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Valuations and their generalizations for UP-algebras

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Abstract: In this paper, we introduce the notions of a weak pseudo-valuation, a 0-weak pseudo-valuation, a weak valuation, a near pseudo-valuation, a near valuation, a pseudo-valuation, and a valuation and induce a pseudo-metric without triangle inequality, a quasi pseudo-metric, a pseudo-metric, and a metric by some these mappings on a UP-algebra. We also prove that the binary operation defined on a UP-algebra is uniformly continuous under the induced metric by a valuation in some conditions.

Keywords: UP-algebra, weak pseudo-valuation, 0-weak pseudo-valuation, weak valuation, near pseudo-valuation, near valuation, pseudo-valuation, valuation.

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1. Introduction and preliminaries

The fundamental concept of pseudo-valuation was first introduced by Busneag [1] in 1996. He defined a pseudo-valuation on a Hilbert algebra and proved that every pseudo-valuation induces a pseudo metric on a Hilbert algebra. In 2003, Busneag [2] provided several theorems on extensions of pseudo-valuations. In 2007, Busneag [3] introduced the notion of a pseudo-valuation (valuations) on a residuated lattice and proved some theorems of extension for these (using the model of a Hilbert algebra).

In 2010, Doh and Kang [4] introduced the notion of a pseudo-valuation on a BCK/BCI-algebra and studied results based on a pseudo-valuation. Ghorbani [5] defined a congruence relation and gave a quotient structure of a BCI-algebra based on a pseudo-valuation. In 2011, Doh and Kang [6] introduced the notion of a commutative pseudo valuation on a BCK-algebra. Jun *et al.*, [7] introduced the notion of a positive implicative pseudo-valuation on a BCK-algebra and investigated its characterizations. In 2012, Jun *et al.*, [8] introduced the notions of a pseudo-valuation and a valuation on a BCC-algebra. In 2013, Zhan and Jun [9] introduced the notions of a pseudo-valuation and an implicative pseudo-valuation on a R_0-algebra. Lee [10] introduced the notions of a pseudo-valuation and a valuation, and a pseudo-metric is induced by a pseudo-valuation on a BE-algebra. In 2015, Song *et al.*, [11] introduced the notions of a quasi-valuation map based on a subalgebra and an ideal on a BCK/BCI-algebra. In 2017, Yang and Xin [12] introduced the notions of a pseudo pre-valuation, an implicative pseudo pre-valuation, a positive implicative pseudo pre-valuation, and a strong pseudo pre-valuation, and investigated some characterizations of an EQ-algebra. In 2018, Mehrshada and Kouhestanib [13] studied some properties of a pseudo-valuation and their induced metrics on a BCK-algebra and showed that there are many pseudo-valuations on a BCK-algebra.

In 2019, Koam *et al.*, [14] introduced the notion of a pseudo-metric which induced from a pseudo-valuation on a KU-algebra. Ali *et al.*, [15] introduced the notion of a pseudo-valuation and investigated the relationship between a pseudo-valuation and an ideal of a JU-algebra. Romano [16] introduced the notion of a pseudo-valuation on a UP-algebra and analyzed the relationship of these mappings with UP-substructures.

In this paper, we introduce the notions of a weak pseudo-valuation, a 0-weak pseudo-valuation, a weak valuation, a near pseudo-valuation, a near valuation, a pseudo-valuation, and a valuation and induce a pseudo-metric without triangle inequality, a quasi pseudo-metric, a pseudo-metric, and a metric by some these mappings on a UP-algebra. We also prove that the binary operation defined on a UP-algebra is uniformly continuous under the induced metric by a valuation in some conditions.

Before we begin our study, we will give the definition of UP-algebras.

Definition 1. [17] An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra*, where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms:

(UP-1) $(\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(UP-2) $(\forall x \in A)(0 \cdot x = x)$,
(UP-3) $(\forall x \in A)(x \cdot 0 = 0)$, and
(UP-4) $(\forall x, y \in A)(x \cdot y = 0, y \cdot x = 0 \Rightarrow x = y)$.

From [17], we know that the notion of UP-algebras is a generalization of KU-algebras (see [18]).

Example 1. [19] Let X be a universal set and let $\Omega \in \mathcal{P}(X)$ where $\mathcal{P}(X)$ means the power set of X. Let $\mathcal{P}_\Omega(X) = \{A \in \mathcal{P}(X) \mid \Omega \subseteq A\}$. Define a binary operation $\cdot$ on $\mathcal{P}_\Omega(X)$ by putting $A \cdot B = B \cap (A^C \cup \Omega)$ for all $A, B \in \mathcal{P}_\Omega(X)$ where A^C means the complement of a subset A. Then $(\mathcal{P}_\Omega(X), \cdot, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type 1 with respect to* Ω. Let $\mathcal{P}^\Omega(X) = \{A \in \mathcal{P}(X) \mid A \subseteq \Omega\}$. Define a binary operation $*$ on $\mathcal{P}^\Omega(X)$ by putting $A * B = B \cup (A^C \cap \Omega)$ for all $A, B \in \mathcal{P}^\Omega(X)$. Then $(\mathcal{P}^\Omega(X), *, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type 2 with respect to* Ω. In particular, $(\mathcal{P}(X), \cdot, \emptyset)$ is a UP-algebra and we shall call it the *power UP-algebra of type 1*, and $(\mathcal{P}(X), *, X)$ is a UP-algebra and we shall call it the *power UP-algebra of type 2*.

Example 2. [20] Let $\mathbb{N}_0$ be the set of all natural numbers with zero. Define two binary operations $\circ$ and $\bullet$ on $\mathbb{N}_0$ by

$$(\forall x, y \in \mathbb{N}_0)\left(x \circ y = \begin{cases} y & \text{if } x < y, \\ 0 & \text{otherwise} \end{cases}\right)$$

and

$$(\forall x, y \in \mathbb{N}_0)\left(x \bullet y = \begin{cases} y & \text{if } x > y \text{ or } x = 0, \\ 0 & \text{otherwise} \end{cases}\right).$$

Then $(\mathbb{N}_0, \circ, 0)$ and $(\mathbb{N}_0, \bullet, 0)$ are UP-algebras.

Example 3. [21] Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	0	4
2	0	1	0	0	4
3	0	1	2	0	4
4	0	1	2	3	0

Then $(A, \cdot, 0)$ is a UP-algebra.

