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Einstein equations for a Finsler-Larange space with canonical N-linear connections

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Abstract: This paper is devoted to study the geometry of Einstein equations of Finsler-Lagrange with (α, β)-metrics. We characterized the Einstein equations of Finsler Lagrange space with Randers metric, by using canonical N-metrical connection.

Keywords: Finsler Space, Lagrange space, Cartan connection, (α, β)-metrics, Ricci curvature, scalar curvature, torsion tensor.

MSC: 53B20, 53C20, 53C60.

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1. Introduction

The geometry of Lagrange spaces is applied to the description of classical general relativity and electrodynamics. First, the Einstein equations are given in a new form, where the geometrical objects related to the internal variables are separated from those related to the external variables. After this, several special Lagrange spaces are analyzed. The almost Riemannian Lagrange spaces are rather simple for explicit calculations and they recover all classical results of general relativity and electrodynamics.

The theory of Finsler spaces with (α, β)-metrics was introduced by Matsumoto [1]. The natural extension of this theory is based on the canonial Cartan nonlinear connection N [2].

In [3], Bucataru studied the Finsler space with (α, β)-metrics have nonholonomic frames which are useful for unifying theories in theoretical physics.

The notion of Lorentz nonlinear connection N was introduced by Hassan, which depends only on the metric $L(\alpha, \beta)$, so the spaces $FL^n = (M, L(\alpha, \beta), N)$ are called the Finsler-Lagrange spaces with (α, β)-metrics. This theory has been applied in the study of gravitational and electromagnetic [4,5].

The present paper organized the Euler-Lagrange spaces with (α, β)-metrics and Lorentz equations. Also, Einstein equations for Lagrange space with (α, β)-metrics, in particularly Randers metric by means of canonical N-metrical connection is presented.

2. Preliminaries

The present section deals with some fundamental concepts and facts of Finsler-Lagrange geometry [6–8].

2.1. Finsler-Lagrange space with (α, β)-metrics

Let $F^n = (M, F(x, y))$ be a Finsler space with (α, β)-metric and $F(x, y)$ be a fundamental function of the form

$$F(x, y) = \hat{F}(\alpha(x, y), \beta(x, y)),$$

where, $\hat{F}$ is a differentiable function of two variables:

$$\alpha^2(x, y) = a_{ij}(x) y^i y^j,$$
$$\beta(x, y) = b_i(x) y^i.$$

The notion α in the above equation represent the pseudo-Riemannian metric on the base manifold M which gives the gravitational part of $F(x, y)$ whereas β is the eletromagnetic 1-form on M.

Denoting $L(\alpha(x,y),\beta(x,y)) = \hat{F}(\alpha(x,y),\beta(x,y))$, gives $L^n = (M, L)$ Lagrange space with fundamental metric tensor $g_{ij}(x,y)$ of the form

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}.$$

Which further modified as

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_{-1}(b_i l_j + b_j l_i) + \rho_{-2} l_i l_j, \tag{1}$$

where $b_i = \frac{\partial\beta}{\partial y^i}$, $l_i = a_{ij}y^j = \alpha\frac{\partial\alpha}{\partial y^j}$, ρ, ρ_0, ρ_{-1} and ρ_{-2} are invariants of the space L^n:

$$\rho = \frac{1}{2\alpha}L_\alpha, \qquad \rho_0 = \frac{1}{2}L_{\beta\beta}, \qquad \rho_{-1} = \frac{1}{2}L_{\alpha\beta}, \qquad \rho_{-2} = \frac{1}{2\alpha^2}(L_{\alpha\alpha} - \frac{1}{\alpha}L_\alpha) \tag{2}$$

with

$$L_\alpha = \frac{\partial L}{\partial\alpha}, \qquad L_\beta = \frac{\partial L}{\partial\beta}, \qquad L_{\alpha\alpha} = \frac{\partial^2 L}{\partial\alpha^2} \qquad L_{\beta\beta} = \frac{\partial^2 L}{\partial\beta^2}, \qquad L_{\alpha\beta} = \frac{\partial^2 L}{\partial\alpha\partial\beta}.$$

