On a generalization of KU-algebras pseudo-KU algebras

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Abstract: As a generalization of KU-algebras, the notion of pseudo-KU algebras is introduced in 2020 by the author (D. A. Romano. Pseudo-UP algebras, An introduction. Bull. Int. Math. Virtual Inst., 10(2)(2020), 349-355). Some characterizations of pseudo-KU algebras are established in that article. In addition, it is shown that each pseudo-KU algebra is a pseudo-UP algebra. In this paper it is a concept developed of pseudo-KU algebras in more detail and it has identified some of the main features of this type of universal algebras such as the notions of pseudo-subalgebras, pseudo-ideals, pseudo-filters and pseudo homomorphisms. Also, it has been shown that every pseudo-KU algebra is a pseudo-BE algebra. In addition, a congruence was constructed on a pseudo-KU algebra generated by a pseudo-ideal and shown that the corresponding factor-structure is and pseudo-KU algebra as well.

Keywords: KU-algebra, Pseudo-KU algebra, pseudo-UP algebra, pseudo-BE algebra, pseudo-ideals, pseudo-homomorphism.

MSC: 62D05.

1. Introduction

The concept of pseudo-BCK algebras was introduce in [1] by Georgescu and Iorgulescu as an extension of BCK-algebras. The notion of pseudo-BCI algebras was introduced and analyzed in [2] by Dudek and Jun as a generalization of BCI-algebras. The concept of pseudo-BE algebras was introduced in 2013 and their properties were explored by Borzooei et al., in [3]. These algebraic structures has been in the focus of many authors (for example, see [4–10]). Pseudo BL-algebras are a non-commutative generalization of BL-algebras introduced in [11]. Pseudo BL-algebras are intensively studied by many authors (for example, [12–14]).

Prabpayak and Leerawat 2009 in [15,16] introduced a new algebraic structure which is called KU-algebras. They studied ideals and congruences in KU-algebras. They also introduced the concept of homomorphism of KU-algebras and investigated some related properties. Moreover, they derived some straightforward consequences of the relations between quotient KU-algebras and isomorphism. Many authors took part in the study of this algebraic structure (for example: [17,18]).

A detailed listing of the researchers and their contributions to these activities it can be found in [19]. Here, we will highlight the contribution of [20]. In [21], Kim and Kim introduced the concept of BE-algebras as a generalization of dual BCK-algebras. This class of algebra was also studied by Rezaei and Saeid 2012 in article [22]. In the article [20], the authors (Rezaei, Saeod and Borzooei) proved that a KU-algebra is equivalent to a commutative self-distributive BE-algebra. (A BE-algebra A is a self-distributive if \( x \cdot (y \cdot z) = (z \cdot y) \cdot (x \cdot z) \) for all \( x, y, z \in A \)). Additionally, they proved that every KU-algebra is a BE-algebra ([20], Theorem 3.4), every Hilbert algebra is a KU-algebra ([20], Theorem 3.5) and a self-distributive KU-algebra is equivalent to a Hilbert algebra ([20]). Iampan constructed PU-algebra as a generalization of KU-algebra in [19] in 2017 and showed that each KU-algebra is a PU-algebra.

In article [23], the author designed the concepts of pseudo-UP ([23], Definition 3.1) and pseudo-KU algebras ([23], Definition 4.1) and showed that each pseudo-KU algebra is a pseudo-UP algebra ([23], Theorem 4.1). However, the term 'pseudo KU-algebra' and mark 'PKU' has already been used in [24] for different purposes. It should be noted here that this term 2019 has been renamed to 'JU-algebra' ([25]). Although introducing the term 'pseudo-KU algebra' as a name for a structure constructed in the manner described here and using the abbreviation 'pKU' for this algebra can lead to confusion, we did it for needs of article [23] and of this paper.
In this paper we develop the concept in more detail of pseudo-KU algebras and we identify some of the main features of this type of universal algebras. The paper was designed as follows: After the Section 2, which outlines the necessary previous terms, Section 3 introduces the concept of pseudo-KU algebra and analyzes some of its important properties. In Section 4, the concept of pseudo-KU algebras is linked to the concepts of pseudo-UP and pseudo-BE algebras. Section 5 deals with some substructures of this class of algebras such as pseudo-subalgebras, pseudo-ideals and pseudo-filters. Finally, in Section 6, the concepts of pseudo-homomorphisms and congruences on pseudo-KU algebras are analyzed.

2. Preliminaries

In this section we will describe some elements of KU-algebras from the literature [15,16] necessary for our intentions in this text.

Definition 1. ([15]) An algebra \( A = (A, \cdot, 0) \) of type \((2, 0)\) is called a KU-algebra where \( A \) is a nonempty set, ‘\( \cdot \)’ is a binary operation on \( A \), and 0 is a fixed element of \( A \) (i.e. a nullary operation) if it satisfies the following axioms:

(KU-1) \((\forall x,y,z \in A)((x \cdot y) \cdot ((y \cdot z) \cdot (x \cdot z)) = 0)\),

(KU-2) \((\forall x \in A)(0 \cdot x = x)\),

(KU-3) \((\forall x \in A)(x \cdot 0 = 0)\), and

(KU-4) \((\forall x,y \in A)((x \cdot y = 0 \land y \cdot x = 0) \implies x = y)\).

On a KU-algebra \( A = (A, \cdot, 0) \), we define the KU-ordering \( \leq \) on \( A \) as follows ([15], pp. 56):

\((\forall x,y \in A)(x \leq y \iff y \cdot x = 0)\).

