

Article

# Foreign vector measures

Abalo Douhadji<sup>1,\*</sup> and Yaovi Awussi<sup>2</sup>

<sup>1</sup> Department of Mathematics, University of Lomé, PObox 1515, Lomé, Togo.

<sup>2</sup> Department of Mathematics, Mathematics and Applications Laboratory, University of Lomé, PObox 1515, Lomé, Togo.

\* Correspondence: douhadjiabalo@gmail.com

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**Abstract:** We study the foreign measures in general by proving all operations possibilities with their characteristic relation  $\perp$  and deduce that the set of foreign vector measures is a subset of bounded vector measures; stable par linear combination.

**Keywords:** Vector measure, Haar measure, foreign measure, absolute continuity.

**MSC:** 28C10, 28A05, 22C05, 03E17.

## 1. Introduction

In 2019, Monfort [1] wrote an article on the importance of foreign positive measure. Radon-Nikodym-Lebesgue showed that any positive,  $\sigma$ -finite measure can be decomposed uniquely to two measures; absolutely continue measure and the foreign measure [2–4]. Recently, in [5], Douhadji and Awussi showed that if we take  $K(G, E)$ , the space of continuous function with compact support on  $G$ , any vector measure  $m$  on  $K(G, E)$  decomposes uniquely into:

$$m = m_a + m_s \quad \text{with} \quad m_a \ll \mu \quad \text{and} \quad m_s \perp \mu$$

where  $m_a \ll \mu$  means that  $m_a$  is absolutely continue with respect to Haar measure  $\mu$  [4,6].

In this paper, we prove that the set of foreign vector measures endowed with addition and their characteristic relation  $\perp$  is stable par linear combination. This paper is organized as follows: in Section 2, we give some definitions related to vector measure and foreign measures, then we give some essentials theorems which help us to arrive to the expected theorem. In the Section 3 we present our main result.

## 2. Preliminaries

In this section, we give basic definitions of concepts concerning a vector measure, foreign measure and some important results. We fixed the following notions:

- $E$  and  $F$  are Banach spaces;
- $G$  a locally compact group;
- $K(G, E)$  the space of continuous function with compact support on  $G$ ;
- $M(G)$  the vector space of bounded vector measure;
- $\mathfrak{M}(G)$  the subset of  $M(G)$  composed of foreign vector measures.

Our work in this article is to show the possible operations in the set  $\mathfrak{M}(G)$ .

**Definition 1.** Let  $G$  be a locally compact group and  $K(G; E)$  be the space of  $E$  valued functions with compact support on  $G$ . A vector measure on  $G$  with respect to Banach spaces  $E$  and  $F$  is a linear map:

$$\begin{aligned} m : K(G; E) &\rightarrow F \\ f &\mapsto m(f) \end{aligned}$$

such as  $\forall K$  compact in  $G \exists a_K > 0, \|m(f)\|_F \leq a_K \|f\|_\infty$ , where  $\|\cdot\|_F$  is the norm on Banach spaces  $F$  and  $\|f\|_\infty = \sup\{\|f(t)\|_E, t \in G\}$  is the norm on  $K(G; E)$ .

**Definition 2.** [2] A map  $m$  is called a finitely additive vector measure or simply vector measure if whenever  $A$  and  $B$  are disjoint subset of  $G$  then:

$$m(A \cup B) = m(A) + m(B). \quad (1)$$

If  $(A_i)_{1 \leq i \leq n}$  are pairwise disjoint subsets of  $G$ , we have:

$$m\left(\bigsqcup_{i=1}^{n=\infty} A_i\right) = \sum_{i=1}^{\infty} m(A_i).$$

We note that [2]:

$$m(A) = \int 1_A dm.$$

Let  $m$  be a vector measure. The variation of  $m$  is the nonnegative function  $\|m\|$  whose value on a set  $A \subset G$  is given by:

$$\|m\|(A) = \sup \sum_{B \in \pi} \|m(B)\|, \quad (2)$$

where the supremum is taken over all partitions  $\pi$  of  $A$ .