The totally symmetric Cartan tensor defined by

$$C_{ijk} = \frac{1}{4}\frac{\partial^3 L}{\partial y^i \partial y^j \partial y^k} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}.$$

By means of g_{ij} from (1) taking into account the formulae from [9], one obtains

$$2C_{ijk} \;=\; \sigma_{(i,j,k)}(\rho_{-1}a_{ij}b_k + \rho_{-2}a_{ij}l_k + \frac{1}{3}r_{-1}b_ib_jb_k + r_{-2}b_ib_jl_k + r_{-3}b_il_jl_k + \frac{1}{3}r_{-4}l_il_jl_k), \tag{3}$$

where $\sigma_{(i,j,k)}$ means the cyclic sum in the indices i, j, k.

2.2. Variational problem and Lorentz non-linear Connection

Let $L : TM \longrightarrow \mathbb{R}$ be a regular Lagrangian and $c : t \in [0,1] \longrightarrow (x^i(t)) \in U \subset M$ be a regular curve. The functional defined as follows:

$$I(c) = \int_0^1 L(\alpha(x,y),\beta(x,y))dt,$$

gives Euler-Lagrange equations as:

$$E_i(L) = \frac{\partial L}{\partial x^i} - \frac{d}{dt}(\frac{\partial L}{\partial y^i}) = 0, \quad y^i = \frac{dx^i}{dt}. \tag{4}$$

The co-vector $E_i(L)$ can also be expressed as:

$$E_i(L) = E_i(\alpha^2) + 2\frac{\rho_{-1}}{\rho}E_i(\beta) + 2\frac{d\alpha}{dt}\frac{\partial\alpha}{\partial y^i}. \tag{5}$$

If c is an extremal curve, *i.e.*, c is a solution of Euler-Lagrange equation (4), then along c the energy of a Lagrangian L is:

$$E_L = y^i\frac{\partial L}{\partial y^i} - L.$$

Now, let us fix the parametrization of the curve c by a natural parameter $t = s$, with respect to the Riemannian metric $\alpha^2(x, dx/dt)$ given by:

$$ds^2 = \alpha^2(x, \frac{dx}{dt})dt^2. \tag{6}$$

Thus, along the extremal curve c parameterized by arc lengths $t = s$, we have $\alpha^2(x, dx/ds) = 1$ and $d\alpha/ds = 0$, $dL/ds = 0$, which implies that $d\beta/ds = 0$, $dL_\alpha/ds = 0$, $dL_\beta/ds = 0$.

Since $E_i(\beta)$ is given by:

$$E_i(\beta) = F_{ij}(x)\frac{dx^j}{ds}, \quad F_{ij} = \frac{\partial b_j}{\partial x^i} - \frac{\partial b_i}{\partial x^j}. \tag{7}$$

Then, Miron and Hassan [5] obtained the following theorems:

Theorem 1. *Consider the natural parametrization $t = s$, the Euler-Lagrange equations of the Lagrangian $L(\alpha, \beta)$ are given by*

$$E_i(\alpha^2) + 2\frac{\rho_{-1}}{\rho}F_{ij}(x)y^j = 0, \ \ y^i = \frac{\partial x^i}{\partial s}. \tag{8}$$

Choosing $\gamma^i_{jk}(x)$ as the Christoffel symbols of pseudo-Riemannian metric α^2 and

$$\sigma(x,y) = \frac{\rho_{-1}}{\rho}, \quad F^i_j(x) = a^{ih}(x)F_{hj}(x),$$

we have the following result.

Theorem 2. *The Euler-Lagrange equation (8) are equivalent to the Lorentz equations as:*

$$\frac{d^2x^i}{ds2} + \gamma^i_{jk}(x)\frac{dx^j}{ds}\frac{dx^k}{ds} = \sigma(x, \frac{dx}{ds})F^i_j(x)\frac{dx^j}{ds}. \tag{9}$$

If Euler-Lagrange equations $E_i(L)$=0, then we determine a canonical semispray S as:

$$S = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i},$$

where $2G^i(x,y) = \gamma^i_{jk}(x) - \sigma(x,y)F^i_j(x)y^j$. Then, the integral curve of S are given by the Lorentz equation (9).