Lemma 1. In a KU-algebra \( A \), the following properties hold:

1. \((\forall x \in A)(x \leq x)\),
2. \((\forall x,y \in A)((x \leq y \land y \leq x) \implies x = y)\),
3. \((\forall x,y,z \in A)((x \leq y \land y \leq z) \implies x \leq z)\),
4. \((\forall x,y,z \in A)(x \leq y \implies z \cdot x \leq z \cdot y)\),
5. \((\forall x,y,z \in A)(x \leq y \implies y \cdot z \leq x \cdot z)\),
6. \((\forall x,y \in A)(x \cdot y \leq y)\) and
7. \((\forall x \in A)(0 \leq x)\).

Definition 2. ([15]) Let \( S \) be a non-empty subset of a KU-algebra \( A \).

(a) The subset \( S \) is said to be a KU-subalgebra of \( A \) if \((S, \cdot, 0)\) is a KU-algebra.

(b) The subset \( S \) is said to be an ideal of \( A \) if it satisfies the following conditions:

(I1) \(0 \in S\), and

(I2) \((\forall x,y,z \in A)((x \cdot (y \cdot z) \in S \land y \in S) \implies x \cdot z \in S)\).

As shown in [18], this kind of algebra satisfies one specific equality.

Lemma 2 ([18]). In a KU-algebra \( A \), the following holds:

(KU-5) \((\forall x,y,z \in A)((y \cdot (z \cdot x) = y \cdot (z \cdot x))\).

In the light of the previous equality, condition (I2) is transformed into condition:

(I3) \((\forall x,y \in A)((x \cdot y \in S \land x \in S) \implies y \in S)\).

Indeed, if we put \( x = 0, y = x \) and \( z = y \) in (I2), we immediately obtain (I3) by (KU-2). Conversely, let (I3) be a valid formula and let \( x, y, z \in A \) be arbitrary elements such that \( x \cdot (y \cdot z) \in J \) and \( y \in J \). Then \( y \cdot (x \cdot z) \in J \) by (KU-5). Thus \( x \cdot z \in J \) by (I3).

From (I3) it immediately follows:

Lemma 3. Let \( S \) be an ideal in a KU-algebra \( A \). Then

(I4) \((\forall x,y \in A)((x \leq y \land y \in S) \implies x \in S)\).

We can introduce the concept of filters in KU-algebra if formula (I3) serves as a motivation.
Definition 3. The subset \( F \) is said to be a filter of \( A \) if it satisfies the following conditions:

(F1) \( 0 \in F \), and

(F3) \( (\forall x, y \in A)((x \cdot y \in F \land y \in F) \implies x \in F) \).

A filter in KU-algebra, designed in this way, has the following property:

Lemma 4. Let \( F \) be a filter in a KU-algebra \( A \). Then

(F4) \( (\forall x, y \in A)((x \leq y \land x \in F) \implies y \in F) \).

3. Concept of pseudo-KU algebra

Definition 4. ([23]) An algebra \( \mathfrak{A} = ((A, \leq), \cdot, *, 0) \) of type \((2, 2, 0)\) is called a pseudo-KU algebra if it satisfies the following axioms:

(pKU-1): \( (\forall x, y, z \in A)((y \cdot x) \leq ((x \cdot z) \cdot (y \cdot z)) \land (y \cdot x) \leq ((x \cdot z) \cdot (y \cdot z))) \),

(pKU-2): \( (\forall x \in A)((0 \cdot x = x) \land (0 \cdot x = x)) \),

(pKU-3): \( (\forall x \in A)(x \leq 0) \),

(pKU-4): \( (\forall x, y \in A)((x \leq y \land y \leq x) \implies x = y) \), and

(pKU-5): \( (\forall x, y \in A)((x \leq y \implies x \cdot y = 0) \land (x \leq y \implies x \cdot y = 0)) \).

Remark 1. We emphasize that in pseudo-KU algebra the relation of the order is determined inversely with respect to the definition of the order in the KU-algebra.

Lemma 5. If \( \mathfrak{A} \) is a pseudo-KU algebra, then

(pKU-6) \( (\forall x \in A)((x \cdot x = 0) \land (x \cdot x = 0)) \).

Proof. If we put \( x = 0 \), \( y = 0 \), and \( z = x \) in the formula (pKU-1), we get

\[
(0 \cdot 0) \cdot ((0 \cdot x) \cdot (0 \cdot x)) = 0 \land (0 \cdot 0) \cdot ((0 \cdot x) \cdot (0 \cdot x)) = 0.
\]

From where we get

\[
x \cdot x = 0 \land x \cdot x = 0
\]

with respect to (pKU-2). \( \square \)

Proposition 1. If \( \mathfrak{A} \) is a pseudo-KU algebra, then

(11) \( (\forall x, y, z \in A)(x \leq y \implies ((y \cdot z \leq x \cdot z) \land (y \cdot z \leq x \cdot z))) \) and

(12) \( (\forall x, y, z \in A)(x \leq y \implies ((z \cdot x \leq z \cdot y) \land (z \cdot x \leq z \cdot y))) \).

Proof. Let \( x, y, z \in A \) such that \( x \leq y \). Then \( x \cdot y = 0 = x \cdot y \). If we put \( x = y \) and \( y = x \) in (pKU-1), we get

\[
0 = (x \cdot y) \cdot ((y \cdot z) \cdot (x \cdot z)) = 0 \cdot ((y \cdot z) \cdot (x \cdot z)) = (y \cdot z) \cdot (x \cdot z).
\]

So, we have \( y \cdot z \leq x \cdot z \). Similarly, we have

\[
0 = (x \cdot y) \cdot ((y \cdot z) \cdot (x \cdot z)) = 0 \cdot ((y \cdot z) \cdot (z \cdot x)) = (y \cdot z) \cdot (z \cdot x)\]

and \( y \cdot z \leq z \cdot x \).

