some sufficient conditions for the existence of positive solutions for boundary value problem of sixth-order elastic beam equation (SBVP):

$$-u^{(6)}(t) = q(t)f(t, u(t), u'(t), u''(t), u'''(t), u^{(4)}(t), u^{(5)}(t)), \quad 0 < t < 1. \tag{1}$$

$$u(0) = u'(1) = u''(0) = u'''(1) = u^{(4)}(0) = u^{(5)}(1) = 0, \tag{2}$$

where $q : [0,1] \to [0,\infty)$, $f : [0,1] \times [0,\infty) \times [0,\infty) \times (-\infty,0] \times (-\infty,0] \times [0,\infty) \times [0,\infty) \to [0,\infty)$, are continuous.

This article is organized as follows. In Section 2, we present some definitions that will be used to prove the main results. In Section 3, we prove our main results which consists of existence theorems for positive solution of the BVP (1-2) without imposing any nonnegativity condition on f. Also, we establish some existence criteria of at least one positive solution by using the Leray-Schauder nonlinear alternative and Leray-Schauder fixed point theorem. Finally, in Section 4, as an application, we give an example to illustrate the results we obtained.

2. Preliminaries

In this section, we present some definitions, Leray-Schauder nonlinear alternative and Leray-Schauder fixed point theorem.

Definition 1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone of E if it satisfies the following two conditions:
(1) $x \in P, \lambda > 0$ implies $\lambda x \in P$,
(2) $x \in P, -x \in P$ implies $x = 0$.

Definition 2. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Definition 3. Suppose P is a cone in a Banach space E. The map α is a nonnegative continuous concave functional on P provided $\alpha : P \to [0,\infty)$ is continuous and

$$\alpha(rx + (1-r)y) \geq r\alpha(x) + (1-r)\alpha(y)$$

for all $x, y \in P$ and $r \in [0,1]$.

Similarly, a map β is nonnegative continuous convex functional on P provided $\beta : P \to [0,\infty)$ is continuous and

$$\beta(rx + (1-r)y) \leq r\beta(x) + (1-r)\beta(y)$$

for all $x, y \in P$ and $r \in [0,1]$.

We shall use the well-known Leray-Schauder fixed point theorem and Leray-Schauder nonlinear alternative to search for positive solution of the problem (1-2).

Theorem 1. *[17,18] Let E be Banach space and Ω be a bounded open subset of E, $0 \in \Omega$. Let $T : \overline{\Omega} \to E$ be a completely continuous operator. Then, either*
(i) there exists $u \in \partial\Omega$ and $\lambda > 1$ such that $T(u) = \lambda u$, or
(ii) there exists a fixed point $u^* \in \overline{\Omega}$.

3. Mains results

In this section, we shall impose growth conditions on f, which allow us to apply Leray-Schauder nonlinear alternative, and Leray-Schauder fixed point theorem to establish the existence of at least one positive solution to the SBVP (1-2), and we assume that $q(t) \equiv 1$.

Lemma 1. *Let $E = \{u \in C^5([0,1]) : u(0) = u'(1) = u''(0) = u'''(1) = u^{(4)}(0) = 0\}$ be the Banach space equipped with the maximum norm*

$$\|u\| = \max\{|u|_0, |u'|_0, |u''|_0, |u'''|_0, |u^{(4)}|_0, |u^{(5)}|_0\},$$

where $|u|_0 = \max_{0\le t\le 1} |u(t)|$. *Then for any* $u \in E$, *we have*

$$\|u\| = |u^{(5)}|_0 \text{ and } |u|_0 \le \frac{2}{15}\|u\|,\ |u^{'}|_0 \le \frac{5}{24}\|u\|,\ |u^{''}|_0 \le \frac{1}{3}\|u\|,\ |u^{'''}|_0 \le \frac{1}{2}\|u\|,\ |u^{(4)}|_0 \le \|u\|.$$

Proof. Let $G(t,s)$ be the Green's function of fifth-order homogeneous boundary value problem

$$u^{(5)}(t) = 0, \quad 0 < t < 1,$$

with $u(0) = u^{'}(1) = u^{''}(0) = u^{'''}(1) = u^{(4)}(0) = 0$. Then

$$G(t,s) = \frac{1}{24}\begin{cases} (t^4+s^4) + [6s^2t - 12s^2 + 4(2-t^2)]s, & 0 \le s \le t \le 1, \\ [4s^3 + 4t^2 s - 12s^2 + 4(2-t^2)]t, & 0 \le t \le s \le 1. \end{cases} \tag{3}$$