For more examples of UP-algebras, see [19,22–25].
In a UP-algebra $A = (A, \cdot, 0)$, the following assertions are valid (see [17,24]).

$$(\forall x \in A)(x \cdot x = 0), \tag{1}$$
$$(\forall x, y, z \in A)(x \cdot y = 0, y \cdot z = 0 \Rightarrow x \cdot z = 0), \tag{2}$$
$$(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (z \cdot x) \cdot (z \cdot y) = 0), \tag{3}$$
$$(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (y \cdot z) \cdot (x \cdot z) = 0), \tag{4}$$
$$(\forall x, y \in A)(x \cdot (y \cdot x) = 0), \tag{5}$$
$$(\forall x, y \in A)((y \cdot x) \cdot x = 0 \Leftrightarrow x = y \cdot x), \tag{6}$$
$$(\forall x, y \in A)(x \cdot (y \cdot y) = 0), \tag{7}$$
$$(\forall a, x, y, z \in A)((x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0), \tag{8}$$

$$(\forall a, x, y, z \in A)((((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0), \tag{9}$$
$$(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot z) = 0), \tag{10}$$
$$(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow x \cdot (z \cdot y) = 0), \tag{11}$$
$$(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (x \cdot (y \cdot z)) = 0), \text{ and} \tag{12}$$
$$(\forall a, x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot (a \cdot z)) = 0). \tag{13}$$

From [17], the binary relation $\leq$ on a UP-algebra $A = (A, \cdot, 0)$ is defined as follows:

$$(\forall x, y \in A)(x \leq y \Leftrightarrow x \cdot y = 0). \tag{14}$$

In UP-algebras, 5 types of special subsets are defined as follows:

Definition 2. [17,26–28] A nonempty subset S of a UP-algebra $A = (A, \cdot, 0)$ is called

(1) a *UP-subalgebra* of A if $(\forall x, y \in S)(x \cdot y \in S)$.
(2) a *near UP-filter* of A if it satisfies the following conditions:
  (i) the constant 0 of A is in S, and
  (ii) $(\forall x, y \in A)(y \in S \Rightarrow x \cdot y \in S)$.
(3) a *UP-filter* of A if it satisfies the following conditions:
  (i) the constant 0 of A is in S, and
  (ii) $(\forall x, y \in A)(x \cdot y \in S, x \in S \Rightarrow y \in S)$.
(4) a *UP-ideal* of A if it satisfies the following conditions:
  (i) the constant 0 of A is in S, and
  (ii) $(\forall x, y, z \in A)(x \cdot (y \cdot z) \in S, y \in S \Rightarrow x \cdot z \in S)$.
(5) a *strong UP-ideal* (renamed from a strongly UP-ideal) of A if it satisfies the following conditions:
  (i) the constant 0 of A is in S, and
  (ii) $(\forall x, y, z \in A)((z \cdot y) \cdot (z \cdot x) \in S, y \in S \Rightarrow x \in S)$.

Iampan [26,27] proved that the notion of UP-subalgebras is a generalization of near UP-filters, near UP-filters is a generalization of UP-filters, UP-filters is a generalization of UP-ideals, and UP-ideals is a generalization of strong UP-ideals. Furthermore, they proved that the only strong UP-ideal of a UP-algebra A is A.

2. Valuations and their generalizations

In this section, we introduce the notions of a weak pseudo-valuation, a 0-weak pseudo-valuation, a weak valuation, a near pseudo-valuation, a near valuation, a pseudo-valuation, and a valuation on a UP-algebra. From now on, unless another thing is stated, we take $A = (A, \cdot, 0)$ as a UP-algebra.

Definition 3. A real-valued function φ on A is called a *pseudo-valuation* on A if it satisfies the following conditions:

$$\varphi(0) = 0, \tag{15}$$
$$(\forall x, y \in A)(\varphi(y) \leq \varphi(x \cdot y) + \varphi(x)). \tag{16}$$

A pseudo-valuation φ on A is called a *valuation* on A if it satisfies the following condition:

$$(\forall x \in A)(\varphi(x) = 0 \Rightarrow x = 0). \tag{17}$$

Definition 4. A real-valued function φ on A is called a *near pseudo-valuation* on A if it satisfies the condition (15) and the following condition:

$$(\forall x, y \in A)(\varphi(x \cdot y) \leq \varphi(y)). \tag{18}$$

A near pseudo-valuation φ on A is called a *near valuation* on A if it satisfies the condition (17).

Definition 5. A real-valued function φ on A is called a *weak pseudo-valuation* on A if

$$(\forall x, y \in A)(\varphi(x \cdot y) \leq \varphi(x) + \varphi(y)). \tag{19}$$

A weak pseudo-valuation φ on A is called a 0-*weak pseudo-valuation* on A if it satisfies the condition (15). A 0-weak pseudo-valuation φ on A is called a *weak-valuation* on A if it satisfies the condition (17).

Theorem 1. *Every nonnegative constant real-valued function on A is a weak pseudo-valuation.*

Proof. Let φ be a nonzero constant real-valued function on A. Then there exists a nonnegative real number c such that $\varphi(x) = c$ for all $x \in A$. Let $x, y \in A$. Then

$$\varphi(x \cdot y) = c \leq c + c = \varphi(x) + \varphi(y).$$

Hence, φ is a weak pseudo-valuation on A. □

The following example shows that the converse of Theorem 1 is not true in general.

Example 4. Let $A = \{0, 1, 2, 3, 4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	3	0
2	0	1	0	3	4
3	0	1	2	0	4
4	0	4	2	3	0

Let φ be a real-valued function on A defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.2 & 0.6 & 0.7 & 0.5 & 0.2 \end{pmatrix}.$$

Then φ is a weak pseudo-valuation on A. But φ is not a nonnegative constant real-valued function on A because $\varphi(0) = 0.2 \neq 0.6 = \varphi(1)$.

Theorem 2. *The zero constant real-valued function on A is a* 0*-weak pseudo-valuation (resp., near pseudo-valuation, pseudo-valuation).*

Proof. It is straightforward from the definitions of 0-weak pseudo-valuations, near pseudo-valuations, and pseudo-valuations. □

If A has more than one element, then the zero constant real-valued function on A is not a weak valuation, a near valuation, and a valuation.

The following example shows that the converse of Theorem 2 is not true in general.

Example 5. Let $A = \{0, 1, 2, 3, 4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	1	0	3	4
3	0	1	0	0	4
4	0	1	2	3	0

Let φ be a real-valued function on A defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0.5 & 0.4 & 0.7 & 0.1 \end{pmatrix}.$$

Then φ is a pseudo-valuation on A. But φ is not a zero constant real-valued function on A because $\varphi(0) = 0 \neq 0.5 = \varphi(1)$.

Example 6. From Example 2, we have $(\mathbb{N}_0, \circ, 0)$ is a UP-algebra. Let φ be a real-valued function on $\mathbb{N}_0$ defined by $\varphi(0) = 0$ and if $x \leq y$ where $\leq$ is the standard ordering among real numbers, then $0 < \varphi(x) \leq \varphi(y)$ for all positive numbers x, y. Let $x, y \in \mathbb{N}_0$.

Case 1: $x < y$. Then $\varphi(x \circ y) = \varphi(y)$. Thus $\varphi(x \circ y) + \varphi(x) = \varphi(y) + \varphi(x) \geq \varphi(y)$.

Case 2: $x \geq y$. Then $\varphi(x \circ y) = \varphi(0) = 0$. Thus $\varphi(x \circ y) + \varphi(x) = 0 + \varphi(x) = \varphi(x)$. If $y = 0$, then $\varphi(y) = 0$. Thus $\varphi(x \circ y) + \varphi(x) = \varphi(x) \geq 0 = \varphi(y)$. If $x = 0$, then $y = 0$. By the above proof, we have $\varphi(x \circ y) + \varphi(x) = \varphi(x) \geq 0 = \varphi(y)$. If $x > 0$ and $y > 0$, it follows from the definition of φ that $\varphi(x \circ y) + \varphi(x) = \varphi(x) \geq \varphi(y)$.

Hence, φ is a valuation on $\mathbb{N}_0$.

Example 7. From Example 2, we have $(\mathbb{N}_0, \circ, 0)$ is a UP-algebra. Let φ be a real-valued function on $\mathbb{N}_0$ defined by $\varphi(x) = x$ for all $x \in \mathbb{N}_0$. By Example 6, we have φ is a valuation on $\mathbb{N}_0$.