Now, let us consider the non-linear connection N with the coefficients as:

$$N^i_j = \gamma^i_{jk}(x)y^k - \sigma(x)F^i_j(x)$$

Thus, the variation of autoparallel curves of a non-linear connections work should be progressed in 2003.

Since the autoparallel curves of N are given by the Lorentz equation (8), we call it as the Lorentz non-linear connection of the metric L and so FL^n is the Finsler- Lagrange (α, β)-metric $L(\alpha, \beta)$ and the Lorentz non-linear connection N. The semispray S associated to N has the coefficients as:

$$2G^i = N^i_j y^j. \tag{10}$$

2.3. Properties of the Lorentz non-linear connection

(i) The Berwald connection $B\Gamma(N) = (B^i_{jk}(x,y), 0)$ of N has the coefficients

$$B^i_{jk}(x,y) = \gamma^i_{jk}(x) - \dot{\sigma}_k F^i_j(x),$$

where $\dot{\sigma}_k = \frac{\partial\sigma}{\partial y^k}$

(ii) The weak torsion of N is

$$L^i_{jk} = \dot{\sigma}_j F^i_k(x) - \dot{\sigma}_k F^i_j(x).$$

Clearly, if $b_i = grad_i\varphi(x)$, then $L^i_{jk} = 0$.

(iii) The adapted bases are

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i\frac{\partial}{\partial y^j}.$$

(iv) The integrability tensor

$$R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j},$$

of N is

$$R^i_{jk} = y^h\rho^i_{hjk}(x) + \sigma_j F^i_k - \sigma_k F^i_j - \sigma(F^i_{j|k} - F^i_{k|j}),$$

where $\sigma_j = \frac{\partial \sigma}{\partial x^j}$ and '$|$' is the covariant derivative with respect to the Levi-Civita connection of α^2 and $\rho^i_{hjk}(x)$ is the curvature tensor of the Levi-Civita connection.

(v) The Lorentz non-linear connection N is integrable if and only if the d-tensor of integrability R^i_{jk} vanishes.

(vi) The dual basis $(dx^i, \delta y^i)$ of $(\delta/\delta x^i, \partial/\partial y^i)$ is determined by

$$\begin{aligned} \delta y^i &= dy^i + N^i_j dx^j, \\ &= dy^i + \gamma^i_{jk} y^k dx^j - \sigma F^i_j dx^j, \\ &= \delta y^i - \sigma F^i_j dx^j. \end{aligned}$$

(vii) The autoparallel curves of Lorentz non-linear connection are given by the system of differential equations:

$$\frac{\delta y^i}{dt}, \quad y^i = \frac{dx^i}{dt}.$$

(viii) In the parametrization S with $\alpha^2(x, dx/ds) = 1$, the property (vii) are the Lorentz equation (8).

(ix) The exterior differential of 1-forms δy^i of the form

$$d\delta y^i = \frac{1}{2} R^i_{jk} dx^k \wedge dx^j + B^i_{jk} \delta y^k \wedge dx^j.$$

2.4. Canonical N-metrical connection

The metric N-linear connection is called the canonical N-linear connection or the Cartan connection of the Lagrange space. The space $FL^n = (M, L(\alpha, \beta), N)$ has a canonical N-linear connection $C\Gamma(N)$ with the coefficients (L^i_{jk}, C^i_k) given by the generalized Christoffel symbols are:

$$\left.\begin{aligned} L^i_{jk} &= \tfrac{1}{2} g^{is} \left(\tfrac{\delta g_{sk}}{\delta x^j} + \tfrac{\delta g_{js}}{\delta x^k} - \tfrac{\delta g_{jk}}{\delta x^s} \right) \\ C^i_{jk} &= \tfrac{1}{2} g^{is} \left(\tfrac{\partial g_{sk}}{\partial y^j} + \tfrac{\partial g_{js}}{\partial y^k} - \tfrac{\partial g_{jk}}{\partial y^s} \right) \end{aligned}\right\} \tag{11}$$