On the other hand, if we put \( z = y \) and \( y = z \) in (pKU-1), we have

\[
0 = (z \cdot x) \cdot ((x \cdot y) \cdot (z \cdot y)) = (z \cdot x) \cdot ((0 \cdot (z \cdot y)) = (z \cdot x) \cdot (z \cdot y).
\]

This means \( z \cdot x \leq z \cdot y \). It can be similarly proved that it is \( z \cdot x \leq z \cdot y \). \( \square \)

In 2011, Mostafa, Naby and Yousef proved Lemma 2.2 in [18]. In the following Proposition, we show that analogous equality is also valid in pseudo-KU algebras.

Proposition 2. In pseudo-KU algebra \( \mathfrak{A} \), then

(pKU) \( (\forall x, y, z \in A)(x \cdot (y \cdot z) = y \cdot (x \cdot z) \land x \cdot (y \cdot z) = y \cdot (x \cdot z)) \).
is valid formula.

**Proof.** If we put \( y = 0 \) in (pKU-1), we have

\[
0 \cdot x \leq (x \cdot z) \ast (0 \cdot z).
\]

Then, we have \( x \leq (x \cdot z) \ast z \). From here it follows

\[
((x \cdot z) \ast z) \cdot (y \ast z) \leq x \cdot (y \ast z)
\]

by (11). On the other hand, if we put \( x = z \cdot z \) in (pKU-1), we get

\[
y \ast (x \cdot z) \leq ((x \cdot z) \ast z) \cdot (y \ast z) \leq x \cdot (y \ast z).
\]

Since the variables \( x, y, z \in A \) are free variables, if we put \( x = y \) and \( y = x \), we get an inverse inequality. From here it follows (pKU) by (pKU-4).

The other equality can be proved in an analogous way. □

4. Correlation of pseudo-KU algebras with other types of pseudo algebras

The notion of pseudo-UP algebra as a generalization of the concept of UP-algebras was introduced and analyzed in [23].

**Definition 5. ([23])** A pseudo-UP algebra is a structure \( \mathfrak{A} = ((A, \leq, \cdot, *, 0)) \), where \( \leq \) is a binary relation on a set \( A \), \( \cdot, * \) and \( \cdot, * \) are internal binary operations on \( A \) and \( 0 \) is an element of \( A \), verifying the following axioms:

- (pUP-1) \( (\forall x, y, z \in A)(y \cdot z \leq (x \cdot y) \ast (x \cdot z) \land y \ast z \leq (x \ast y) \ast (x \ast z)); \)
- (pUP-4) \( (\forall x, y \in A)((x \leq y \land y \leq x) \implies x = y); \)
- (pUP-5) \( (\forall x, y \in A)((y \ast 0) \ast x = x \land (y \ast 0) \ast x = x) \) and
- (pUP-6) \( (\forall x, y \in A)((x \leq y \iff x \cdot y = 0) \land (x \leq y \iff x \ast y = 0)). \)

The following theorem is an important result of pseudo-KU algebras for study in the connections between pseudo-UP algebras and pseudo-KU algebras.

**Theorem 1.** Any pseudo-KU algebra is a pseudo-UP algebra.

**Proof.** It only needs to show (pUP-1). By Proposition 2, we have that any pseudo-KU algebra satisfies (pUP-1). □

Pseudo-BE algebra is defined by the follows:

**Definition 6. ([3])** An algebra \( A = (A, \cdot, *, 1) \) of type \((2,2,0)\) is called a pseudo BE-algebra if satisfies in the following axioms:

- (pBE-1) \( (\forall x \in A)(x \cdot x = 1 \land x \ast x = 1); \)
- (pBE-2) \( (\forall x \in A)(x = 1 \land x \ast 1 = 1); \)
- (pBE-3) \( (\forall x \in A)(1 \cdot x = x \land 1 \ast x = x); \)
- (pBE-4) \( (\forall x, y, z \in A)(x \cdot (y \ast z) = y \ast (x \cdot z)); \) and
- (pBE-5) \( (\forall x, y \in A)(x \cdot y = 1 \iff x \ast y = 1). \)

If we replace 1 with 0 in (BE-1), (BE-2), (BE-3) and (BE-5) and prove that the formula (pBE-4) is a valid formula in a pseudo-KU algebra \( A \), we have proved that every pseudo-KU algebra \( A \) is a pseudo-BE algebra.

**Theorem 2.** Any pseudo-KU algebra is a pseudo-BE algebra.

**Proof.** It is sufficient to prove that the formula (pBE-4) is a valid formula in any pseudo-KU algebra. If we put \( y = 0 \) in the left-hand side of the formula (pKU-1), we get \( 0 \cdot x \leq ((x \cdot z) \ast (0 \cdot z)). \) It means \( x \leq (x \cdot z) \ast z \). From here follows

\[
((x \cdot z) \ast z) \cdot (y \ast z) \leq x \cdot (y \ast z),
\]
by the left part of formula (11). On the other hand, if we put \( x = x \cdot z \) in the right-hand side of the formula (pKU-1), we get

\[
y \ast (x \cdot z) \leq ((x \cdot z) \ast z) \cdot (y \ast z).
\]

Which together with the previous inequality gives

\[
y \ast (x \cdot z) \leq x \cdot (y \ast z).
\]

From this inequality by substituting the variables \( x \) and \( y \), we obtain the necessary reverse inequality

\[
x \cdot (y \ast z) \leq y \ast (x \cdot z).
\]

From these two inequalities follows the validity of the formula (pBE-4) in any pseudo-KU algebra by the axiom (pKU-4).

Since the formula previously proven is important below, we point it out in particular.

**Proposition 3.** In any pseudo-KU algebra \( \mathfrak{A} \),

(pKU-7) \( (\forall x, y, z \in A) (x \cdot (y \ast z) = y \ast (x \cdot z)) \)

is a valid formula.