Example 8. From Example 2, we have $(\mathbb{N}_0, \bullet, 0)$ is a UP-algebra. Let φ be a real-valued function on $\mathbb{N}_0$ defined by $\varphi(0) = 0$ and if $x \leq y$ where $\leq$ is the standard ordering among real numbers, then $\varphi(x) \geq \varphi(y) > 0$ for all positive numbers x, y. Let $x, y \in \mathbb{N}_0$.

Case 1: $x > y$ or $x = 0$. Then $\varphi(x \bullet y) = \varphi(y)$. Thus $\varphi(x \bullet y) + \varphi(x) = \varphi(y) + \varphi(x) \geq \varphi(y)$.

Case 2: $x \leq y$ and $x \neq 0$. Then $y \neq 0$ and $\varphi(x \bullet y) = \varphi(0) = 0$. Since $x > 0$ and $y > 0$, it follows from the definition of φ that $\varphi(x \bullet y) + \varphi(x) = 0 + \varphi(x) = \varphi(x) \geq \varphi(y)$.

Hence, φ is a valuation on $\mathbb{N}_0$.

Example 9. From Example 2, we have $(\mathbb{N}_0, \bullet, 0)$ is a UP-algebra. Let c be a positive real number and φ be a real-valued function on $\mathbb{N}_0$ defined by $\varphi(0) = 0$ and $\varphi(x) = c$ for all positive number x. By Example 8, we have φ is a valuation on $\mathbb{N}_0$.

Example 10. Let $A = \{0, 1, 2, 3, 4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	1	0	0	0
3	0	1	2	0	0
4	0	1	2	3	0

Let φ be a real-valued function on A defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0.5 & 0.7 & 0.3 & 0.2 \end{pmatrix}.$$

Then φ is a valuation on A.

Example 11. Let $A = \{0, 1, 2, 3, 4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	1	0	3	4
3	0	1	2	0	4
4	0	0	0	0	0

Let φ be a real-valued function on A defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0.2 & 0 & 0.3 & 0.5 \end{pmatrix}.$$

Then φ is a pseudo-valuation on A.

Example 12. Let $A = \{0, 1, 2, 3, 4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	3
2	0	0	0	3	3
3	0	1	1	0	1
4	0	0	1	0	0

Let φ be a real-valued function on A defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0.3 & 0.9 & 0.2 & 0.5 \end{pmatrix}.$$

Then φ is a near pseudo-valuation on A.

Example 13. Let $A = \{0, 1, 2, 3, 4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	2
2	0	0	0	0	2
3	0	0	1	0	2
4	0	0	0	0	0

Let φ be a real-valued function on A defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0.5 & 0.4 & 0.7 & 0.1 \end{pmatrix}.$$

Then φ is a 0-weak pseudo-valuation on A.

Example 14. Let $A = \{0, 1, 2, 3, 4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	0
2	0	1	0	3	4
3	0	1	0	0	4
4	0	4	2	3	0

Let φ be a real-valued function on A defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.2 & 0.4 & 0.9 & 0.3 & 0.1 \end{pmatrix}.$$

Then φ is a weak pseudo-valuation on A.

Example 15. Let $A = \{0, 1, 2, 3, 4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	0	0
2	0	1	0	4	0
3	0	1	0	0	0
4	0	1	2	3	0

Let φ be a real-valued function on A defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0.2 & 0.3 & 0.8 & 0.4 \end{pmatrix}.$$

Then φ is a near valuation on A.

Example 16. Let $A = \{0, 1, 2, 3, 4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	2
2	0	0	0	3	2
3	0	1	1	0	2
4	0	0	0	3	0

Let φ be a real-valued function on A defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0.2 & 0.7 & 0.2 & 0.5 \end{pmatrix}.$$

Then φ is a weak valuation on A.

Theorem 3. *Every near pseudo-valuation φ on A satisfies the following condition:*

$$(\forall x \in A)(\varphi(x) \geq 0). \tag{20}$$

Proof. Let $x \in A$. Then $0 = \varphi(0) = \varphi(x \cdot x) \leq \varphi(x)$. □

Theorem 4. *Every valuation on A is a pseudo-valuation.*

Proof. It is straightforward from the definitions of valuations and pseudo-valuations. □

Theorem 5. *Every pseudo-valuation on A is a near pseudo-valuation.*

Proof. Let φ be a pseudo-valuation on A. Let $x, y \in A$. Then $\varphi(x \cdot y) \leq \varphi(y \cdot (x \cdot y)) + \varphi(y) = \varphi(0) + \varphi(y) = 0 + \varphi(y) = \varphi(y)$. Hence, φ is a near pseudo-valuation on A. □

Theorem 6. *Every near pseudo-valuation on A is a* 0*-weak pseudo-valuation.*

Proof. Let φ be a near pseudo-valuation on A. Clearly, $\varphi(0) = 0$. Let $x, y \in A$. Then $\varphi(x \cdot y) \leq \varphi(y) \leq \varphi(x) + \varphi(y)$. Hence, φ is a 0-weak pseudo-valuation on A. □

Theorem 7. *Every* 0*-weak pseudo-valuation on A is a weak pseudo-valuation.*

Proof. It is straightforward from the definitions of 0-weak pseudo-valuations and weak pseudo-valuations. □

Theorem 8. *Every valuation on A is a near valuation.*

Proof. It is straightforward from the definitions of valuations and near valuations and Theorem 5. □

Theorem 9. *Every near valuation on A is a weak valuation.*

Proof. It is straightforward from the definitions of near valuations and weak valuations and Theorem 6. □

Theorem 10. *Every weak valuation on A is a* 0*-weak pseudo-valuation.*

Proof. It is straightforward from the definitions of weak valuations and 0-weak pseudo-valuations. □

Theorem 11. *Every near valuation on A is a near pseudo-valuation.*

Proof. It is straightforward from the definitions of near valuations and near pseudo-valuations. □

The following example shows that the converse of Theorem 4 is not true in general.

Example 17. Let $A = \{0, 1, 2, 3, 4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	0
2	0	1	0	3	4
3	0	1	2	0	4
4	0	1	2	3	0

Let φ be a real-valued function on A defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0.5 & 0 & 0.3 & 0.2 \end{pmatrix}.$$

Then φ is a pseudo-valuation on A. But φ is not a valuation on A because there exists $2 \neq 0$ and $\varphi(2) = 0$.

The following example shows that the converse of Theorem 5 is not true in general.

Example 18. Let $A = \{0, 1, 2, 3, 4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	1	3	3
2	0	0	0	3	3
3	0	0	1	0	1
4	0	0	0	0	0

Let φ be a real-valued function on A defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0.2 & 0.5 & 0.1 & 0.4 \end{pmatrix}.$$

Then φ is a near pseudo-valuation on A. But φ is not a pseudo-valuation on A because $\varphi(1) = 0.2 \nleq 0.1 = 0 + 0.1 = \varphi(0) + \varphi(3) = \varphi(3 \cdot 1) + \varphi(3)$.

The following example shows that the converse of Theorem 6 is not true in general.