By (3), it is easy to see that

$$G(t,s) \ge 0,\ \frac{\partial G(t,s)}{\partial t} \ge 0,\ \frac{\partial^2 G(t,s)}{\partial t^2} \le 0,\ \frac{\partial^3 G(t,s)}{\partial t^3} \le 0,\ \frac{\partial^4 G(t,s)}{\partial t^4} \ge 0, \tag{4}$$

and

$$\begin{aligned}
\int_0^1 |G(t,s)|ds &= \int_0^1 G(t,s)ds = \frac{1}{120}t^5 - \frac{1}{12}t^3 + \frac{5}{24}t,\\
\int_0^1 |\frac{\partial G(t,s)}{\partial t}|ds &= \int_0^1 \frac{\partial G(t,s)}{\partial t}ds = \frac{1}{24}t^4 - \frac{1}{4}t^2 + \frac{5}{24},\\
\int_0^1 |\frac{\partial^2 G(t,s)}{\partial t^2}|ds &= -\int_0^1 \frac{\partial^2 G(t,s)}{\partial t^2}ds = -\frac{1}{6}t^3 + \frac{1}{2}t,\\
\int_0^1 |\frac{\partial^3 G(t,s)}{\partial t^3}|ds &= -\int_0^1 \frac{\partial^3 G(t,s)}{\partial t^3}ds = -\frac{1}{2}t^2 + \frac{1}{2},\\
\int_0^1 |\frac{\partial^4 G(t,s)}{\partial t^4}|ds &= \int_0^1 \frac{\partial^4 G(t,s)}{\partial t^4}ds = t.
\end{aligned}$$

From which we get

$$\begin{aligned}
\max_{0\le t\le 1}\int_0^1 |G(t,s)|ds &= \frac{2}{15},\\
\max_{0\le t\le 1}\int_0^1 |\frac{\partial G(t,s)}{\partial t}|ds &= \frac{5}{24},\\
\max_{0\le t\le 1}\int_0^1 |\frac{\partial^2 G(t,s)}{\partial t^2}|ds &= \frac{1}{3},\\
\max_{0\le t\le 1}\int_0^1 |\frac{\partial^3 G(t,s)}{\partial t^3}|ds &= \frac{1}{2},\\
\max_{0\le t\le 1}\int_0^1 |\frac{\partial^4 G(t,s)}{\partial t^4}|ds &= 1.
\end{aligned}$$

Let $u \in E$ and $\|u\| = r$. Then

$$\begin{aligned}
u(t) &= \int_0^1 G(t,s)[u^{(5)}(s)]ds,\\
u^{'}(t) &= \int_0^1 \frac{\partial G(t,s)}{\partial t}[u^{(5)}(s)]ds,\\
u^{''}(t) &= \int_0^1 \frac{\partial^2 G(t,s)}{\partial t^2}[u^{(5)}(s)]ds,\\
u^{'''}(t) &= \int_0^1 \frac{\partial^3 G(t,s)}{\partial t^3}[u^{(5)}(s)]ds,\\
u^{(4)}(t) &= \int_0^1 \frac{\partial^4 G(t,s)}{\partial t^4}[u^{(5)}(s)]ds.
\end{aligned}$$

Thus

$$\begin{aligned}
|u|_0 &\leq \max_{0\leq t\leq 1}\int_0^1 |G(t,s)||u^{(5)}(s)|ds \leq |u^{(5)}|_0 \max_{0\leq t\leq 1}\int_0^1 |G(t,s)|ds = \frac{2}{15}|u^{(5)}|_0,\\
|u^{'}|_0 &\leq \max_{0\leq t\leq 1}\int_0^1 |\frac{\partial G(t,s)}{\partial t}||u^{(5)}(s)|ds \leq |u^{(5)}|_0 \max_{0\leq t\leq 1}\int_0^1 |\frac{\partial G(t,s)}{\partial t}|ds = \frac{5}{24}|u^{(5)}|_0,\\
|u^{''}|_0 &\leq \max_{0\leq t\leq 1}\int_0^1 |\frac{\partial^2 G(t,s)}{\partial t^2}||u^{(5)}(s)|ds \leq |u^{(5)}|_0 \max_{0\leq t\leq 1}\int_0^1 |\frac{\partial^2 G(t,s)}{\partial t^2}|ds = \frac{1}{3}|u^{(5)}|_0,\\
|u^{'''}|_0 &\leq \max_{0\leq t\leq 1}\int_0^1 |\frac{\partial^3 G(t,s)}{\partial t^3}||u^{(5)}(s)|ds \leq |u^{(5)}|_0 \max_{0\leq t\leq 1}\int_0^1 |\frac{\partial^3 G(t,s)}{\partial t^3}|ds = \frac{1}{2}|u^{(5)}|_0,\\
|u^{(4)}|_0 &\leq \max_{0\leq t\leq 1}\int_0^1 |\frac{\partial^4 G(t,s)}{\partial t^4}||u^{(5)}(s)|ds \leq |u^{(5)}|_0 \max_{0\leq t\leq 1}\int_0^1 |\frac{\partial^4 G(t,s)}{\partial t^4}|ds = |u^{(5)}|_0.
\end{aligned}$$