We know that

$$\|m\|(A) = 0 \Leftrightarrow m(A) = 0. \quad (3)$$

Let  $\mu$  be positive and  $\sigma$  finite and  $\nu$  complex and  $\sigma$  finite. Now, we give the definition of foreign positive measure.

**Definition 3.** [1,3] Two measures  $\nu$  and  $\mu$  are foreign (or singular or separate) if there exists a partition  $(E_1, E_2)$  of  $E$  such as  $|\mu|(E_1) = 0$  and  $|\nu|(E_2) = 0$ . Note that  $\mu \perp \nu$ .

**Proposition 1.** [1,3] Two measures  $\nu$  and  $\mu$  are foreign if there exists measurable set  $A$  such as

$$|\mu|(A) = 0 \text{ and } |\nu|(\bar{A}) = 0. \quad (4)$$

**Proposition 2.** [1,3] Two measures  $\nu$  and  $\mu$  are foreign if there exists measurable set  $A$  such as  $A$  carries  $\mu$  and  $\bar{A}$  carries  $\nu$ .

The following theorem proved in [5] allows us to use the previous properties for vector measures.

**Theorem 1.** Let  $E$  and  $F$  are two Banach spaces and  $G$  a locally compact group. Let  $\nu \in K(G, E)$  be a complex measure which is  $\sigma$ -finite and  $w \in F$  a nonzero vector. The mapping

$$\begin{aligned} m : K(G, E) &\rightarrow F \\ f &\mapsto w\nu(f) \text{ is a vector measure.} \end{aligned}$$

Now, we are able to give the following theorem:

**Theorem 2.** Let  $m_1$  and  $m_2$  belong to  $\mathfrak{M}(G)$ , then

$$m_1 \perp m_2 \iff \exists A \subset G \text{ } \|m_1\|(A) = 0 \text{ and } \|m_2\|(\bar{A}) = 0. \quad (5)$$

**Proof.** Suppose  $m_1 \perp m_2$ , then from Theorem 1, we have  $m_1(f) = w\nu_1(f)$  and  $m_2(f) = w\nu_2(f)$ ;  $\forall f \in K(G; E)$ ;  $w \neq 0$ . So  $\nu_1 \perp \nu_2$ . Now by Proposition 1, there exists  $A \subset G$  such as

$$\nu_1(A) = 0 \text{ and } \nu_2(\bar{A}) = 0,$$

which is equivalent to

$$w\nu_1(A) = 0 \text{ and } w\nu_2(\bar{A}) = 0,$$

which is equivalent to

$$m_1(A) = 0 \text{ and } m_2(\bar{A}) = 0.$$

Now from (2), we have  $\|m_1\|(A) = \sup\{\|m_1(B)\|; B \subset A\}$ , so  $\|m_1\|(A) = 0$ . Similarly we can have  $\|m_2\|(\bar{A}) = 0$ .  $\square$

In light of Theorem 2, we are entitled to define the set  $\mathfrak{M}(G)$  as follows:

$$\mathfrak{M}(G) = \{m \in M(G), \exists A, \forall m' \in M(G), \|m\|(A) = 0 \text{ and } \|m'(A)\| = 0\}.$$

Equivalently

$$\mathfrak{M}(G) = \{m \in M(G) | \forall m' \in M(G), m \perp m'\}.$$

From here we can see that  $\perp$  is the characteristic condition for foreign vector measures.

**Theorem 3.** Any linear combination of vector measures is a vector measure.

**Proof.** To make it simple, let's take two vector measures  $m_1$  and  $m_2$  and show that  $\alpha m_1 + \beta m_2 \in M(G)$  with  $\alpha$  and  $\beta$  two scalars. From Definition 2, we have:

$$m_1 \in M(G) \implies \forall K_1 \text{ compact } \subset G \exists a_{K_1} > 0, \|m_1(f)\|_F \leq a_{K_1} \|f\|_\infty.$$

$$m_2 \in M(G) \implies \forall K_2 \text{ compact } \subset G \exists a_{K_2} > 0, \|m_2(f)\|_F \leq a_{K_2} \|f\|_\infty.$$