The 1-form connection $C\Gamma(N)$ is:

$$\omega^i_j = L^i_{jk} dx^k + C^i_{jk} \delta y^k.$$

By the property (ix), the structure equations of $C\Gamma(N)$ expressed in the following theorem [10]:

Theorem 3. *The structure equations of the canonical N-linear metrical connection $C\Gamma(N)$ of the space FL^n are as follows:*

$$\begin{aligned} d(dx^i) - dx^k \wedge \omega^i_k &= -{}^1\Omega^i, \\ d(\delta y^i) - \delta y^k \wedge \omega^i_k &= -{}^2\Omega^i, \\ d\omega^i_k - \omega^k_j \wedge \omega^i_k &= -\Omega^i_j, \end{aligned}$$

where ${}^1\Omega$, ${}^2\Omega$ are the 2-forms of torsion.

Here

$$\begin{aligned} {}^1\Omega^i &= C^i_{jk} dx^j \wedge \delta y^k, \\ {}^2\Omega^i &= \frac{1}{2} R^i_{jk} dx^j \wedge dx^k + P^i_{jk} dx^j \wedge \delta y^k \end{aligned}$$

and Ω is the 2-form of curvature.

$$\Omega^i_j = \frac{1}{2} R^i_{jkh} dx^k \wedge dx^h + P^i_{jkh} dx^k \wedge \delta y^h + \frac{1}{2} S^i_{jkh} \delta y^k \wedge \delta y^h,$$

where R^i_{jk} is tensor given in property (iv), $P^i_{jk} = B^i_{jk} - L^i_{jk}$ and R^i_{jkh}, P^i_{jkh}, S^i_{jkh} are the curvature tensor of $C\Gamma(N)$.

In this study, we use the metric N-linear connection $D\Gamma(N) = (\bar{L}^i_{jk}, \bar{C}^i_{jk})$ and has a given d-tensor of torsion $\bar{\mathrm{T}}^i_{jk}$ and $\bar{S}^i_{jk}$ as follows:

$$\left.\begin{aligned} \bar{L}^i_{jk} &= L^i_{jk} + \tfrac{1}{2} g^{ih}\left(g_{jr}\bar{\mathrm{T}}^r_{kh} + g_{jh}\bar{\mathrm{T}}^r_{jh} - g_{hr}\bar{\mathrm{T}}^r_{kj}\right), \\ \bar{C}^i_{jk} &= C^i_{jk} + \tfrac{1}{2} g^{ih}\left(g_{jr}S^r_{kh} + g_{jh}S^r_{jh} - g_{hr}S^r_{kj}\right), \end{aligned}\right\} \tag{12}$$

where (L^i_{jk}, C^i_{jk}) are the local coefficients of the canonical metric N-linear connection $C\Gamma(N)$ and $\bar{\mathrm{T}}^i_{jk}, \bar{S}^i_{jk}$ simply by $(\mathrm{T}^i_{jk}, S^i_{jk})$.

2.5. Einstein Equations on TM

Let TM be endowed with a non-linear connection N, an h-v metric structure G and a metrical N-connection $D\Gamma(N)$ with a priori given torsions $(\mathrm{T}^i_{jk}, S^i_{jk})$.

Given an h-v metric G on TM becomes a pseudo-Riemannian manifold of dimension $2n$. The Einstein equations written for the connection $D\Gamma(N)$ on TM as:

$$Ric(D) - \frac{1}{2} Sc(D) G = k\mathrm{T}, \tag{13}$$

where $Ric(D)$ is the Ricci tensor field and $Sc(D)$ is the scalar curvature of $D\Gamma(N)$, k is constant and T is the energy-momentum tensor field.