**5. Some substructures in pseudo-KU algebras**

**5.1. Concept of pseudo-subalgebras**

**Definition 7.** A nonempty subset \( S \) of a pseudo-KU algebra \( A \) is a pseudo-subalgebra in \( A \) if

\[
(\forall x, y \in A)((x \in S \land y \in S) \implies (x \cdot y \in S \land x \ast y \in S)).
\]

holds.

Putting \( y = x \) in the previous definition, it immediately follows:

**Lemma 6.** If \( S \) is a pseudo-subalgebra of a pseudo-KU algebra \( \mathfrak{A} \), then \( 0 \in S \).

**Proof.** Let \( S \) be a pseudo-subalgebra of a pseudo-KU algebra \( \mathfrak{A} \). It means that \( S \) is a nonempty subset of \( A \). Then there exists an element \( y \in S \). Thus \( 0 = y \cdot y = y \ast y \in S \) by Definition 7.

It is clear that subsets \( \{0\} \) and \( A \) are pseudo-subalgebras of a pseudo-KU algebras \( \mathfrak{A} \). So, the family \( \mathcal{S}(A) \) of all pseudo-subalgebras of a pseudo-KU algebra \( \mathfrak{A} \) is not empty. Without major difficulties, the following theorem can be proved.

**Theorem 3.** The family \( \mathcal{S}(A) \) of all pseudo-subalgebras of a pseudo-KU algebra \( \mathfrak{A} \) forms a complete lattice.

**5.2. Concept of pseudo-ideals**

**Definition 8.** The subset \( J \) is said to be a pseudo-ideal of a pseudo-KU algebra \( A \) if it satisfies the following conditions:

(pJ1) \( 0 \in J \),

(pJ3a) \( (\forall x, y \in A)((x \cdot y \in J \land x \in J) \implies y \in J) \) and

(pJ3b) \( (\forall x, y \in A)((x \ast y \in J \land x \in J) \implies y \in J) \).

**Proposition 4.** Let \( J \) be a nonempty subset of a pseudo-KU algebra \( \mathfrak{A} \). Then the condition (pJ3a) is equivalent to the condition:

(pJ4a) \( (\forall x, y, z \in A)((x \ast (y \cdot z) \in J \land y \in J) \implies x \ast z \in J) \).

**Proof.** Putting \( x = y \) and \( y = x \ast z \) in the condition (pJ3a), it immediately follows

\[
(\forall x, y, z \in A)((y \ast (x \cdot z) \in J \land y \in J) \implies x \ast z \in J).
\]
Thus

\((\forall x, y, z \in A)((y \ast (x \cdot z) \in J \land y \in J) \implies x \ast z \in J)\)

by (pKU-7).

Conversely, let (pJ4a) be. Let us choose \(x = 0, y = x\) and \(z = y\) in (pJ4a). We get \((0 \ast (x \cdot y) \in J \land x \in J) \implies 0 \ast y \in J\). Thus (pJ3a) by (pKU-2).

**Corollary 4.** Let \(J\) be a pseudo-ideal in a pseudo-KU-algebra \(A\). Then

\((13) (\forall x, y \in A)(x \in J \implies x \ast y \in J)\).

**Proof.** Putting \(z = y\) in (pJ4a), with respect to (pKU-6), (pKU-3) and (pJ1), we obtain (13).

**Proposition 5.** Let \(J\) be a nonempty subset of a pseudo-KU algebra \(A\). Then the condition (pJ3b) is equivalent to the condition

\((pJ4b) (\forall x, y, z \in A)((x \cdot (y \ast z) \in J \land y \in J) \implies x \cdot z)\).

**Proof.** If we put \(x = y\) and \(y = x \cdot z\) in (pJ3b), we get

\((y \ast (x \cdot z) \in J \land y \in J) \implies x \cdot z \in J)\).

Hence

\((x \cdot (x \cdot z) \in J \land y \in J) \implies x \cdot z \in J)\)

by (pKU-7).

Conversely, if we put \(x = 0, y = x\) and \(z = y\) in (pJ4b), we get

\((0 \cdot (x \cdot y) \in J \land x \in J) \implies 0 \cdot y \in J)\).

Thus (pJ3b) with respect to (pKU-2).

**Corollary 5.** Let \(J\) be a pseudo-ideal in a pseudo-KU-algebra \(A\). Then

\((14) (\forall x, y \in A)(y \in J \implies x \cdot y \in J)\).

**Proof.** Putting \(z = y\) in (pJ4b), with respect to (pKU-6), (pKU-3) and (pJ1), we obtain (14).

The following important statement describes the connection between conditions (pJ3a) and (pJ3b).

**Proposition 6.** Let \(J\) be a pseudo-ideal of a pseudo-KU algebra \(A\). Then

\((pJ3a) \iff (pJ3b)\).

**Proof.** \((pJ3a) \iff (pJ3b)\). Suppose (pJ3a) holds and let \(x \ast y \in J\) and \(x \in J\). How obvious it is that the following

\(x \ast ((x \cdot y) \ast y) = 0 \iff x \cdot ((x \cdot y) \ast x) = 0 \iff (x \cdot y) \ast (x \cdot y) = 0\)

is valid, we have

\((x \in J \land x \cdot ((x \ast u) \cdot y) = 0 \in J) \implies (x \ast y) \cdot y \in J)\).

Now

\((x \ast y \in J \land (x \ast y) \cdot y \in J) \implies y \in J)\).

We have proved that (pJ3b) is a valid implication.

\((pJ3b) \implies (pJ3a)\). Let (pJ3b) be a valid formula and let \(x, y \in A\) be such that \(x \in J\) and \(x \cdot y \in J\). As above, from

\(x \ast ((x \cdot y) \ast y) = 0 \iff x \cdot ((x \cdot y) \ast y) = 0 \iff (x \cdot y) \ast (x \cdot y) = 0\)

it follows

\((x \in J \land x \ast ((x \cdot y) \ast y) = 0 \in J) \implies (x \cdot y) \ast y \in J)\).