Now

$$\begin{aligned} \|(\alpha m_1 + \beta m_2)(f)\|_F &= \|\alpha m_1(f) + \beta m_2(f)\|_F \\ &\leq \|\alpha m_1(f)\|_F + \|\beta m_2(f)\|_F \\ &\leq |\alpha| \|m_1(f)\|_F + |\beta| \|m_2(f)\|_F \\ &\leq |\alpha| a_{K_1} \|f\|_\infty + |\beta| a_{K_2} \|f\|_\infty \\ &\leq (|\alpha| a_{K_1} + |\beta| a_{K_2}) \|f\|_\infty. \end{aligned}$$

By setting  $A_K = |\alpha| a_{K_1} + |\beta| a_{K_2}$  and  $K = K_1 \cap K_2$ , we have

$$\|(\alpha m_1 + \beta m_2)(f)\| \leq A_K \|f\|_\infty, \quad \forall f \in K.$$

Hence the result.  $\square$

### 3. Main Result

In this section, we prove all the operations with the operator  $\perp$  characterizing the foreign vector measures. We take  $\mathfrak{M}(G)$  provided with  $+$  and  $\perp$ .

**Theorem 4.** Let  $m_1, m_2, m_3 \in \mathfrak{M}(G)$ , then

1.  $m_1 \perp m_1 \implies m_1 = 0$ .
2.  $m_1 \perp m_2 \iff m_2 \perp m_1$ .
3.  $m_1 \perp m_2 \iff \|m_1\| \perp \|m_2\|$ .
4.  $m_1 \perp m_3$  and  $m_2 \perp m_3 \implies (\alpha m_1 + \beta m_2) \perp m_3$  for all  $\alpha$  and  $\beta$  scalars.

**Proof.** 1. If  $m_1 \perp m_1$ , then there exist  $A \subset G$  such that  $\|m_1\|(A) = 0$  and  $\|m_1\|(\bar{A}) = 0$ . Suppose  $m_1 \neq 0$ . Since

$$m_1(\bar{A}) = m_1(G \setminus A) = m_1(G) - m_1(A)$$

implies

$$0 = m_1(G) - m_1(A)$$

implies

$$m_1(G) = m_1(A) \neq 0$$

and

$$\|m_1\|(A) \neq 0$$

which is absurd. Hence  $m_1 = 0$ .

2. Suppose that  $m_1 \perp m_2$

$$\begin{aligned} m_1 \perp m_2 &\Leftrightarrow \exists A \subset G \mid \|m_1\|(A) = 0 \text{ and } \|m_2\|(\overline{A}) = 0 \\ &\Leftrightarrow \|m_2\|(\overline{A}) = 0 \text{ and } \|m_1\|(A) = 0 \\ &\Leftrightarrow \|m_2\|(B) = 0 \text{ and } \|m_1\|(\overline{B}) = 0 \quad \text{by setting } B = \overline{A} \\ &\Leftrightarrow m_2 \perp m_1 \end{aligned}$$

3. Suppose that  $m_1 \perp m_2$

$$\begin{aligned} m_1 \perp m_2 &\Leftrightarrow \exists A \subset G \mid \|m_1\|(A) = 0 \text{ and } \|m_2\|(\overline{A}) = 0 \\ &\Leftrightarrow \|m_1\| \perp \|m_2\| \text{ according to (4) Proposition 1.} \end{aligned}$$

4. Suppose that  $m_1 \perp m_3$  and  $m_2 \perp m_3$ ;  $\alpha$  and  $\beta$  scalars

$$\begin{aligned} m_1 \perp m_3 &\Leftrightarrow \exists A \subset G \mid \|m_1\|(A) = 0 \text{ and } \|m_3\|(\overline{A}) = 0 \\ m_2 \perp m_3 &\Leftrightarrow \exists B \subset G \mid \|m_2\|(B) = 0 \text{ and } \|m_3\|(\overline{B}) = 0 \\ m_3(\overline{A \cap B}) &= m_3(\overline{A \cup B}) = m_3(\overline{A}) + m_3(\overline{B}) = 0 \text{ according to the hypothesis.} \end{aligned}$$