Example 19. From Example 13, let φ be a real-valued function on A defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0.1 & 0.2 & 0.5 & 0.1 \end{pmatrix}.$$

Then φ is a 0-weak pseudo-valuation on A. But φ is not a near pseudo-valuation on A as $\varphi(1 \cdot 4) = \varphi(2) = 0.2 \nleq 0.1 = \varphi(4)$.

The following example shows that the converse of Theorem 7 is not true in general.

Example 20. From Example 14, let φ be a real-valued function on A defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.2 & 0.1 & 0.4 & 0.3 & 0.1 \end{pmatrix}.$$

Then φ is a weak pseudo-valuation on A. But φ is not a 0-weak pseudo-valuation on A because $\varphi(0) = 0.2$.

The following example shows that the converse of Theorem 8 is not true in general.

Example 21. Let $A = \{0, 1, 2, 3, 4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	0	0
2	0	1	0	4	0
3	0	1	2	0	0
4	0	1	2	3	0

Let φ be a real-valued function on A defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0.2 & 0.7 & 0.9 & 0.5 \end{pmatrix}.$$

Then φ is a near valuation on A. But φ is not a valuation on A because $\varphi(4) = 0.5 \nleq 0.2 = 0 + 0.2 = \varphi(0) + \varphi(1) = \varphi(1 \cdot 4) + \varphi(1)$.

The following example shows that the converse of Theorem 9 is not true in general.

Example 22. Let $A = \{0, 1, 2, 3, 4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	0	2
3	0	1	1	0	2
4	0	0	0	0	0

Let φ be a real-valued function on A defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0.2 & 0.5 & 0.2 & 0.3 \end{pmatrix}.$$

Then φ is a weak valuation on A. But φ is not a near valuation on A because $\varphi(2 \cdot 4) = \varphi(2) = 0.5 \nleq 0.3 = \varphi(4)$.

The following example shows that the converse of Theorem 10 is not true in general.

Example 23. Let $A = \{0, 1, 2, 3, 4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	0	1
3	0	1	1	0	1
4	0	0	0	0	0

Let φ be a real-valued function on A defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0.10 & 0.45 & 0.75 & 0 \end{pmatrix}.$$

Then φ is a 0-weak pseudo-valuation on A. But φ is not a weak valuation on A because $\varphi(4) = 0$.

The following example shows that the converse of Theorem 11 is not true in general.

Example 24. Let $A = \{0,1,2,3,4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	0	0
2	0	1	0	0	0
3	0	1	2	0	0
4	0	1	2	4	0

Let φ be a real-valued function on A defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0.7 & 0.9 & 0.1 & 0 \end{pmatrix}.$$

Then φ is a near pseudo-valuation on A. But φ is not a near valuation on A because there exists $4 \neq 0$ and $\varphi(4) = 0$.

The following two examples show that a pseudo-valuation and a near valuation are not identical in general.

Example 25. Let $A = \{0,1,2,3,4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	0
2	0	1	0	0	4
3	0	1	2	0	4
4	0	1	2	3	0

Let φ be a real-valued function on A defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0.75 & 0.50 & 0 & 0.50 \end{pmatrix}.$$

Then φ is a pseudo-valuation on A. But φ is not a near valuation on A because there exists $3 \neq 0$ and $\varphi(3) = 0$.

Example 26. Let $A = \{0,1,2,3,4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	1	3	3
2	0	0	0	3	3
3	0	0	1	0	1
4	0	0	1	0	0

Let φ be a real-valued function on A defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0.25 & 0.30 & 0.40 & 0.90 \end{pmatrix}.$$

Then φ is a near valuation on A. But φ is not a pseudo-valuation on A because $\varphi(4) = 0.9 \nleq 0.7 = 0.4 + 0.3 = \varphi(3) + \varphi(2) = \varphi(2 \cdot 4) + \varphi(2)$.

The following two examples show that a near pseudo-valuation and a weak valuation are not identical in general.

Example 27. Let $A = \{0,1,2,3,4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	0	0
2	0	1	0	3	4
3	0	1	2	0	4
4	0	1	2	3	0

Let φ be a real-valued function on A defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0.70 & 0.30 & 0.35 \end{pmatrix}.$$

Then φ is a near pseudo-valuation on A. But φ is not a weak valuation on A because there exists $1 \neq 0$ and $\varphi(1) = 0$.

Example 28. Let $A = \{0,1,2,3,4,5,6\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	0	0	2	3	2	3	6
2	0	1	0	3	1	5	3
3	0	1	2	0	4	1	2
4	0	0	0	3	0	3	3
5	0	0	2	0	2	0	2
6	0	1	0	0	1	1	0

Let φ be a real-valued function on A defined by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0.1 & 0.2 & 0.3 & 0.5 & 0.2 & 0.1 \end{pmatrix}.$$

Then φ is a weak valuation on A. But φ is not a near pseudo-valuation as $\varphi(3 \cdot 6) = \varphi(2) = 0.2 \nleq 0.1 = \varphi(6)$.

Theorem 12. *Every weak pseudo-valuation φ on A satisfies the condition (20).*

Proof. For all $x \in A$, $\varphi(0) = \varphi(x \cdot x) \leq \varphi(x) + \varphi(x)$. In particular, $\varphi(0) \leq \varphi(0) + \varphi(0)$, so $0 \leq \varphi(0)$. Hence, $0 \leq \varphi(0) \leq \varphi(x) + \varphi(x) = 2\varphi(x)$ for all $x \in A$, that is, $0 \leq \varphi(x)$ for all $x \in A$. □

Corollary 13. *Every 0-weak pseudo-valuation (resp., near pseudo-valuation, pseudo-valuation, valuation, weak valuation, near valuation) φ on A satisfies the condition (20).*

Lemma 1. *If φ is a real-valued function on A satisfying the condition (18) and the following condition:*

$$(\forall x \in A)(\varphi(a) \leq \varphi(x)) \Rightarrow \varphi(a) = 0, \tag{21}$$

then it satisfies the conditions (15) and (20).

Proof. For all $x \in A$, $\varphi(0) = \varphi(x \cdot x) \leq \varphi(x)$. By (21), we have $\varphi(0) = 0$, that is, φ satisfies the condition (15). Thus φ is a near pseudo-valuation on A. By Corollary 13, we have φ satisfies the condition (20). □

Theorem 14. *If φ is a real-valued function on A satisfying the conditions (18) and (21) and the following condition:*

$$(\forall x, y, z \in A)(\varphi(x \cdot z) \leq \varphi(x \cdot y) + \varphi(y \cdot z)), \tag{22}$$

then φ is a pseudo-valuation on A.