Now, \(x \cdot y \in J\) and \((x \cdot y) \ast y\) it follows \(y \in J\). This proves the validity of the formula (pJ3a).
Proposition 7. Any pseudo-ideal in a pseudo-KU-algebra $\mathfrak{A}$ is a pseudo-subalgebra in $\mathfrak{A}$.

Proof. The proof of this proposition follows from (13) and (14).

Theorem 6. The family $\mathfrak{J}(A)$ of all pseudo-ideals in a pseudo-KU algebra $\mathfrak{A}$ forms a complete lattice and $\mathfrak{J}(A) \subseteq \mathfrak{S}(A)$ holds.

Proof. Let $\{I_i\}_{i \in I}$ be a family of pseudo-ideals in a pseudo-KU algebra $\mathfrak{A}$. Clearly $0 \in \bigcap_{i \in I} I_i$ is valid. Let $x, y \in A$ be elements such that $x \cdot y \in \bigcap_{i \in I} I_i$, $x \cdot y \in \bigcap_{i \in I} I_i$, and $x \in \bigcap_{i \in I} I_i$. Then $x \cdot y \in I_i$, $x \cdot y \in I_i$, and $x \in F_i$ for any $i \in I$. Thus $y \in I_i$ because $I_i$ is a pseudo-ideal in $\mathfrak{A}$ and $x \in \bigcap_{i \in I} I_i$. So, $\bigcap_{i \in I} I_i$ is a pseudo-ideal in $\mathfrak{A}$.

If $\mathfrak{X}$ is the family of all pseudo-ideals of $\mathfrak{A}$ that contain the union $\bigcup_{i \in I} I_i$, then $\mathfrak{X}$ is also a pseudo-ideal in $\mathfrak{A}$ that contains $\bigcup_{i \in I} I_i$ by previous evidence.

To round out this subsection we need the following lemma.

Lemma 7. Let $J$ be a pseudo-ideal in a pseudo-KU algebra $\mathfrak{A}$. Then

$$(15) (\forall x, y \in A)((x \leq y \land x \in J) \implies y \in J).$$

Proof. The proof of this proposition follows from (pJ3a) (or (pJ3b)) with respect to (pKU-6) and (pJ1).

Theorem 7. Let $J$ be a subset of a pseudo-KLI algebra $\mathfrak{A}$ such that $0 \in J$. Then, $J$ is a pseudo-ideal in $\mathfrak{A}$ if and only if the following holds

$$(pJ5) (\forall x, y, z \in A)((x \in A \land y \in A \land x \leq y \cdot z) \implies z \in J).$$

Proof. Let $J$ be a pseudo-ideal in $\mathfrak{A}$ and let $x, y, z \in A$ such that $x \in J$, $y \in J$ and $x \leq y \cdot z$. Then $x \cdot (y \cdot z) = 0 \in J$. Thus $y \cdot z \in J$ by (pJ3a) and again, from here and $y \in J$ it follows $z \in J$. So, we have shown that (pJ5) is a valid formula.

Opposite, suppose that (pJ5) is a valid in $\mathfrak{A}$. Let us show that $J$ is a pseudo-ideal and $\mathfrak{A}$. Let $x, y \in A$ be such that $x \in J$ and $y \in J$. Then $x \cdot y \in J$ by Proposition 6. On the other hand, from $x \cdot ((x \cdot y) \cdot y) = 0$, i.e. from $x \leq (x \cdot y) \cdot y$ it follows $y \in J$ by hypothesis. So, the set $J$ is a pseudo-ideal in $\mathfrak{A}$.

For a relation on the set $A$ we say that it is a quasi-order relation on $A$ if it is reflexive and transitive. It is easy to prove that if $\sigma$ is a quasi-order relation on $A$, then the relation $\sigma \cap \sigma^{-1}$ is an equivalence on $A$.

Theorem 8. Let $J$ be a pseudo-ideal in a pseudo-KLI algebra $\mathfrak{A}$. Then the relation $' \preceq '$, defined by

$$(\forall x, y \in A)(x \preceq y \iff x \cdot y \in J),$$

is a quasi-order in the set $A$ left compatible and right reverse compatible with the internal operations in $\mathfrak{A}$.

Proof. Since $x \cdot x = 0 \in J$ is valid in $\mathfrak{A}$ for any $x \in A$, it is clear that $' \preceq '$ is a reflexive relation in the set $A$.

Let $x, y, z \in A$ be arbitrary elements such that $x \leq y$ and $y \leq z$. This means $x \cdot y \in J$ and $y \cdot z \in J$. From inequality (pKU-1) in the form $x \cdot y \leq (y \cdot z) \cdot (x \cdot z)$ and $x \cdot y \in J$ it follows $(y \cdot z) \cdot (x \cdot z) \in J$ according to (15). From here and from $y \cdot z \in J$ it follows $x \cdot z \in J$ according to (pJ3a). Hence, the relation $' \preceq '$ is transitive. So, this relation is a quasi-order in $A$.

Let $x, y, z \in A$ be such $x \preceq y$. Then $x \cdot y \in J$ and $x \cdot y \in J$.

(i) If we put $x = y$ and $y = x$ in the left part of the formula (pKU-1), we get $x \cdot y \leq (y \cdot z) \cdot (x \cdot z)$. Now, from here and from $x \cdot y \in J$ it follows $y \cdot z \cdot (x \cdot z) \in J$ by (15). Thus $y \cdot z \cdot (x \cdot z) \in J$ by Proposition 6. Finally, we have $y \cdot z \preceq x \cdot z$. So, the relation $' \preceq '$ is reverse right compatible with the internal operation $' \cdot '$ in $\mathfrak{A}$.