So, $|u^{(5)}|_0 = \|u\| = r$ and the proof is completed. □

Theorem 2. *Suppose that* $f \in C([0,1]\times[0,\infty)\times[0,\infty)\times(-\infty,0]\times(-\infty,0]\times[0,\infty)\times[0,\infty),[0,\infty))$ *and* $f(t,0,0,0,0,0,0)\neq 0,\ t\in[0,1]$. *Suppose there exist nonnegative functions* $a_i \in L^1[0,1],\ i=0,1,2,3,4,5,6,$ *such that*

$$B = \frac{2}{15}\int_0^1 a_0(s)ds + \frac{5}{24}\int_0^1 a_1(s)ds + \frac{1}{3}\int_0^1 a_2(s)ds + \frac{1}{2}\int_0^1 a_3(s)ds + \int_0^1 a_4(s)ds + \int_0^1 a_5(s)ds < 1, \tag{5}$$

and for any $(t,u_0,u_1,u_2,u_3,u_4,u_5)\in[0,1]\times[0,\frac{2}{15}\rho]\times[0,\frac{5}{24}\rho]\times[-\frac{1}{3}\rho,0]\times[-\frac{1}{2}\rho,0]\times[0,\rho]\times[0,\rho]$, *f satisfies*

$$f(t,u_0,u_1,u_2,u_3,u_4,u_5) \leq a_0(t)u_0 + a_1(t)u_1 - a_2(t)u_2 - a_3(t)u_3 + a_4(t)u_4 + a_5(t)u_5 + a_6(t), \tag{6}$$

where $\rho = A(1-B)^{-1}$, $A=\int_0^1 a_6(s)ds$. *Then problem (1-2) has at least one positive solution* $u^*\in C^6([0,1])$ *such that* $\frac{15}{2}\max_{0\leq t\leq 1}u^*(t) \leq \frac{24}{5}\max_{0\leq t\leq 1}(u^*)^{'}(t) \leq 3\max_{0\leq t\leq 1}[-(u^*)^{''}(t)] \leq 2\max_{0\leq t\leq 1}[-(u^*)^{'''}(t)] \leq \max_{0\leq t\leq 1}(u^*)^{(4)}(t) \leq \max_{0\leq t\leq 1}(u^*)^{(5)}(t) \leq \rho$.

Proof. Since $f(t,0,0,0,0,0,0)\neq 0$ and $|f(t,0,0,0,0,0,0)|\leq a_6(t),\ t\in[0,1]$, we have $A=\int_0^1 a_6(s)ds>0$, so, it follows from (5) that $\rho>0$. From Equation (1) and boundary condition $u^{(5)}(1)=0$, we have

$$u^{(5)}(t) = \int_t^1 f(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau),u^{(4)}(\tau),u^{(5)}(\tau))d\tau,$$

which implies that

$$u(t) = \int_0^1 G(t,s)\int_s^1 f(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau),u^{(4)}(\tau),u^{(5)}(\tau))d\tau ds,\ t\in[0,1],$$

where $G(t,s)$ is defined by (3). Let $\Omega_\rho=\{u\in E,\|u\|<\rho\}$, then Ω_ρ is a bounded closed convex set of E and $0\in\Omega_\rho$. For $u\in\Omega_\rho$, define the operator T by

$$(Tu)(t) = \int_0^1 G(t,s)\int_s^1 f(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau),u^{(4)}(\tau),u^{(5)}(\tau))d\tau ds. \tag{7}$$

Then

$$\begin{aligned}
(Tu)^{'}(t) &= \int_0^1 \frac{\partial G(t,s)}{\partial t}\int_s^1 f(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau),u^{(4)}(\tau),u^{(5)}(\tau))d\tau ds,\\
(Tu)^{''}(t) &= \int_0^1 \frac{\partial^2 G(t,s)}{\partial t^2}\int_s^1 f(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau),u^{(4)}(\tau),u^{(5)}(\tau))d\tau ds\\
(Tu)^{'''}(t) &= \int_0^1 \frac{\partial^3 G(t,s)}{\partial t^3}\int_s^1 f(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau),u^{(4)}(\tau),u^{(5)}(\tau))d\tau ds,\\
(Tu)^{(4)}(t) &= \int_0^1 \frac{\partial^4 G(t,s)}{\partial t^4}\int_s^1 f(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau),u^{(4)}(\tau),u^{(5)}(\tau))d\tau ds,\\
(Tu)^{(5)}(t) &= \int_t^1 f(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau),u^{(4)}(\tau),u^{(5)}(\tau))d\tau, \quad t\in[0,1].
\end{aligned}$$