Proof. Assume that φ is a real-valued function on A satisfying the conditions (18), (21), and (22). By Lemma 1, we have $\varphi(x) \geq 0$ for all $x \in A$ and $\varphi(0) = 0$. Let $x, y \in A$. Then

$$\begin{aligned}\varphi(y) &= \varphi(0 \cdot y)\\ &\leq \varphi(0 \cdot x) + \varphi(x \cdot y)\\ &= \varphi(x) + \varphi(x \cdot y)\\ &= \varphi(x \cdot y) + \varphi(x).\end{aligned}$$

Hence, φ is a pseudo-valuation on A. □

Definition 6. Let φ be a real-valued function on A. Define the subset $\operatorname{Ker}\varphi$ of A by

$$\operatorname{Ker}\varphi := \{x \in A \mid \varphi(x) = 0\}.$$

Theorem 15. *If φ is a pseudo-valuation on A, then $\operatorname{Ker}\varphi$ is a UP-filter of A.*

Proof. Assume that φ is a pseudo-valuation on A. By (15), we have $0 \in \operatorname{Ker}\varphi$. Let $x, y \in A$ be such that $x \cdot y \in \operatorname{Ker}\varphi$ and $x \in \operatorname{Ker}\varphi$. Then $\varphi(x \cdot y) = 0$ and $\varphi(x) = 0$. By (16), we have

$$\varphi(y) \leq \varphi(x \cdot y) + \varphi(x) = 0 + 0 = 0.$$

It follows from Corollary 13 that $\varphi(y) = 0$, so $y \in \operatorname{Ker}\varphi$. Hence, $\operatorname{Ker}\varphi$ is a UP-filter of A. □

Theorem 16. *If φ is a near pseudo-valuation on A, then $\operatorname{Ker}\varphi$ is a near UP-filter of A.*

Proof. Assume that φ is a near pseudo-valuation on A. By (15), we have $0 \in \operatorname{Ker}\varphi$. Let $x \in A$ and $y \in \operatorname{Ker}\varphi$. Then $\varphi(y) = 0$. By (18), we have

$$\varphi(x \cdot y) \leq \varphi(y) = 0.$$

It follows from Corollary 13 that $\varphi(x \cdot y) = 0$, so $x \cdot y \in \operatorname{Ker}\varphi$. Hence, $\operatorname{Ker}\varphi$ is a near UP-filter of A. □

Theorem 17. *If φ is a 0-weak pseudo-valuation on A, then $\operatorname{Ker}\varphi$ is a UP-subalgebra of A.*

Proof. Assume that φ is a 0-weak pseudo-valuation on A. By (15), we have $0 \in \operatorname{Ker}\varphi$. Let $x, y \in A$ be such that $x \in \operatorname{Ker}\varphi$ and $y \in \operatorname{Ker}\varphi$. Then $\varphi(x) = 0$ and $\varphi(y) = 0$. By (19), we have

$$\varphi(x \cdot y) \leq \varphi(x) + \varphi(y) = 0 + 0 = 0,$$

It follows from Corollary 13 that $\varphi(x \cdot y) = 0$, so $x \cdot y \in \operatorname{Ker}\varphi$. Hence, $\operatorname{Ker}\varphi$ is a UP-subalgebra of A. □

Theorem 18. *If φ is a weak pseudo-valuation on A, then a nonempty subset $\operatorname{Ker}\varphi$ is a UP-subalgebra of A.*

Proof. Assume that φ is a weak pseudo-valuation on A. Let $\operatorname{Ker}\varphi \neq \emptyset$. Let $x, y \in \operatorname{Ker}\varphi$. Then $\varphi(x) = 0$ and $\varphi(y) = 0$. By (19), we have

$$\varphi(x \cdot y) \leq \varphi(x) + \varphi(y) = 0 + 0 = 0.$$

It follows from Corollary 13 that $\varphi(x \cdot y) = 0$, so $x \cdot y \in \operatorname{Ker}\varphi$. Hence, $\operatorname{Ker}\varphi$ is a UP-subalgebra of A. □

Proposition 1. *Let φ be a pseudo-valuation on A. Then*

(1) $(\forall a, b, x \in A)(a \leq b \cdot x \Rightarrow \varphi(x) \leq \varphi(a) + \varphi(b))$,
(2) $(\forall x, y \in A)(x \leq y \Rightarrow \varphi(y) \leq \varphi(x))$,
(3) $(\forall x, y, z \in A)(\varphi(x \cdot z) \leq \varphi(x \cdot y) + \varphi(y \cdot z))$, *and*
(4) $(\forall x, y, z \in A)(z \leq x \cdot y \Rightarrow \varphi(z) \leq \varphi(x) + \varphi(y))$.

Proof. (1) Let $a, b, x \in A$ be such that $a \leq b \cdot x$. Then $a \cdot (b \cdot x) = 0$. Thus

$$\begin{aligned}\varphi(x) &\leq \varphi(b \cdot x) + \varphi(b) \\ &\leq \varphi(a \cdot (b \cdot x)) + \varphi(a) + \varphi(b) \\ &= \varphi(0) + \varphi(a) + \varphi(b) \\ &= 0 + \varphi(a) + \varphi(b) \\ &= \varphi(a) + \varphi(b).\end{aligned}$$

(2) Let $x, y \in A$ be such that $x \leq y$. Then $x \cdot y = 0$. Thus

$$\begin{aligned}\varphi(y) &\leq \varphi(x \cdot y) + \varphi(x) \\ &= \varphi(0) + \varphi(x) \\ &= 0 + \varphi(x) \\ &= \varphi(x).\end{aligned}$$

(3) Let $x, y, z \in A$. By (UP-1), we have $y \cdot z \leq (x \cdot y) \cdot (x \cdot z)$. Thus

$$\begin{aligned}\varphi(x \cdot z) &\leq \varphi(y \cdot z) + \varphi(x \cdot y) \\ &= \varphi(x \cdot y) + \varphi(y \cdot z).\end{aligned}$$

(4) Let $x, y, z \in A$ be such that $z \leq x \cdot y$. Then $z \cdot (x \cdot y) = 0$. Thus

$$\begin{aligned}\varphi(z) &\leq \varphi(y \cdot z) + \varphi(y) \\ &\leq \varphi(x \cdot (y \cdot z)) + \varphi(x) + \varphi(y) \\ &= \varphi(0) + \varphi(x) + \varphi(y) \\ &= 0 + \varphi(x) + \varphi(y) \\ &= \varphi(x) + \varphi(y).\end{aligned}$$

□

Theorem 19. *If φ is a 0-weak pseudo-valuation on A satisfying the following condition:*

$$(\forall x, y \in A)(x \cdot y \neq 0 \Rightarrow \varphi(x) = 0), \tag{23}$$

then φ is a near pseudo-valuation on A.

Proof. Assume that φ is a 0-weak pseudo-valuation on A satisfying the condition (23). Clearly, $\varphi(0) = 0$. Let $x, y \in A$. If $x \cdot y = 0$, it follows from Corollary 13 that

$$\varphi(x \cdot y) = \varphi(0) = 0 \leq \varphi(y).$$

If $x \cdot y \neq 0$, it follows from (23) that $\varphi(x) = 0$. By (19), we have

$$\varphi(x \cdot y) \leq \varphi(x) + \varphi(y) = 0 + \varphi(y) = \varphi(y).$$

Hence, φ is a near pseudo-valuation on A. □

Theorem 20. *If φ is a real-valued function on A satisfying the following condition:*

$$(\forall x, y, z \in A)(z \leq x \cdot y \Rightarrow \varphi(z) \leq \varphi(x) + \varphi(y)), \tag{24}$$

then φ is a weak pseudo-valuation on A.

Proof. Assume that φ is a real-valued function on A satisfying the condition (24). Let $x, y \in A$. By (1), we have $x \cdot y \leq x \cdot y$. By (24), we have $\varphi(x \cdot y) \leq \varphi(x) + \varphi(y)$. Hence, φ is a weak pseudo-valuation on A. □

Theorem 21. *If φ is a real-valued function on A satisfying the condition (15) and the following condition:*

$$(\forall x, y, z \in A)(z \leq x \cdot y \Rightarrow \varphi(z) \leq \varphi(y)), \tag{25}$$

then φ is a zero constant real-valued function on A. Moreover, φ is a 0-weak pseudo-valuation, a near pseudo-valuation, and a pseudo-valuation on A.