In local coordinates, Miron and Anastasiei stated as [11]:

Theorem 4. *The Einstein equations of the Lagrange space $L^n = (M, L)$ corresponding to the metric N-linear connection $D\Gamma(N) = (L^i_{jk}, C^i_{jk})$ with the coefficients (12) have the following form:*

$$\begin{aligned} R_{ij} &= \frac{1}{2}(R+S)g_{ij} = k\mathrm{T}_{ij}, \\ S_{ij} &= \frac{1}{2}(R+S)g_{ij} = k\mathrm{T}_{(i)(j)}, \\ {}^1P^i_j &= k\mathrm{T}_{(i)j}, \ \ {}^2P^i_j = -k\mathrm{T}_{i(j)}, \end{aligned}$$

where T_{ij}, $\mathrm{T}_{(i)(j)}$, $\mathrm{T}_{i(j)}$ are d-tensor fields.

3. The notion of Randers metric

The preliminaries theories has a remarkable particular case, that is based on the Randers metric.

$$F(x,y) = \alpha(x,y) + \beta(x,y). \tag{14}$$

The Lagrange space $L^n = (M, L)$ with

$$L(\alpha(x,y), \beta(x,y)) = \hat{F}^2(\alpha(x,y), \beta(x,y)) = (\alpha+\beta)^2. \tag{15}$$

The invariants (2) of Randers metric are given by:

$$\rho = \frac{\alpha+\beta}{\alpha}, \ \rho_0 = 1, \ \rho_{-1} = \frac{1}{\alpha}, \ \rho_{-2} = -\frac{\beta}{\alpha^3}.$$

Using the formula (1), we obtain the fundamental metric tensor g_{ij},

$$g_{ij} = \frac{\alpha+\beta}{\alpha} a_{ij} + b_i b_j + \frac{1}{\alpha}(b_i l_j + b_j l_i) - \frac{\beta}{\alpha^3} l_i l_j. \tag{16}$$

Its contravariant counterpart g^{ij} as:

$$g^{ij} = \frac{1}{\rho}\left(a^{ij} - \frac{(b^i l^j + b^j l^i)}{\alpha+\beta} + \frac{b^2}{(\alpha+\beta)^2} y^i y^j\right). \tag{17}$$

And we know g_{ij} is positively defined if $b^2 < 1$. The Cartan tensor C_{ijk} (3) given by:

$$C_{ijk} = \sigma_{(i,j,k)} \frac{1}{2} \left(\frac{1}{\alpha} a_{ij} b_k - \frac{\beta}{\alpha^3} a_{ij} l_k - \frac{1}{\alpha^3} b_i l_j l_k + \frac{\beta}{\alpha^4} l_i l_j l_k \right). \tag{18}$$

Clearly, we see that $C_{ijk} \neq 0$. Thus, we have

Theorem 5. *The Cartan tensor C_{ijk} of Randers metric is non zero (different from zero).*

Moreover, the Randers metric is not reducible to a Riemannian metric. For this metric (15), the Euler-Lagrange equation in the natural parametrization given by:

$$E_i(\alpha^2) + 2\sigma F_{ij} y^j = 0, \;\; y^i = \frac{dx^i}{ds}, \tag{19}$$

where $\sigma = \frac{\rho_{-1}}{\rho} = \frac{1}{2}\frac{L_\beta}{\rho} = \alpha$.

From Theorem 2, we have the result

Theorem 6. *The Euler-Lagrange equations of Randers metric $L = (\alpha + \beta)^2$ is the natural parametrization $\alpha(x, dx/ds) = 1$ and are given by the Lorentz equation*

$$\frac{d^2x^i}{ds^2} + \gamma^i_{jk}(x) \frac{dx^j}{ds} \frac{dx^k}{ds} = \alpha F^i_j(x) \frac{dx^j}{ds} \tag{20}$$

Thus, the coefficients of the canonical semispray and non-linear connection N are:

$$2G^i(x,y) = \gamma^i_{jk}(x) y^j y^k - \alpha(x,y) F^i_j(x) y^j,$$
$$N^i_j = \gamma^i_{jk}(x) y^k - \alpha F^i_j(x).$$

The weak torsion of N is $L^i_{jk} = 0$ and the metric N-linear connection $D\Gamma(N) = (\bar{L}^i_{jk}, \bar{C}^i_{jk})$ is given in (12) coincide with those of the Cartan connection. Moreover, taking into account that, with respect to the canonical N-linear connection N, we have $\frac{\delta F}{\delta x^i} = 0$.