So, $(Tu)(0) = (Tu)'(1) = (Tu)''(0) = (Tu)'''(1) = (Tu)^{(4)}(0) = (Tu)^{(5)}(1) = 0$. Therefore, $T : \Omega_\rho \to E$. By Ascoli-Arzela Theorem, it is easy to know that this operator $T : \Omega_\rho \to E$ is a completely continuous operator. So, the problem (1-2) has a solution $u = u(t)$ if and only if u solves the operator equation $Tu = u$.

Suppose there exists $u \in \partial\Omega_\rho$, $\lambda > 1$ such that $Tu = \lambda u$. Noticing that $\|u\| = \rho$, it follows from Lemma that

$$|u|_0 \le \frac{2}{15}\rho, \ \ |u'|_0 \le \frac{5}{24}\rho, \ \ |u''|_0 \le \frac{1}{3}\rho, \ \ |u'''|_0 \le \frac{1}{2}\rho, \ \ |u^{(4)}|_0 \le \rho, \ \ |u^{(5)}|_0 = \rho.$$

Thus from (5), (6) and (7), we have

$$\begin{aligned}
\lambda\rho &= \lambda\|u\| = \|Tu\| = \max_{0\le t\le 1}|u^{(5)}(t)| \\
&= \max_{0\le t\le 1}\left|\int_t^1 f(s,u(s),u'(s),u''(s),u'''(s),u^{(4)}(s),u^{(5)}(s))ds\right| \\
&= \max_{0\le t\le 1}\int_t^1 f(s,u(s),u'(s),u''(s),u'''(s),u^{(4)}(s),u^{(5)}(s))ds \\
&= \int_0^1 f(s,u(s),u'(s),u''(s),u'''(s),u^{(4)}(s),u^{(5)}(s))ds \\
&\le \int_0^1 \left[a_0(s)u(s) + a_1(s)u'(s) - a_2(s)u''(s) - a_3(s)u'''(s) + a_4(s)u^{(4)}(s) + a_5(s)u^{(5)}(s) + a_6(s)\right] ds \\
&\le \int_0^1 \left[\frac{2}{15}a_0(s)\rho + \frac{5}{24}a_1(s)\rho + \frac{1}{3}a_2(s)\rho + \frac{1}{2}a_3(s)\rho + a_4(s)\rho + a_5(s)\rho + a_6(s)\right] ds \\
&= \left[\frac{2}{15}\int_0^1 a_0(s)ds + \frac{5}{24}\int_0^1 a_1(s)ds + \frac{1}{3}\int_0^1 a_2(s)ds + \frac{1}{2}\int_0^1 a_3(s)ds + \int_0^1 a_4(s)ds + \int_0^1 a_5(s)ds\right]\rho \\
&\quad + \int_0^1 a_6(s)ds \\
&= B\rho + A = B\rho + (1-B)\rho = \rho,
\end{aligned}$$

a contradiction. So, by Theorem 1, T has a fixed point $u^* \in E$, which is a solution of the problem (1-2). Noticing that $f(t,0,0,0,0,0,0) \ne 0$, we assert that $u = 0$ is not a solution of the (1-2), therefore, $|u^*|_0 > 0$. It follows from (4) that $u^*(t)$ is nondecreasing and concave on $[0,1]$, thus $u^*(t) \ge t|u^*|_0 > 0$ for $t \in [0,1]$, i.e., $u^*(t)$ is a positive solution of the problem (1-2). This completes the proof. □

Lemma 2. *The Green's function of the sixth-order homogeneous equation $-u^{(6)}(t) = 0$, $t \in [0,1]$, with boundary conditions (2) is*

$$G(t,s) = \frac{1}{120}\begin{cases} [s^2(s^2+10t^2-20t)+20t(2-t^2)+5t^4]s, \ 0\le s\le t\le 1, \\ \\ [t^2(t^2+10s^2-20s)+20s(2-s^2)+5s^4]t, \ 0\le t\le s\le 1, \end{cases} \tag{8}$$

and for any $t,s \in [0,1]$,

$$G(t,s)\ge 0, \ \frac{\partial G(t,s)}{\partial t}\ge 0, \ \frac{\partial^2 G(t,s)}{\partial t^2}\le 0, \ \frac{\partial^3 G(t,s)}{\partial t^3}\le 0, \ \frac{\partial^4 G(t,s)}{\partial t^4}\ge 0, \ \frac{\partial^5 G(t,s)}{\partial t^5}\ge 0. \tag{9}$$