Proof. Assume that φ is a real-valued function on A satisfying the conditions (15) and (25). Let $x \in A$. By (UP-3), we have $x \leq 0 \cdot 0$. By (25) and (15), we have $\varphi(x) \leq \varphi(0) = 0$. By (UP-3) and (1), we have $0 \leq 0 = x \cdot x$. By (15) and (25), we have $0 = \varphi(0) \leq \varphi(x)$. Thus $\varphi(x) = 0$ for all $x \in A$, that is, φ is a zero constant real-valued function on A. By Theorem 2, we have φ is a 0-weak pseudo-valuation, a near pseudo-valuation, and a pseudo-valuation on A. □

Theorem 22. *If φ is a real-valued function on A satisfying the condition (24) and the following condition:*

$$\varphi(0) \leq 0, \tag{26}$$

then φ is a zero constant real-valued function on A. Moreover, φ is a 0-weak pseudo-valuation, a near pseudo-valuation, and a pseudo-valuation on A.

Proof. Assume that φ is a real-valued function on A satisfying the conditions (24) and (26).

By (UP-3), we have $0 \leq 0 \cdot 0$. By (24), we have $\varphi(0) \leq \varphi(0) + \varphi(0)$ and so $0 \leq \varphi(0)$. By (26), we have $\varphi(0) = 0$, that is, φ satisfies the condition (15). Let $x \in A$. By (UP-3), we have $x \leq 0 \cdot 0$. By (24) and (15), we have $\varphi(x) \leq \varphi(0) + \varphi(0) = 0 + 0 = 0$. By (UP-3), we have $0 \leq x \cdot 0$. By (15) and (24), we have $0 = \varphi(0) \leq \varphi(x) + \varphi(0) = \varphi(x) + 0 = \varphi(x)$. Thus $\varphi(x) = 0$ for all $x \in A$, that is, φ is a zero constant real-valued function on A. By Theorem 2, we have φ is a 0-weak pseudo-valuation, a near pseudo-valuation, and a pseudo-valuation on A. □

3. Metrics and their generalizations

In this section, we induce a pseudo-metric without triangle inequality, a quasi pseudo-metric, a pseudo-metric, and a metric by some these mappings on a UP-algebra. We also prove that the binary operation defined on a UP-algebra is uniformly continuous under the induced metric by a valuation in some conditions.

Definition 7. Let A be a nonempty set. A real-valued function d on $A \times A$ is called

(1) a *metric* on A if it satisfies the following conditions:

(M-1) $(\forall x, y \in A)(d(x, y) \geq 0)$,
(M-2) $(\forall x \in A)(d(x, x) = 0)$,
(M-3) $(\forall x, y \in A)(d(x, y) = d(y, x))$,
(M-4) $(\forall x, y, z \in A)(d(x, z) \leq d(x, y) + d(y, z))$, and
(M-5) $(\forall x, y \in A)(d(x, y) = 0 \Rightarrow x = y)$.

(2) a *pseudo-metric* on A if it satisfies the conditions (M-1), (M-2), (M-3), and (M-4),
(3) a *pseudo-metric without triangle inequality* on A if it satisfies the conditions (M-1), (M-2), and (M-3), and
(4) a *quasi pseudo-metric* on A if it satisfies the conditions (M-1), (M-2), and (M-4).

Theorem 23. *Let φ be a 0-weak pseudo-valuation (resp., weak valuation, near pseudo-valuation, near valuation, pseudo-valuation, valuation) on A and r be a positive real number. Then the real-valued function d_φ^r on $A \times A$ defined by*

$$(\forall (x, y) \in A \times A)(d_\varphi^r(x, y) = \frac{\varphi(x \cdot y) + \varphi(y \cdot x)}{r}) \tag{27}$$

is a pseudo-metric without triangle inequality on A, called a pseudo-metric without triangle inequality induced by a 0-weak pseudo-valuation (resp., weak valuation, near pseudo-valuation, near valuation, pseudo-valuation, valuation) φ.

Proof.(M-1) By Corollary 13, we have $d_\varphi^r(x, y) = \dfrac{\varphi(x \cdot y) + \varphi(y \cdot x)}{r} \geq 0$ for all $x, y \in A$.

(M-2) Let $x \in A$. By (1) and (15), we have $d_\varphi^r(x, x) = \dfrac{\varphi(x \cdot x) + \varphi(x \cdot x)}{r} = \dfrac{\varphi(0) + \varphi(0)}{r} = \dfrac{0 + 0}{r} = 0$.

(M-3) Let $x, y \in A$.
Then $d^r_\varphi(x, y) = \dfrac{\varphi(x \cdot y) + \varphi(y \cdot x)}{r} = \dfrac{\varphi(y \cdot x) + \varphi(x \cdot y)}{r} = d^r_\varphi(y, x)$.
Hence, d^r_φ is a pseudo-metric without triangle inequality on A. □

Theorem 24. *Let φ be a pseudo-valuation (resp., valuation) on A and r be a positive real number. Then the real-valued function $d^r_\varphi : A \times A \to \mathbb{R}$ which is defined in (27) is a pseudo-metric on A, called a pseudo-metric induced by a pseudo-valuation (resp., valuation) φ.*

Proof. By Theorem 23, it suffices to prove (M-4).

(M-4) Let $x, y, z \in A$. Then

$$\begin{aligned} d^r_\varphi(x, y) + d^r_\varphi(y, z) &= \frac{\varphi(x \cdot y) + \varphi(y \cdot x)}{r} + \frac{\varphi(y \cdot z) + \varphi(z \cdot y)}{r} \\ &= \frac{\varphi(x \cdot y) + \varphi(y \cdot x) + \varphi(y \cdot z) + \varphi(z \cdot y)}{r} \\ &= \frac{\varphi(x \cdot y) + \varphi(y \cdot z)}{r} + \frac{\varphi(z \cdot y) + \varphi(y \cdot x)}{r} \\ &\geq \frac{\varphi(x \cdot z)}{r} + \frac{\varphi(z \cdot x)}{r} \\ &= \frac{\varphi(x \cdot z) + \varphi(z \cdot x)}{r} \\ &= d^r_\varphi(x, z). \end{aligned}$$

Hence, d^r_φ is a pseudo-metric on A. □

Theorem 25. *Let φ be a valuation on A and r be a positive real number. Then the real-valued function $d^r_\varphi : A \times A \to \mathbb{R}$ which is defined in (27) is a metric on A, called a metric induced by a valuation φ.*

Proof. By Theorem 24, it suffices to prove (M-5).

