Stochastic dynamic for an extensible beam equation with localized nonlinear damping and linear memory

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Abstract: In this paper, we concerned to prove the existence of a random attractor for the stochastic dynamical system generated by the extensible beam equation with localized non-linear damping and linear memory defined on bounded domain. First we investigate the existence and uniqueness of solutions, bounded absorbing set, then the asymptotic compactness. Longtime behavior of solutions is analyzed. In particular, in the non-autonomous case, the existence of a random attractor attractors for solutions is achieved.

Keywords: Beam equation, memory, nonlinear damping, Random Dynamical System, random attractor.

MSC: Primary 35Q35, Secondary 35B40, 35B41, 35B45.

1. Introduction

We consider the following extensible beam equation with localized non-linear damping and linear memory on a bounded domain:

\[
\begin{aligned}
\begin{cases}
\frac{\partial^2 u}{\partial t^2} + \Delta^2 u - k(0)(1 + \int_\Omega |\nabla u|^2 dx)\Delta u - \int_0^\infty k'(s)\Delta u(t - s)ds + a(x)g(u_t) + f(u) = q(x, t) + \sum_{j=1}^m h_j \mathcal{W}_j(t), \\
u = \frac{\partial u}{\partial n} = 0, \quad x \in \partial \Omega, \quad t \in \mathbb{R}, \\
u(\tau, x) = u_0(\tau, x), \quad u_t(\tau, x) = u_1(\tau, x), \quad x \in \Gamma, \quad \tau \in \mathbb{R},
\end{cases}
\end{aligned}
\]

where \(\Gamma\) is a bounded domain of \(\mathbb{R}^n\), \(k(0), k(\infty) > 0\) and \(k'(s) \leq 0\) for every \(s \in \mathbb{R}^+\), \(\epsilon\) is a positive constant. The given function \(g(x, t) \in L^2_{loc}(\Omega, L^2(\Gamma))\) is an external force depending on \(t\), \(h_j \in H^2(\Gamma)\) and \(W(t)\) is an independent two sided real-valued Wiener processes on probability space. The function \(a(x)\) satisfies

\[
a(x) \in L^\infty(\Gamma), a(x) \geq a_0 > 0, \quad \text{in } \Gamma
\]

where \(a_0\) is constant. The function \(f \in C^1(\mathbb{R})\) satisfies

\[
\begin{aligned}
(A_1) & : |f'(s)| \leq C_1(1 + |s|^{\gamma - 1}), \forall s \in \mathbb{R}, \\
(A_2) & : \liminf_{|s| \to \infty} \frac{|f(s)|}{s} > -\lambda_1, \\
(A_3) & : F(s) = \int_0^s f(r)dr \geq C_2(|s|^{\gamma + 1} - 1), \\
(A_4) & : sf(s) \geq C_4(F(s) - 1), \\
(A_5) & : C_2(|s|^{\gamma + 1} - 1) \leq F(s) \leq \frac{1}{C_5}(sf(s) + C_3),
\end{aligned}
\]

where \(C_i\) are positive constants \((i = 1, 2, 3, 4)\), \(1 \leq \gamma \leq \frac{n+2}{n-2}, \quad n \geq 3\) and \(\lambda_1\) is the best constant in the Poincaré-type inequality

\[
\lambda_1 \int_\Omega |u|^2 dx \leq \int_\Omega |\nabla u|^2 dx.
\]
The damping function \(g\) satisfies \(|g'(s)| \geq 0\), \(g(s)\) strictly increasing, and
\[
|h(0)| = 0, \quad 0 < \alpha_1 \leq |h'(s)| \leq \alpha_2 < \infty.
\] (4)

As like to \([1,2]\), we define a new variable
\[
\eta(x,t,s) = u(x,t) - u(x,t-s), \quad \eta_t = \frac{\partial}{\partial t} \eta, \quad \eta_s = \frac{\partial}{\partial s} \eta.
\] (5)

Let \(\mu(s) = k'(s)\). Equation (1) transforms into the following system:
\[
\begin{cases}
\begin{align*}
u_{tt} + \Delta^2 u - (1 + k(0)) \int_\Omega |\nabla u|^2 \, dx \Delta u - \int_0^\infty \mu(s) \Delta \eta(s) \, ds + a(x) g(u(t)) + f(u) &= q(x,t) + \kappa \sum_{j=1}^m h_j \bar{W}_j; \\
\eta_t &= -\eta_s + u_t; \\
u(t,x) &= 0, \quad x \in \partial \Gamma, \quad t > 0; \\
\eta(x,t,s) &= 0, \quad x \in \partial \Gamma, \quad t > 0, \quad s \in \mathbb{R}^+; \\
u(\tau,x) &= u_0(x), \quad u_t(\tau,x) = u_1(x), \quad x \in \Gamma; \\
\eta(\tau,x,s) &= \eta_0(x,s) = u_0(x) - u_0(x,-s), \quad x \in \Gamma, \quad s \in \mathbb{R}^+.
\end{align*}
\end{cases}
\] (6)

The following hypotheses are necessary to obtain our main results, infer to \([3–5]\).