(ii) If we put $x = y$ and $y = x$ in the right part of the formula (pKU-1), we get $x \cdot y \leq (y \cdot z) \cdot (x \cdot z)$. Then $(y \cdot z) \cdot (x \cdot z) \in J$ by (15). Thus $y \cdot z \preceq x \cdot z$. Therefore, the relation $' \preceq '$ is reverse right compatible with the internal operation $' \cdot '$ in $\mathfrak{A}$.

(iii) Let us put $y = z$ and $z = y$ in the left part of the formula (pKU-1). We get $(z \cdot x) \cdot ((x \cdot y) \cdot (z \cdot y)) = 0 \in J$. From here and from $x \cdot y \in J$ it follows $(z \cdot x) \cdot (z \cdot y) \in J$ by (pJ4a). Thus $z \cdot x \preceq z \cdot y$. So, the relation $' \preceq '$ is left compatible with the operation $' \cdot '$.
(iv) Let us put \( y = z \) and \( z = y \) in the right part of the formula (pKU-1). We get \( (z \ast x) \cdot ((x \ast y) \cdot (z \ast y)) = 0 \in J \). From here and from \( x \ast y \in J \) it follows \( (z \ast x) \cdot (z \ast y) \in J \) by (pK4b). Thus \( z \ast x \preceq z \ast y \). So, the relation \( \preceq \) is left compatible with the operation \( \ast \). \( \square \)

5.3. Concept of pseudo-filters

**Definition 9.** A non-empty subset \( F \) of a pseudo-KU algebra \( \mathfrak{A} \) is called a pseudo-filter of \( A \) if it satisfies in the following axioms:

\[
(pF1) \ 0 \in F; \\
(pF3) \ (\forall x,y \in A)((x \cdot y \in F \land x \ast y \in F \land y \in F) \implies x \in F).
\]

\( \{0\} \) and \( A \) are pseudo-filters of \( \mathfrak{A} \). So, the family \( \mathfrak{F}(A) \) of all pseudo-filters in a pseudo-KU algebra \( \mathfrak{A} \) is not empty.

It is obviously the following is valid

**Lemma 8.** Let \( F \) be a pseudo-filter in a pseudo-KU algebra \( \mathfrak{A} \). Then

\[
(16) \ (\forall x,y \in A)((x \leq y \land y \in F) \implies x \in F).
\]

**Theorem 9.** The family \( \mathfrak{F}(A) \) of all pseudo-ideals in a pseudo-KU algebra \( \mathfrak{A} \) forms a complete lattice.

**Proof.** Let \( \{F_i\}_{i \in I} \) be a family of pseudo-filters in a pseudo-KU algebra \( \mathfrak{A} \). Clearly \( 0 \in \bigcap_{i \in I} F_i \) is valid. Let \( x,y \in A \) be elements such that \( x \cdot y \in \bigcap_{i \in I} F_i, x \ast y \in \bigcap_{i \in I} F_i \) and \( y \in \bigcap_{i \in I} F_i \). Then \( x \cdot y \in F_i, x \ast y \in F_i \) and \( y \in F_i \) for any \( i \in I \). Thus \( x \in F_i \) because \( F_i \) is a pseudo-filter in \( \mathfrak{A} \) and \( x \in \bigcap_{i \in I} F_i \). So, \( \bigcap_{i \in I} F_i \) is a pseudo-filter in \( \mathfrak{A} \).

If \( \mathfrak{X} \) is the family of all pseudo-filters of \( \mathfrak{A} \) that contain the union \( \bigcup_{i \in I} F_i \), then \( \bigcap \mathfrak{X} \) is also a pseudo-filter in \( \mathfrak{A} \) that contains \( \bigcup_{i \in I} F_i \) by previous evidence.

If we put \( \bigcap_{i \in I} F_i = \bigcap_{i \in I} F_i \) and \( \bigcup_{i \in I} F_i = \bigcap \mathfrak{X} \), then \( \mathfrak{F}(A), \bigcap, \bigcup \) is a complete lattice. \( \square \)

6. Concept of pseudo-homomorphisms

**Definition 10.** \((A, \preceq_A, \cdot, \ast_A, 0_A) \) and \((B, \preceq_B, \cdot, \ast_B, 0_B) \) be pseudo-KU algebras. A mapping \( f : A \rightarrow B \) of pseudo-KU algebras is called a pseudo-homomorphism if

\[
(\forall x,y \in A)((x \cdot_A y) =_B f(x) \cdot_B f(y) \land (x \ast_A y) =_B f(x) \ast_B f(y)).
\]

**Remark 2.** Note that if \( f : A \rightarrow B \) is a pseudo homomorphism, then \( f(0_A) = 0_B \). Indeed, if we chose \( y = x \), from the previous formula we immediately get \( f(0_A) =_B 0_B \) with respect (pKU-6).

From here it immediately follows:

**Lemma 9.** Any pseudo-homomorphism between pseudo-KU algebras is isotone mapping.

**Proof.** Let \( f : A \rightarrow B \) be a pseudo-homomorphism between pseudo-KU algebras and let \( x,y \in A \) be such \( x \preceq_A y \). Then \( x \cdot_A y =_A 0_A \). Thus \( 0_B =_B f(x \cdot_A y) =_B f(x) \cdot_B f(y) \). This means \( f(x) \preceq_B f(y) \). \( \square \)

**Lemma 10.** Let \( f : A \rightarrow B \) be a pseudo-homomorphism between pseudo-KU algebras. Then the set \( \text{Ker}(f) =_A \{x \in A : f(x) =_B 0_B\} \) is a pseudo-ideal in \( \mathfrak{A} \).