The torsion tensor of $D\Gamma(N)$ are:

$$\mathrm{T}^i_{jk} = 0, \;\; R^i_{jk}, \;\; C^i_{jk}, \;\; P^i_{jk} = N^i_{jk} - L^i_{kj}, \;\; S^i_{jk} = 0. \tag{21}$$

In the following and using the properties form [2], we get

$$P^i_{jk} y^k = 0, \;\; P^i_{jk} y^j = 0. \tag{22}$$

4. Einstein equations of Lagrange space with Randers metric

Now, we express equation (13) in the basis $(\delta/\delta x^i, \partial/\partial y^i)$, i.e., adapted to the decomposition of $T_u TM$, $u \in TM$ into horizontal and vertical subspaces.

Set $(X_\alpha) = (X_i, X_{(i)})$, where $X_i = \delta/\delta x^i$ and $X_{(i)} = \partial/\partial y^i$. The indices i will run from 1 to $2n$ and (i) will run from $n+1$ to $2n$. The local vector fields (X_a) provides a nonholonomic basis given by

$$[X_b, X_c] = W^a_{bc} X_a,$$

which satisfies the following Vranceanu indentities [12]

$$\sum_{(abc)} [X_a(W^d_{bc}) + W^e ab W^d_{ce}] = 0.$$

Let $D_{X_c} X_b = \Gamma^a_{bc} X_a$, then the basis (X_a) the torsion T of the N-linear connection D has the components:

$$\mathrm{T}^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb} + W^a_{bc}.$$

In the basis (X_a) the curvature R of the N-linear connection D has the components:

$$R^a_{bcd} = X_d \Gamma^a_{bc} - X_c \Gamma^a_{bd} + \Gamma^e_{bc}\Gamma^a_{ed} - \Gamma^e_{bd}\Gamma^a_{ec} + \Gamma^a_{be}\Gamma^e_{cd}.$$

The torsion and curvature components given by:

$$\mathrm{T}(X_c, X_b) = \mathrm{T}^a_{bc} X_a, \;\; R(X_d, X_c)X_b = R^a_{bcd} X_a.$$

In the adapted basis (X_a) the Bianchi identities of D of the form:

$$\sum_{abc}(D_a R^e_{dbc} + R^e_{dav}\mathrm{T}^v_{bc}) = 0,$$
$$\sum_{abc}(D_a \mathrm{T}^d_{bc} + \mathrm{T}^e_{ab}\mathrm{T}^d_{ec} - R^d_{abc}) = 0,$$

where $D_a = DX_a$.

If in these equations the components with respect to $X_i = \delta/\delta x^i$ and $X_{(i)} = \partial/\partial y^i$ are separated, it comes out that among the coefficients Γ^a_{bc}, we have

$$\Gamma^i_{jk} = L^i_{jk}, \;\; \Gamma^{(i)}_{(j)(k)} = C^i_{jk}.$$

This is advantage created by the choice of the basis (X_a) as well as by the fact that D is an N-linear connection.