Theorem 3. *Assume that $f \in C([0,1]\times[0,\infty)\times[0,\infty)\times(-\infty,0]\times(-\infty,0]\times[0,\infty)\times[0,\infty),[0,\infty))$ and $f(t,0,0,0,0,0,0)\ne 0$, $t\in[0,1]$. Suppose that there exists positive number $d > 0$ such that*

$$\begin{aligned}
&\max\left\{f(t,u_0,u_1,u_2,u_3,u_4,u_5) : (t,u_0,u_1,u_2,u_3,u_4,u_5)\in[0,1]\times[0,d]\times\left[0,\frac{96}{61}d\right]\right. \\
&\left.\times\left[-\frac{150}{61}d,0\right]\times\left[-\frac{240}{61}d,0\right]\times\left[0,\frac{360}{61}d\right]\times\left[0,\frac{720}{61}d\right]\right\}\le\frac{720}{61}d.
\end{aligned} \tag{10}$$

Then the problem (1-2) has at least one positive solution $u^* \in C^6([0,1])$ such that

$$0\le u^*(t)\le d, \ 0\le (u^*)'(t)\le\frac{96}{61}d, \ -\frac{150}{61}d\le (u^*)''(t)\le 0,$$

$$-\frac{240}{61}d \le (u^*)'''(t) \le 0,\ 0 \le (u^*)^{(4)}(t) \le \frac{360}{61}d,\ 0 \le (u^*)^{(5)}(t) \le \frac{720}{61}d,\ t \in [0,1].$$

Proof. From (8) and after direct computations, we easily get

$$\begin{aligned}
\int_0^1 |G(t,s)|ds &= \int_0^1 G(t,s)ds = -\frac{1}{720}t^6 + \frac{1}{120}t^5 - \frac{1}{18}t^3 + \frac{2}{15}t,\\
\int_0^1 |\frac{\partial G(t,s)}{\partial t}|ds &= \int_0^1 \frac{\partial G(t,s)}{\partial t}ds = -\frac{1}{120}t^5 + \frac{1}{24}t^4 - \frac{1}{6}t^2 + \frac{2}{15},\\
\int_0^1 |\frac{\partial^2 G(t,s)}{\partial t^2}|ds &= -\int_0^1 \frac{\partial^2 G(t,s)}{\partial t^2}ds = \frac{1}{24}t^4 - \frac{1}{6}t^3 + \frac{1}{3}t,\\
\int_0^1 |\frac{\partial^3 G(t,s)}{\partial t^3}|ds &= -\int_0^1 \frac{\partial^3 G(t,s)}{\partial t^3}ds = \frac{1}{6}t^3 - \frac{1}{2}t^2 + \frac{1}{3},\\
\int_0^1 |\frac{\partial^4 G(t,s)}{\partial t^4}|ds &= \int_0^1 \frac{\partial^4 G(t,s)}{\partial t^4}ds = -\frac{1}{2}t^2 + t,\\
\int_0^1 |\frac{\partial^5 G(t,s)}{\partial t^5}|ds &= \int_0^1 \frac{\partial^5 G(t,s)}{\partial t^5}ds = 1 - t.
\end{aligned}$$

So,

$$\begin{aligned}
\max_{0\le t\le 1}\int_0^1 |G(t,s)|ds &= \frac{61}{720},\\
\max_{0\le t\le 1}\int_0^1 |\frac{\partial G(t,s)}{\partial t}|ds &= \frac{2}{15},\\
\max_{0\le t\le 1}\int_0^1 |\frac{\partial^2 G(t,s)}{\partial t^2}|ds &= \frac{5}{24},\\
\max_{0\le t\le 1}\int_0^1 |\frac{\partial^3 G(t,s)}{\partial t^3}|ds &= \frac{1}{3},\\
\max_{0\le t\le 1}\int_0^1 |\frac{\partial^4 G(t,s)}{\partial t^4}|ds &= \frac{1}{2},\\
\max_{0\le t\le 1}\int_0^1 |\frac{\partial^5 G(t,s)}{\partial t^5}|ds &= 1.
\end{aligned}$$