Let be  $\alpha \neq 0$  and  $\beta \neq 0$

$$\|\alpha m_1\|(A) = 0 \text{ and } \|\beta m_2\|(B) = 0.$$

Let be  $\Omega \subset A \cap B$  then  $m_1(\Omega) = 0$  and  $m_2(\Omega) = 0$

$$\|\alpha m_1 + \beta m_2\|(A \cap B) = \sup_{\Omega \subset A \cap B} \{ \|(\alpha m_1 + \beta m_2)(\Omega)\| \} = 0.$$

We have:  $\|\alpha m_1 + \beta m_2\|(A \cap B) = 0$  and  $\|m_3\|(\overline{A \cap B}) = 0$ .

□

### Theorem 5.

If  $m_1, m_2, m_3 \in \mathfrak{M}(G)$  and  $m_1 \perp m_3, m_2 \perp m_3$  then  $(\alpha m_1 + \beta m_2) \perp m_3 \forall \alpha$  and  $\beta$  nonzero scalars.

**Proof.** Suppose that  $m_1, m_2, m_3 \in \mathfrak{M}(G)$  and  $m_1 \perp m_3, m_2 \perp m_3$ . Also since  $\alpha$  and  $\beta$  nonzero scalars, so  $\alpha m_1 + \beta m_2 \in \mathfrak{M}(G)$  according to the Theorem 3.

Now, we show that  $\alpha m_1 + \beta m_2 \perp m_3$ , by proving that  $(\alpha m_1 + \beta m_2) \perp m_3$ .

According to the hypothesis

$$m_1 \perp m_3 \Leftrightarrow \exists A \subset G \mid \|m_1\|(A) = 0 \text{ and } \|m_3\|(\overline{A}) = 0,$$

then

$$m_2 \perp m_3 \Leftrightarrow \exists B \subset G \mid \|m_2\|(B) = 0 \text{ and } \|m_3\|(\overline{B}) = 0.$$

We get easily:

$$\begin{aligned} m_3(\overline{A \cap B}) &= m_3(\overline{A \cup B}) \quad (1) \\ &= m_3(\overline{A}) + m_3(\overline{B}) \\ &= 0 \text{ according to the hypothesis.} \\ \Rightarrow \|m_3\|(\overline{A \cap B}) &= 0 \end{aligned} \tag{6}$$

Let  $\alpha \neq 0$  and  $\beta \neq 0$  with  $\|\alpha m_1\|(A) = 0$  and  $\|\beta m_2\|(B) = 0$

Let  $\Omega \subset A \cap B$ , then if

$$\begin{aligned}\Omega \subset A \cap B &\Rightarrow \Omega \subset A \text{ then } m_1(\Omega) = 0, \\ \Omega \subset A \cap B &\Rightarrow \Omega \subset B \text{ then } m_2(\Omega) = 0.\end{aligned}$$

$$\begin{aligned}\|\alpha m_1 + \beta m_2\|(A \cap B) &= \sup\{|\alpha m_1 + \beta m_2(\Omega)|; \Omega \subset A \cap B\} \\ &= \sup\{|\alpha m_1(\Omega) + \beta m_2(\Omega)|; \Omega \subset A \cap B\} \quad (2) \\ &= 0\end{aligned}$$

By choosing  $A \cap B$ , we have

$$\|\alpha m_1 + \beta m_2\|(A \cap B) = 0 \text{ and } \|m_3\|(\overline{A \cap B}) = 0 \text{ (according to (10)).}$$

Then  $(\alpha m_1 + \beta m_3) \perp m_3$ .

□

Even if the characteristic relation  $\perp$  is symmetrical or permutable by Theorem 4(2), we find it necessary to inform you that, it is neither associative nor transitive. Its demonstration is tedious we prefer to offer it in our next article.

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