(M-5) Let $x, y \in A$ be such that $d^r_\varphi(x, y) = 0$. Then

$$0 = d^r_\varphi(x, y) = \frac{\varphi(x \cdot y) + \varphi(y \cdot x)}{r} = \frac{\varphi(x \cdot y)}{r} + \frac{\varphi(y \cdot x)}{r}.$$

It follows from Corollary 13 that $\varphi(x \cdot y) = 0$ and $\varphi(y \cdot x) = 0$. By (17), we have $x \cdot y = 0$ and $y \cdot x = 0$. By (UP-4), we have $x = y$. Hence, d^r_φ is a metric on A.
□

Theorem 26. *Let φ be a 0-weak pseudo-valuation (resp., weak valuation, near pseudo-valuation, near valuation, pseudo-valuation, valuation) on A and r be a positive real number. Then the real-valued function $D^r_\varphi : A \times A \to \mathbb{R}$ defined by*

$$(\forall (x, y) \in A \times A)(D^r_\varphi(x, y) = \frac{\varphi(x \cdot y) \times \varphi(y \cdot x)}{r}) \tag{28}$$

is a pseudo-metric without triangle inequality on A, called a pseudo-metric without triangle inequality induced by a 0-weak pseudo-valuation (resp., weak valuation, near pseudo-valuation, near valuation, pseudo-valuation, valuation) φ.

Proof.(M-1) By Corollary 13, we have $D^r_\varphi(x, y) = \dfrac{\varphi(x \cdot y) \times \varphi(y \cdot x)}{r} \geq 0$ for all $x, y \in A$.

(M-2) Let $x \in A$. By (1) and (15), we have $D^r_\varphi(x, x) = \dfrac{\varphi(x \cdot x) \times \varphi(x \cdot x)}{r} = \dfrac{\varphi(0) \times \varphi(0)}{r} = \dfrac{0 \times 0}{r} = 0$.

(M-3) Let $x, y \in A$. Then $D^r_\varphi(x, y) = \dfrac{\varphi(x \cdot y) \times \varphi(y \cdot x)}{r} = \dfrac{\varphi(y \cdot x) \times \varphi(x \cdot y)}{r} = D^r_\varphi(y, x)$.
Hence, D^r_φ is a pseudo-metric without triangle inequality on A. □

Theorem 27. *Let φ be a pseudo-valuation (resp., valuation) on A and r be a positive real number. Then the real-valued function $A^r_\varphi : A \times A \to \mathbb{R}$ defined by*

$$(\forall (x,y) \in A \times A)(A^r_\varphi(x,y) = \frac{\varphi(x \cdot y)}{r}) \tag{29}$$

is a quasi pseudo-metric on A, called a quasi pseudo-metric induced by a pseudo-valuation (resp., valuation) φ.

Proof.(M-1) By Corollary 13, we have $A^r_\varphi(x,y) = \dfrac{\varphi(x \cdot y)}{r} \geq 0$ for all $x, y \in A$.

(M-2) Let $x \in A$. By (1) and (15), we have $A^r_\varphi(x,x) = \dfrac{\varphi(x \cdot x)}{r} = \dfrac{\varphi(0)}{r} = \dfrac{0}{r} = 0$.

(M-4) Let $x, y, z \in A$. Then

$$\begin{aligned} A^r_\varphi(x,y) + A^r_\varphi(y,z) &= \frac{\varphi(x \cdot y)}{r} + \frac{\varphi(y \cdot z)}{r} \\ &= \frac{\varphi(x \cdot y) + \varphi(y \cdot z)}{r} \\ &\geq \frac{\varphi(x \cdot z)}{r} \\ &= A^r_\varphi(x,z). \end{aligned}$$

Hence, A^r_φ is a quasi pseudo-metric on A. □

Proposition 2. *Let φ be a pseudo-valuation on A. Then*

$$(\forall x, y, a, \in A)(d^r_\varphi(x,y) \geq d^r_\varphi(a \cdot x, a \cdot y)). \tag{30}$$

Proof. Let $x, y, a \in A$. By (UP-1) and Proposition 1 (2), we have $\varphi((a \cdot y) \cdot (a \cdot x)) \leq \varphi(y \cdot x)$ and $\varphi((a \cdot x) \cdot (a \cdot y)) \leq \varphi(x \cdot y)$. Thus

$$\begin{aligned} d^r_\varphi(x,y) &= \frac{\varphi(x \cdot y) + \varphi(y \cdot x)}{r} \\ &\geq \frac{\varphi((a \cdot x) \cdot (a \cdot y)) + \varphi((a \cdot y) \cdot (a \cdot x))}{r} \\ &= d^r_\varphi(a \cdot x, a \cdot y). \end{aligned}$$

□

The following lemma is easily proved.

Lemma 2. *Let a_1, a_2, b_1, b_2 be real numbers. Then*

(1) $\max\{a_1, b_1\} + \max\{a_2, b_2\} \geq \max\{a_1 + a_2, b_1 + b_2\}$, *and*
(2) $\min\{a_1, b_1\} + \min\{a_2, b_2\} \leq \min\{a_1 + a_2, b_1 + b_2\}$.

Theorem 28. *Let d be a real-valued function on $A \times A$. Define the real-valued functions d° and $d_\circ$ on $(A \times A) \times (A \times A)$ by*

$$(\forall (x,y),(a,b) \in A \times A)(d^\circ((x,y),(a,b)) = \max\{d(x,a), d(y,b)\}) \tag{31}$$

and

$$(\forall (x,y),(a,b) \in A \times A)(d_\circ((x,y),(a,b)) = \min\{d(x,a), d(y,b)\}). \tag{32}$$

Then

(1) if d satisfies the condition (M-1), then d° and $d_\circ$ satisfy the condition (M-1);
(2) if d satisfies the condition (M-2), then d° and $d_\circ$ satisfy the condition (M-2);
(3) if d satisfies the condition (M-3), then d° and $d_\circ$ satisfy the condition (M-3);
(4) if d satisfies the condition (M-4), then d° satisfies the condition (M-4);
(5) if d satisfies the condition (M-5), then d° satisfies the condition (M-5).

Proof. (1) Assume that d satisfies the condition (M-1). Let $(x,y),(a,b) \in A \times A$. Since $d(x,a) \geq 0$ and $d(y,b) \geq 0$, we have

$$d^\circ((x,y),(a,b)) = \max\{d(x,a),d(y,b)\} \geq 0$$

and

$$d_\circ((x,y),(a,b)) = \min\{d(x,a),d(y,b)\} \geq 0.$$

Hence, d° and $d_\circ$ satisfy the condition (M-1).

(2) Assume that d satisfies the condition (M-2). Since $d(x,x) = 0$ and $d(y,y) = 0$, we have

$$d^\circ((x,y),(x,y)) = \max\{d(x,x),d(y,y)\} = \max\{0,0\} = 0$$

and

$$d_\circ((x,y),(x,y)) = \min\{d(x,x),d(y,y)\} = \min\{0,0\} = 0.$$

Hence, d° and $d_\circ$ satisfy the condition (M-2).

(3) Assume that d satisfies the condition (M-3). Since $d(x,a) = d(a,x)$ and $d(y,b) = d(b,y)$, we have

$$d^\circ((x,y),(a,b)) = \max\{d(x,a),d(y,b)\}) = \max\{d(a,x),d(b,y)\} = d^\circ((a,b),(x,y))$$

and

$$d_\circ((x,y),(a,b)) = \min\{d(x,a),d(y,b)\}) = \min\{d(a,x),d(b,y)\} = d_\circ((a,b),(x,y)).$$

Hence, d° and $d_\circ$ satisfy the condition (M-3).