Now, we consider the Banach space $E = C^5([0,1])$ equipped with the norm

$$\|u\| = \max\left\{|u|_0,\ \frac{61}{96}|u'|_0,\ \frac{61}{150}|u''|_0,\ \frac{61}{240}|u'''|_0,\ \frac{61}{360}|u^{(4)}|_0,\ \frac{61}{720}|u^{(5)}|_0\right\}, \tag{11}$$

where $|u|_0 = \max_{0\le t\le 1}|u(t)|$, for $u \in E$ define the operator T by

$$(Tu)(t) = \int_0^1 G(t,s)f(s,u(s),u'(s),u''(s),u'''(s),u^{(4)}(s),u^{(5)}(s))ds. \tag{12}$$

Then

$$(Tu)'(t) = \int_0^1 \frac{\partial G(t,s)}{\partial t}f(s,u(s),u'(s),u''(s),u'''(s),u^{(4)}(s),u^{(5)}(s))ds,\ t \in [0,1],$$

$$(Tu)''(t) = \int_0^1 \frac{\partial^2 G(t,s)}{\partial t^2}f(s,u(s),u'(s),u''(s),u'''(s),u^{(4)}(s),u^{(5)}(s))ds,\ t \in [0,1],$$

$$(Tu)'''(t) = \int_0^1 \frac{\partial^3 G(t,s)}{\partial t^3}f(s,u(s),u'(s),u''(s),u'''(s),u^{(4)}(s),u^{(5)}(s))ds,\ t \in [0,1],$$

$$(Tu)^{(4)}(t) = \int_0^1 \frac{\partial^4 G(t,s)}{\partial t^4}f(s,u(s),u'(s),u''(s),u'''(s),u^{(4)}(s),u^{(5)}(s))ds,\ t \in [0,1],$$

$$(Tu)^{(5)}(t) = \int_0^1 \frac{\partial^5 G(t,s)}{\partial t^5}f(s,u(s),u'(s),u''(s),u'''(s),u^{(4)}(s),u^{(5)}(s))ds,\ t \in [0,1].$$

So, $(Tu)(0) = (Tu)^{'}(1) = (Tu)^{''}(0) = (Tu)^{'''}(1) = (Tu)^{(4)}(0) = (Tu)^{(5)}(1) = 0$. Therefore, by Ascoli-Arzela Theorem, it is easy to see that this operator $T : E \to E$ is a completely continuous operator. Problem (1-2) has a solution $u = u(t)$ if and only if u is a fixed point of operator T defined by (12).

Let

$$\Omega_d = \{u \in E,\ \|u\| < d,\ \ u(t) \geq 0,\ u^{'}(t) \geq 0,\ u^{''}(t) \leq 0,\ u^{'''}(t) \leq 0,\ u^{(4)}(t) \geq 0,\ u^{(5)}(t) \geq 0,\ t \in [0,1]\},$$

then Ω_d is a bounded closed convex set of E. If $u \in \Omega_d$, then by (11), we have

$$|u|_0 \leq d,\ |u^{'}|_0 \leq \frac{96}{61}d,\ |u^{''}|_0 \leq \frac{150}{61}d,\ |u^{'''}|_0 \leq \frac{240}{61}d,\ |u^{(4)}|_0 \leq \frac{360}{61}d,\ |u^{(5)}|_0 \leq \frac{720}{61}d,$$

which implies

$$0 \leq u(t) \leq d,\ 0 \leq u^{'}(t) \leq \frac{96}{61}d,\ -\frac{150}{61}d \leq u^{''}(t) \leq 0,$$
$$-\frac{240}{61}d \leq u^{'''}(t) \leq 0,\ 0 \leq u^{(4)}(t) \leq \frac{360}{61}d,\ 0 \leq u^{(5)}(t) \leq \frac{720}{61}d,\ t \in [0,1].$$

Thus (10) implies

$$f(t,u(t),u^{'}(t),u^{''}(t),u^{'''}(t),u^{(4)}(t),u^{(5)}(t)) \leq \frac{720}{61}d,\ t \in [0,1].$$

Therefore,

$$\begin{aligned} |(Tu)|_0(t) &= \max_{0\leq t\leq 1}\left|\int_0^1 G(t,s)f(s,u(s),u^{'}(s),u^{''}(s),u^{'''}(s),u^{(4)}(s),u^{(5)}(s))ds\right| \\ &\leq \max_{0\leq t\leq 1}\int_0^1 G(t,s)f(s,u(s),u^{'}(s),u^{''}(s),u^{'''}(s),u^{(4)}(s),u^{(5)}(s))ds \\ &\leq \frac{720}{61}d \times \max_{0\leq t\leq 1}\int_0^1 G(t,s)ds = d. \end{aligned}$$