**Proof.** It is obvious \( 0_A \in \text{Ker}(f) \).

Let \( x,y \in A \) be such \( x \cdot_A y \in \text{Ker}(f) \) and \( x \in \text{Ker}(f) \). Then \( f(x) =_B 0_B \) and \( 0 =_B f(x \cdot_A y) =_B f(x) \cdot_B f(y) =_B 0_B \cdot_B f(y) = f(y) \). Thus \( y \in \text{Ker}(f) \).

The implication of \( x \cdot_A y \in \text{Ker}(f) \land x \in \text{Ker}(f) \implies y \in \text{Ker}(f) \) can be proved by analogy with the previous proof. \( \square \)

The following statement is easy to prove:

**Lemma 11.** If \( f : A \rightarrow B \) is a pseudo-homomorphism between pseudo-KU algebras, then \( f(A) \) is a pseudo-subalgebra in \( B \).
Proposition 8. Let \( f : A \rightarrow B \) be a pseudo homomorphism between pseudo-KU algebras \( \mathfrak{A} \) and \( \mathfrak{B} \).

(i) If \( K \) is a pseudo-ideal in \( \mathfrak{B} \), then \( f^{-1}(K) \) is a pseudo-ideal in \( \mathfrak{A} \).

(ii) If \( G \) is a pseudo-filter in \( \mathfrak{B} \), then \( f^{-1}(G) \) is a pseudo-filter in \( \mathfrak{A} \).

Proof. (i) Assume that \( K \) is a pseudo-filter of \( \mathfrak{B} \). Obviously \( 0_A \in f^{-1}(K) \). Let \( x, y \in A \) be such \( x \cdot y \in f^{-1}(K) \) and \( x \in f^{-1}(K) \). Then \( f(x) \cdot_B f(y) = B f(x \cdot_A y) \in K \) and \( f(x) \in K \). It follows that \( f(y) \in K \) by (p3a) since \( K \) is a pseudo-ideal in \( \mathfrak{B} \). Therefore, \( y \in f^{-1}(K) \). Thus, the set \( f^{-1}(K) \) satisfies the implication (p3a). That the set \( f^{-1}(K) \) satisfies the implication (p3b) can be proved in an analogous way. Therefore, the set \( f^{-1}(K) \) is a pseudo-ideal in \( \mathfrak{A} \).

(ii) It is obvious \( 0_A \in f^{-1}(G) \) again. Let \( x, y \in A \) be elements such that \( x \cdot_A y \in f^{-1}(G), x \ast_A y \in f^{-1}(G) \) and \( y \in f^{-1}(G) \). Then \( f(x) \cdot_B f(y) = B f(x \cdot_A y) \in G, f(x) \ast_B f(y) = B f(x \ast_A y) \in G \) and \( f(y) \in G \). Thus \( f(x) \in G \) because \( G \) is a pseudo-filter in \( \mathfrak{B} \). This means \( x \in f^{-1}(G) \). So, the set \( f^{-1}(G) \) is a pseudo-filter in \( \mathfrak{A} \).

In the following definition, we will introduce the concept of congruence on pseudo-KU algebras. Since we have two unitary operations on this algebra, it is possible to determine three different types of congruences.

Definition 11. Let \( \mathfrak{A} = ((A, \leq), \cdot, \ast, 0) \) be a pseudo-KU algebra.

For the equivalence relation \( q \) on the set \( A \) we say that it is a congruence of type ‘-’ on \( \mathfrak{A} \) if it compatible with the operations ‘-’ in \( \mathfrak{A} \) in the following sense

\[
(17) \ (\forall x, y, z \in A) ((x, y) \in q \Rightarrow (x \cdot z, y \cdot z) \in q \land (z \cdot x, z \cdot y) \in q)).
\]

For the equivalence relation \( q \) on the set \( A \) we say that it is a congruence of type ‘\ast’ on \( \mathfrak{A} \) if it compatible with the operations ‘\ast’ in \( \mathfrak{A} \) in the following sense

\[
(18) \ (\forall x, y, z \in A) ((x, y) \in q \Rightarrow (x \ast z, y \ast z) \in q \land (z \ast x, z \ast y) \in q)).
\]

For the equivalence relation \( q \) on the set \( A \) we say that it is a congruence of common type on \( \mathfrak{A} \) if it is compatible with both operations in \( \mathfrak{A} \).

Lemma 12. Let \( q \) be a relation on a pseudo-KU algebra \( \mathfrak{A} \). Then:

(i) The condition (17) is equivalent to the condition

\[
(17a) \ (\forall x, y, u, v \in A) (((x, y) \in q \land (u, v) \in q) \Rightarrow (x \cdot u, y \cdot v) \in q).
\]

(ii) The condition (18) is equivalent to the condition

\[
(18a) \ (\forall x, y, u, v \in A) (((x, y) \in q \land (u, v) \in q) \Rightarrow (x \ast u, y \ast v) \in q).
\]

Proof. (17a) \( \Rightarrow \) (17). If we choose \( v = z \) in (17a), we get the implication \( (x, y) \in q \Rightarrow (x \cdot z, y \cdot z) \in q \). On the other hand, if we put \( x = y = z, u = x \) and \( v = y \) in (17a), we get the implication \( (x, y) \in q \Rightarrow (z \cdot x, z \cdot y) \).

(17) \( \Rightarrow \) (17a). Suppose (17) and let \( x, y, u, v \in A \) such that \( (x, y) \in q \) and \( (u, v) \in q \). Thus \( (x \cdot u, x \cdot v) \in q \) and \( (x \cdot v, y \cdot v) \in q \) by (16). 

Equivalence (18) \( \Leftrightarrow \) (18a) can be proved analogous to the previous proof.