(4) Assume that d satisfies the condition (M-4). Since $d(x,u) + d(u,a) \geq d(x,a)$ and $d(y,v) + d(v,b) \geq d(y,b)$, we have

$$\begin{aligned} d^\circ((x,y),(u,v)) + d^\circ((u,v),(a,b)) &= \max\{d(x,u),d(y,v)\} + \max\{d(u,a),d(v,b)\} \\ &\geq \max\{d(x,u)+d(u,a),d(y,v)+d(v,b)\} \\ &\geq \max\{d(x,a),d(y,b)\} \\ &= d^\circ((x,y),(a,b)). \end{aligned}$$

Hence, d° satisfies the condition (M-4).

(5) Assume that d satisfies the condition (M-5). Let $(x,y),(a,b) \in A \times A$ be such that $d^\circ((x,y),(a,b)) = 0$. Then

$$0 = d^\circ((x,y),(a,b)) = \max\{d(x,a),d(y,b)\}.$$

Thus $0 \geq d(x,a)$ and $0 \geq d(y,b)$. Since $d(x,a) \geq 0$ and $d(y,b) \geq 0$, we have $d(x,a) = 0$ and $d(y,b) = 0$. By (M-5) of d, we have $x = a$ and $y = b$. Thus $(x,y) = (a,b)$. Hence, d° satisfies the condition (M-5).
□

Theorem 29. *Let d be a real-valued function on $A \times A$. Then*

(1) *if d is a pseudo-metric without triangle inequality on A, then d° and $d_\circ$ are pseudo-metrics without triangle inequality on $A \times A$;*
(2) *if d is a quasi pseudo-metric on A, then d° is a quasi pseudo-metric on $A \times A$;*
(3) *if d is a pseudo-metric on A, then d° is a pseudo-metric on $A \times A$;*
(4) *if d is a metric on A, then d° is a metric on $A \times A$.*

Proof. It is straightforward from Theorem 28. □

Corollary 30. *If φ is a 0-weak pseudo-valuation (resp., weak valuation, near pseudo-valuation, near valuation, pseudo-valuation, valuation) on A and r is a positive real number, then $(d^r_\varphi)^\circ, (d^r_\varphi)_\circ, (D^r_\varphi)^\circ$, and $(D^r_\varphi)_\circ$ are pseudo-metrics without triangle inequality on $A \times A$.*

Proof. It is straightforward from Theorems 23, 26 and 29 (1). □

Corollary 31. *If φ is a pseudo-valuation (resp., valuation) on A and r is a positive real number, then $(d^r_\varphi)^\circ$ is a pseudo-metric and $(A^r_\varphi)^\circ$ is a quasi pseudo-metric on $A \times A$.*

Proof. It is straightforward from Theorems 24, 27, 29(2) and 29(3). □

Corollary 32. *If φ is a valuation on A and r is a positive real number, then $(d^r_\varphi)^\circ$ is a metric on $A \times A$.*

Proof. It is straightforward from Theorems 25 and 29(4). □

We recall the definition of a uniformly continuous function of metric spaces:

Definition 8. Let metric spaces (X, d_1) and (Y, d_2), a function $f : X \to Y$ is said to be *uniformly continuous* if for every real number $\varepsilon > 0$ there exists a real number $\delta > 0$ such that

$$(\forall x, y \in X)(d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon).$$

Theorem 33. *If φ is a valuation on A with $\inf\{(d^r_\varphi)^\circ((x,y),(a,b)) \mid (d^r_\varphi)^\circ((x,y),(a,b)) > 0\} > 0$ exists, then the binary operation $\cdot$ on A is uniformly continuous.*

Proof. Assume that φ is a valuation on a finite UP-algebra A with more than one element. By Theorem 25 and Corollary 32, we have (A, d^r_φ) and $(A \times A, (d^r_\varphi)^\circ)$ are metric spaces. Let $\varepsilon > 0$. Choose $\delta = \inf\{(d^r_\varphi)^\circ((x,y),(a,b)) \mid (d^r_\varphi)^\circ((x,y),(a,b)) > 0\}$. Let $(x,y),(a,b) \in A \times A$ be such that $(d^r_\varphi)^\circ((x,y),(a,b)) < \delta$. Then $(d^r_\varphi)^\circ((x,y),(a,b)) = 0$. By (M-5), we have $(x,y) = (a,b)$ and so $x = a$ and $y = b$. Thus, by (M-2), we have $d^r_\varphi(\cdot(x,y), \cdot(a,b)) = d^r_\varphi(\cdot(x,y), \cdot(x,y)) = d^r_\varphi(x \cdot y, x \cdot y) = 0 < \varepsilon$. Hence, the binary operation $\cdot$ on A is uniformly continuous. □

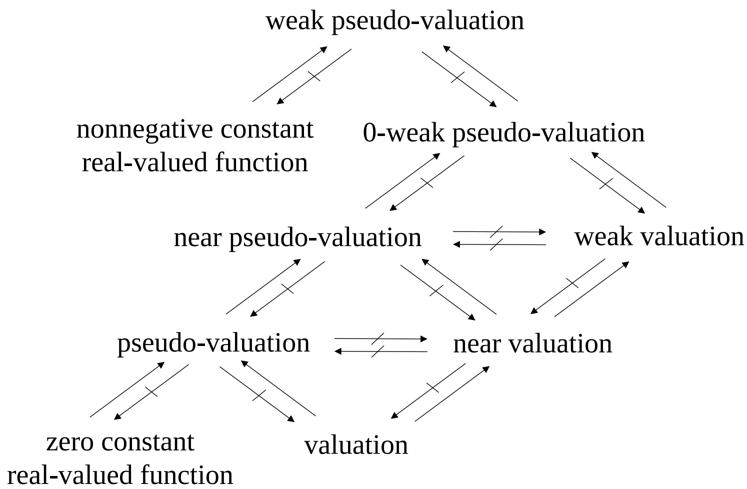

Figure 1. Valuations and their generalizations for UP-algebras

If the UP-algebra $A = \{0\}$, then the valuation φ on A is the zero function. Thus d^r_φ and $(d^r_\varphi)^\circ$ are zero functions. Hence, the binary operation $\cdot$ on A is uniformly continuous.

Theorem 34. *If φ is a valuation on a finite UP-algebra A with more than one element, then the binary operation $\cdot$ on A is uniformly continuous.*

Proof. Assume that φ is a valuation on a finite UP-algebra A with more than one element. By (M-1) and (M-5) and $A \times A$ has more than one element, we have $\{(d^r_\varphi)^\circ((x,y),(a,b)) \mid (d^r_\varphi)^\circ((x,y),(a,b)) > 0\}$ is a finite nonempty subset of $\mathbb{R}$. Thus $\inf\{(d^r_\varphi)^\circ((x,y),(a,b)) \mid (d^r_\varphi)^\circ((x,y),(a,b)) > 0\} = \min\{(d^r_\varphi)^\circ((x,y),(a,b)) \mid (d^r_\varphi)^\circ((x,y),(a,b)) > 0\}$ exists. By Theorem 33, we have the binary operation $\cdot$ on A is uniformly continuous. □

4. Conclusions

In this paper, we have introduced the notions of a weak pseudo-valuation, a 0-weak pseudo-valuation, a weak valuation, a near pseudo-valuation, a near valuation, a pseudo-valuation, and a valuation and induced a pseudo-metric without triangle inequality, a quasi pseudo-metric, a pseudo-metric, and a metric by some these mappings on a UP-algebra. Then, we get the diagram of generalization of these mappings on a UP-algebra as shown in Figure 1 (see Theorems 1, 2, 4, 5, 6, 7, 8, 9, 10, 11).

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