$$\begin{aligned} \left|(Tu)^{'}\right|_0(t) &= \max_{0\leq t\leq 1}\left|\int_0^1 \frac{\partial G(t,s)}{\partial t}f(s,u(s),u^{'}(s),u^{''}(s),u^{'''}(s),u^{(4)}(s),u^{(5)}(s))ds\right| \\ &\leq \max_{0\leq t\leq 1}\int_0^1 \frac{\partial G(t,s)}{\partial t}f(s,u(s),u^{'}(s),u^{''}(s),u^{'''}(s),u^{(4)}(s),u^{(5)}(s))ds \\ &\leq \frac{720}{61}d \times \max_{0\leq t\leq 1}\int_0^1 \frac{\partial G(t,s)}{\partial t}ds = \frac{96}{61}d. \end{aligned}$$

$$\begin{aligned} \left|(Tu)^{''}\right|_0(t) &= \max_{0\leq t\leq 1}\left|\int_0^1 \frac{\partial^2 G(t,s)}{\partial t^2}f(s,u(s),u^{'}(s),u^{''}(s),u^{'''}(s),u^{(4)}(s),u^{(5)}(s))ds\right| \\ &= \max_{0\leq t\leq 1}\int_0^1 \left|\frac{\partial^2 G(t,s)}{\partial t^2}f(s,u(s),u^{'}(s),u^{''}(s),u^{'''}(s),u^{(4)}(s),u^{(5)}(s))\right|ds \\ &\leq \frac{720}{61}d \times \max_{0\leq t\leq 1}\int_0^1 \left|\frac{\partial^2 G(t,s)}{\partial t^2}\right|ds = \frac{150}{61}d. \end{aligned}$$

$$\begin{aligned} \left|(Tu)^{'''}\right|_0(t) &= \max_{0\leq t\leq 1}\left|\int_0^1 \frac{\partial^3 G(t,s)}{\partial t^3}f(s,u(s),u^{'}(s),u^{''}(s),u^{'''}(s),u^{(4)}(s),u^{(5)}(s))ds\right| \\ &= \max_{0\leq t\leq 1}\int_0^1 \left|\frac{\partial^3 G(t,s)}{\partial t^3}f(s,u(s),u^{'}(s),u^{''}(s),u^{'''}(s),u^{(4)}(s),u^{(5)}(s))\right|ds \\ &\leq \frac{720}{61}d \times \max_{0\leq t\leq 1}\int_0^1 \left|\frac{\partial^3 G(t,s)}{\partial t^3}\right|ds = \frac{240}{61}d. \end{aligned}$$

$$\begin{aligned}\left|(Tu)^{(4)}\right|_0(t) &= \max_{0\le t\le 1}\left|\int_0^1 \frac{\partial^4 G(t,s)}{\partial t^4} f(s,u(s),u^{'}(s),u^{''}(s),u^{'''}(s),u^{(4)}(s),u^{(5)}(s))ds\right| \\ &\le \max_{0\le t\le 1}\int_0^1 \frac{\partial^4 G(t,s)}{\partial t^4} f(s,u(s),u^{'}(s),u^{''}(s),u^{'''}(s),u^{(4)}(s),u^{(5)}(s))ds \\ &\le \frac{720}{61}d\times\max_{0\le t\le 1}\int_0^1 \frac{\partial^4 G(t,s)}{\partial t^4}ds = \frac{360}{61}d.\end{aligned}$$

$$\begin{aligned}\left|(Tu)^{(5)}\right|_0(t) &= \max_{0\le t\le 1}\left|\int_0^1 \frac{\partial^5 G(t,s)}{\partial t^5} f(s,u(s),u^{'}(s),u^{''}(s),u^{'''}(s),u^{(4)}(s),u^{(5)}(s))ds\right| \\ &\le \max_{0\le t\le 1}\int_0^1 \frac{\partial^5 G(t,s)}{\partial t^5} f(s,u(s),u^{'}(s),u^{''}(s),u^{'''}(s),u^{(4)}(s),u^{(5)}(s))ds \\ &\le \frac{720}{61}d\times\max_{0\le t\le 1}\int_0^1 \frac{\partial^5 G(t,s)}{\partial t^5}ds = \frac{720}{61}d.\end{aligned}$$

Thus

$$\|Tu\| = \max\left\{|(Tu)|_0,\ \frac{61}{96}|(Tu)^{'}|_0,\ \frac{61}{150}|(Tu)^{''}|_0,\ \frac{61}{240}|(Tu)^{'''}|_0,\ \frac{61}{360}|(Tu)^{(4)}|_0,\ \frac{61}{720}|(Tu)^{(5)}|_0\right\} \le d,$$

i.e., $Tu \in \Omega_d$. So, by Leray-Schauder fixed point Theorem, T has a fixed point $u^* \in \Omega_d$, which is a solution of the problem (1-2). Noticing that $f(t,0,0,0,0,0,0) \ne 0$. So, $u = 0$ is not a solution of the problem (1-2), therefore, $|u^*|_0 > 0$. From (9) we know that $u^*(t)$ is nondecreasing and concave on $[0,1]$, thus $u^*(t) \ge t|u^*|_0 > 0$ for $t \in [0,1]$. So, $u^*(t)$ is a positive solution of the problem (1-2). This completes the proof. □

4. Examples

In order to illustrate the above results, we consider an example.