Let \( f : A \rightarrow B \) be a pseudo homomorphism between pseudo-KU algebras. By direct check without difficulty, it can be proved that the relation \( q_f \), defined by

\[
(\forall x, y \in A)((z, y) \in q_f \Leftrightarrow f(x) =_B f(y)),
\]

is a congruence (all three types) on \( \mathfrak{A} \).

Theorem 10. The relation \( q_f \) is a congruence of type ‘-’, ‘-’ (type ‘\ast’, common type) on the pseudo-KU algebra \( \mathfrak{A} \).

Proof. We will only demonstrate the proof that \( q_f \) is a congruence of type ‘-’ on \( \mathfrak{A} \) because the evidence that \( q_f \) is a congruence of type ‘\ast’ can obtain by analogy with the previous one, and the proof of common type is obtained by combining this two evidences.

Clearly, \( q_f \) is an equivalence relation on the set \( A \). It remains to verify that (16) is a valid formula in \( \mathfrak{A} \). Let \( x, y, u, v \in A \) be such that \( (x, y) \in q_f \) and \( (u, v) \in q_f \). Then \( f(x) =_B f(y) \) and \( f(u) =_B f(v) \). Thus

\[
f(x \cdot_A u) =_B f(x) \cdot_B f(u) =_B f(y) \cdot_B f(v) =_B f(y \cdot_A u).
\]
Hence, \((x \cdot_A u, y \cdot_A v) \in q_f\). We proved that (17a) is a valid formula. So \(q_f\) is a congruence of type \(\prime \cdot \) on \(\mathfrak{A}\). □

**Theorem 11.** Let \(J\) be a pseudo-ideal in a pseudo-KU algebra \(\mathfrak{A}\). Then the relation \(q_J\), defined by \(q_J = \equiv \cap \equiv^{-1}\), is a congruence of common type in \(\mathfrak{A}\).

**Proof.** The relation \(q\) is an equivalence relation on the set \(A\). It is sufficient to prove that \(q\) is compatible with operations in \(\mathfrak{A}\). Since the relation \(\equiv\) is left compatible and right reverse compatible with the internal operations in \(\mathfrak{A}\), by Theorem 8, it is clear that the relation \(q_J\) is a congruence on \(\mathfrak{A}\). □

For a congruence \(q\) on a pseudo-KU algebra \(\mathfrak{A}\) we denote \(\mathfrak{A}/q = \{\mathcal{g} \in A: (x,y) \in q\} = [x]\). Let’s define \(\prime \cdot \) and \(\prime \ast \) in \(\mathfrak{A}/q\) on this way

\[ (\forall x,y \in A)(([x] \cdot y) = [x \cdot y] ) \quad \text{and} \quad (\forall x,y \in A)(([x] \ast y) = [x \ast y]). \]

Without much difficulty it can be verified that the functions \(\prime \cdot \) and \(\prime \ast \), defined in this way, are well-defined internal binary operations in \(\mathfrak{A}/q\). Also, one can check that the set \(\mathfrak{A}/q\) with the operations \(\prime \cdot \) and \(\prime \ast \), determined as above, satisfies all the axioms of Definition 4 except the axiom (pKU-4). However, if we take the relation \(q_J\), defined by an pseudo-ideal \(J\) of a pseudo-KU algebra \(\mathfrak{A}\), then we have

**Theorem 12.** Let \(J\) be a pseudo-ideal in a pseudo-KU algebra \(\mathfrak{A}\). Then the structure \(((\mathfrak{A}/q, \equiv), \prime \cdot, \prime \ast, [0])\), where \(\equiv\) is defined by

\[ (\forall x,y \in A)(([x] \equiv [y]) \iff x \equiv y), \]

is a pseudo-KU algebra, too.

**Proof.** According to the commentary preceding this theorem, to prove this theorem it suffices to show that the structure \(((\mathfrak{A}/q, \equiv), \prime \cdot, \prime \ast, [0])\) satisfies the axiom (pKU-4).

Let \(x, y \in A\) be such \([x] \equiv [y] \) and \([y] \equiv [x] \). Then \(x \equiv y\) and \(y \equiv x\) by definition. Thus \((x,y) \in q_J\) and \([x] = [y]\). □

Let \(f : A \rightarrow B\) be pseudo-homomorphism between pseudo-KU algebras \(((A, \leq_A), \cdot_A, *_A, 0_A)\) and \(((B, \leq_B), \cdot_B, *_B, 0_B)\). Then the set \(f(A)\) is a pseudo-subalgebra of \(\mathfrak{B}\) by and the set \(f = \text{Ker}(f)\) is a pseudo-ideal in \(\mathfrak{B}\) by Lemma 10 and the relation \(q_f\) is a congruence on \(\mathfrak{A}\) by Theorem 10. If \((x,y) \in q_f\) holds soe some \(x, y \in A\), we have \(f(x) = f(y)\). Thus \(f(x \cdot_A y) = f(x) \cdot_B f(y) = f(x) \cdot_B f(x) = 0_B\) i.e. \(x \cdot_A y \in J\). Analogous to the previous one may be shown that \(y \cdot_A x \in J\) holds. Thus, \((x,y) \in q_f \implies (x,y) \in q_J\) is valid.

We end this section with the following theorem. Since this theorem can be proven by direct verification, we will omit evidence for it.

**Theorem 13.** Let \(f : A \rightarrow B\) be pseudo-homomorphism between pseudo-KU algebras \(((A, \leq_A), \cdot_A, *_A, 0_A)\) and \(((B, \leq_B), \cdot_B, *_B, 0_B)\). Then there exists the unique epimorphism \(\pi : A \rightarrow A/q_f\), defined by \(\pi(x) = [x]\) for any \(x \in A\), and the unique monomorphism \(g : A/q_f \rightarrow B\), defined by \(g([x]) = f(x)\) for any \(x \in A\) such that \(f = g \circ \pi\).

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**References**


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