The set of components T^a_{bc} of the torsion field T splits into following:

$$\left.\begin{array}{c} \mathbb{T}^i_{jk} = \mathrm{T}^i_{jk}, \;\; \mathrm{T}^i_{(j)k} = -C^i_{jk}, \;\; \mathbb{T}^i_{j(k)} = -C^i_{jk}, \;\; \mathbb{T}^{(j)(k)} = 0, \\ \mathbb{T}^{(i)}_{jk} = R^i_{jk}, \;\; \mathbb{T}^{(i)}_{(j)k} = -P^i_{kj}, \;\; \mathbb{T}^{(i)}_{j(k)} = P^i_{jk}, \;\; \mathbb{T}^{(i)}_{(j)(k)} = 0. \end{array}\right\} \tag{23}$$

with respect to the basis (X_a), the Ricci tensor field of the N-linear connection $D\Gamma(N)$ has the components

$$\mathbb{R}_{ij} = Rij, \;\; \mathbb{R}_{(i)j} =^1 P_{ij}, \;\; \mathbb{R}_{i(j)} = -^2 P_{ij}, \;\; \mathbb{R}_{(i)(j)} = S_{ij}.$$

By the pseudo-Riemannian metric G has the components G_{ab} given by:

$$G_{ij} = g_{ij}, \;\; G_{i(j)} = 0, \;\; G_{(i)j} = 0, \;\; G_{(i)(j)} = g_{ij},$$
$$G^{ij} = g^{ij}, \;\; G^{i(j)} = 0, \;\; G^{(i)j} = 0, \;\; G^{(i)(j)} = g^{ij},$$

where g_{ij} and g^{ij} are given in (16) and (17) respectively.

Thus, the tensor field $\mathbb{R}^a_b = G^{ac}R_{cb}$ and the scalar curvature $Sc(D)$ have in the frame X_a the components are

$$\mathbb{R}^i_j = R^i_j, \;\; \mathbb{R}^{(i)}_j =^1 P^i_j, \;\; \mathbb{R}^i_{(j)} =^2 P^i_j, \;\; \mathbb{R}^{(i)}_{(j)} = S^i_j, \;\; Sc(D) = R + S,$$

where $R = g^{ij}R_{ij}$ and $S = g^{ij}S_{ij}$.

Theorem 7. *The Einstein equations of the Largrange space with Randers metric corresponding to the metric N-linear connection $D\Gamma(N) = (L^i_{jk}, C^i_k)$ have the following form:*

$$\left.\begin{array}{l} R_{ij} = \frac{1}{2}(R+S)g_{ij}, \\ {}^1P^i_j = 0, \;\; {}^2P^i_j = 0, \\ S_{ij} = \frac{1}{2}(R+S)g_{ij}, \end{array}\right\} \tag{24}$$

where g_{ij} given in (16).

Proof. Making use of the formulae in Theorem (4), one can shows that from theorem and corresponding d-tensor fields in (12), (21), (22) and (23) are equivalent to get (24). □

In vacuum, which corresponds to the case $T_{ij} = 0$, if we multiply this with $G^{ij} = g^{ij}$ the equation (13) of Randers metric can be written in the form:

$$R_{ij} - \frac{1}{2}Sc(D)G_{ij} = 0, \text{ or } R_{ij} - \frac{1}{2}Sc(D)g_{ij} = 0, \tag{25}$$

which implies that $Sc(D) - nSc(D) = o$. Hence, $Sc(D) = 0$ for $n > 1$. Thus, the equation (25) takes the form $R_{ij} = 0$ and immediately, we obtain the following result:

Lemma 8. *For the vacuum state, the Einstein equations of the Lagrange space with Randers metric corresponding to the metric connection $D\Gamma(N) = (L^i_{jk}, C^i_k)$ are as follows:*

$$R_{ij} = 0, \ S_{ij} = 0, \ {}^1P^i_j = 0, \ {}^2P^i_j = 0. \tag{26}$$

5. Conclusion

The development of the geometry of Lagrange spaces, using the fundamental concepts from Analytical Mechanics as: the integral of action, the Euler-Lagrange equations, the law of conservation of energy and symplectic form etc. The geometry of a Lagrange space is mostly derived from the Euler-Lagrange equations. This paper is devoted to derived the Euler Lagrange equations of Randers metric. Then, by using canonical N-metrical connection, characterized the Einstein equations of Finsler Lagrange space with Randers metric.

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