Example 1. Consider the following problem SBVP

$$\begin{aligned}&-u^{(6)} = \tfrac{\sqrt{t}}{26}u + \tfrac{t^{15}}{2}u^{'} - \tfrac{t^{11}}{4}u^{''} - \tfrac{\sqrt[3]{t}}{59}u^{'''} + \tfrac{t^4}{7}u^{(4)} + \tfrac{2\sqrt[5]{t}}{5}u^{(5)} + t^3 + 1, \\ &u(0) = u^{'}(1) = u^{''}(0) = u^{'''}(1) = u^{(4)}(0) = u^{(5)}(1) = 0.\end{aligned} \tag{13}$$

Set

$$f(t,u_0,u_1,u_2,u_3,u_4,u_5) = \frac{\sqrt{t}}{26}u_0 + \frac{t^{15}}{2}u_1 - \frac{t^{11}}{4}u_2 - \frac{\sqrt[3]{t}}{59}u_3 + \frac{t^4}{7}u_4 + \frac{2\sqrt[5]{t}}{5}u_5 + t^3 + 1,$$

and

$$a_0(t) = \frac{\sqrt{t}}{26},\ a_1(t) = t^{15},\ a_2(t) = \frac{t^{11}}{4},\ a_3(t) = \frac{\sqrt[3]{t}}{59},\ a_4(t) = \frac{t^4}{7},\ a_5(t) = \frac{2\sqrt[5]{t}}{5},\ a_6(t) = t^3 + 2.$$

It is easy to prove that $a_i \in L^1[0,1],\ i = 0,1,2,3,4,5,6$, are nonnegative functions, $f(t,0,0,0,0,0,0) = t^3 + 1 \ne 0$. Moreover, we have

$$\begin{aligned}B &= \frac{2}{15}\int_0^1 a_0(s)ds + \frac{5}{24}\int_0^1 a_1(s)ds + \frac{1}{3}\int_0^1 a_2(s)ds + \frac{1}{2}\int_0^1 a_3(s)ds + \int_0^1 a_4(s)ds + \int_0^1 a_5(s)ds, \\ &= \frac{2}{15}\int_0^1 \frac{\sqrt{s}}{26}ds + \frac{5}{24}\int_0^1 s^{15}ds + \frac{1}{3}\int_0^1 \frac{s^{11}}{2}ds + \frac{1}{2}\int_0^1 \frac{\sqrt[3]{s}}{59}ds + \int_0^1 \frac{s^4}{7}ds + \int_0^1 \frac{2\sqrt[5]{s}}{5}ds, \\ &= \frac{2}{585} + \frac{5}{384} + \frac{1}{144} + \frac{3}{472} + \frac{1}{35} + \frac{1}{3} \simeq 0,38592443631 < 1,\end{aligned}$$

and for any

$$(t,u_0,u_1,u_2,u_3,u_4,u_5) \in [0,1]\times[0,\frac{2}{15}\rho]\times[0,\frac{5}{24}\rho]\times[-\frac{1}{3}\rho,0]\times[-\frac{1}{2}\rho,0]\times[0,\rho]\times[0,\rho],$$

and f satisfies

$$f(t,u_0,u_1,u_2,u_3,u_4,u_5) \le a_0(t)u_0 + a_1(t)u_1 - a_2(t)u_2 - a_3(t)u_3 + a_4(t)u_4 + a_5(t)u_5 + a_6(t).$$

where

$$A = \int_0^1 a_6(s)ds = \frac{9}{4}, \ \rho = A(1-B)^{-1} \simeq 3,66404418779.$$

Hence, by Theorem 2, the SBVP (13) has at least one positive solution u^* in $C^6([0,1])$ such that $\frac{15}{2}\max_{0\le t\le 1} u^*(t) \le \frac{24}{5}\max_{0\le t\le 1}(u^*)'(t) \le 3\max_{0\le t\le 1}[-(u^*)''(t)] \le 2\max_{0\le t\le 1}[-(u^*)'''(t)] \le \max_{0\le t\le 1}(u^*)^{(4)}(t) \le \max_{0\le t\le 1}(u^*)^{(5)}(t) \le \rho.$

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