(a) The memory kernel \(\mu\) is required to satisfy the following hypotheses hold:
\[
\begin{align*}
(H_1) & : \quad \mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \\
(H_2) & : \quad \mu(s) \geq 0, \quad \mu'(s) \leq 0, \quad \forall s \in \mathbb{R}^+, \\
(H_3) & : \quad \mu'(s) + k_1 \mu(s) \leq 0, \quad \forall s \in \mathbb{R}^+ \text{ and } \sigma > 0, \\
(H_4) & : \quad m_0 := \int_0^\infty \mu(s) \, ds < \infty.
\end{align*}
\] (7)

(b) We need the following condition on \(q(x,t) \in L^2_{\text{loc}}(\mathbb{R}, L^2(\Gamma))\), there exists a positive constant \(\sigma\) satisfy that
\[
\begin{align*}
(Q_1) & : \quad \int_{-\infty}^\infty e^{\sigma r} \|q(\cdot, r)\|^2 \, dr < \infty, \quad \forall \quad r \in \mathbb{R}, \\
(Q_2) & : \quad \|q(x,t)\|^2 = \sup_{r \in \mathbb{R}} \int \|q(\cdot, r)\|^2 \, ds < \infty \quad \forall \quad r \in \mathbb{R}, \\
(Q_3) & : \quad \lim_{k \to \infty} \int_{-\infty}^\infty \int_{|x| \geq k} e^{\sigma r} |g(x,r)|^2 \, dx \, dr = 0, \quad \forall \tau \in \mathbb{R}.
\end{align*}
\] (8)

The basic concepts and notions of random attractors for the infinite dimensional was recently presented by in \([6–9]\). A random attractor of RDS is a measurable and compact invariant random set attracting all orbits. whilst such an attracting set exists, it is the smallest attracting compact set and the largest invariant set. In recent years, a random attractor for autonomous and non-autonomous stochastic dynamical systems have been studied by many authors, see for example \([10–16]\) and the references therein.

In the deterministic case; that is, \(k = 0\) in (1), the asymptotic behavior of the solution for global attractors an extensible beam equation with localized nonlinear damping with memory has been studied in \([5,17–19]\).

In \([20]\), for the case of \(\mu = 0\) in (1), the authors investigated the existence of random attractor for the stochastic an extensible beam equation with localized nonlinear damping without memory. But, there were no results even for the bounded case. While it is far just our interest in this paper. To the best of our knowledge, the dynamics of system (1) involving but essential difficulties in showing compactness by using the uniform estimates on the tails of solution. Motivated by a similar technique of \([16]\).

The rest of the paper is organized as follows. In Section 2, we recall some basic concepts and properties for general random dynamical systems. In Section 3, we first provide some basic settings about (1) and show that it generates a random dynamical system in proper function space. In Section 4, we prove the existence of a unique random attractor of the random dynamical system by bounded absorbing set and using a compact measurable pullback attracting set.
2. Preliminaries

In this section, we recall some basic concepts related to random attractors for stochastic dynamical systems. The readers are referred to [6–8] for more details. Which are crucial for getting our main results.

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \((X, d)\) be a Polish space with the Borel \(\sigma\)-algebra \(\mathcal{B}(X)\). The distance between \(x \in X\) and \(B \subseteq X\) is denoted by \(d(x, B)\). If \(B \subseteq X\) and \(C \subseteq X\), the Hausdorff semi-distance from \(B\) to \(C\) is denoted by way of \(d(B, C) = \sup_{x \in B} d(x, C)\).

**Definition 2.** Let \(\Xi\) be a collection of random subset of \(X\). If for all \(\tau \in \mathbb{R}, \omega \in \Omega\) and \(t, s \in \mathbb{R}^{+}\), the following conditions are satisfied:

i) \(\Phi(t, \tau, \omega, x) : \mathbb{R}^{+} \times \mathbb{R} \times \Omega \times X \to X\) is a \((\mathcal{B}(\mathbb{R}^{+}) \times \mathcal{F}, \mathcal{B}(\mathbb{R}))\) measurable mapping,

ii) \(\Phi(0, \tau, \omega, x)\) is identity on \(X\),

iii) \(\Phi(t + s, \tau, \omega, x) = \Phi(t, \tau + s, \theta_{s}(\omega), x) \circ \Phi(s, \tau, \omega, x)\),

iv) \(\Phi(t, \tau, \omega, x) : X \to X\) is continuous.

**Definition 3.** Let \(2^{X}\) be the collection of all subsets of \(X\), a set valued mapping \((\tau, \omega) \mapsto \mathcal{D}(t, \omega) : \mathbb{R} \times \Omega \to 2^{X}\) is called measurable with respect to \(\mathcal{F}\) in \(\Omega\) if \(\mathcal{D}(t, \omega)\) is a (usually closed) nonempty subset of \(X\) and the mapping \(\omega \in \Omega \mapsto d(X, B(\tau, \omega))\) is \((\mathcal{F}, \mathcal{B}(\mathbb{R}))\)-measurable for every fixed \(x \in X\) and \(\tau \in \mathbb{R}\). Let \(B = B(t, \omega) \in \mathcal{D}(t, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\) is called a random set.

**Definition 4.** A random bounded set \(B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}\) of \(X\) is called tempered with respect to \(\{\theta(t)\}_{t \in \Omega}\), if for \(p\)-a.e \(\omega \in \Omega\),

\[
\lim_{t \to \infty} e^{-\beta t} d(B(\theta_{-t}\omega)) = 0, \forall \beta > 0,
\]

where \(d(B) = \sup_{x \in B} \|x\|_{X}\).

**Definition 5.** Let \(\mathcal{D}\) be a collection of random subset of \(X\) and \(K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}\), then \(K\) is called an absorbing set of \(\Phi \in \mathcal{D}\) if for all \(\tau \in \mathbb{R}, \omega \in \Omega\) and \(B \in \mathcal{D}\), there exists, \(T = T(\tau, \omega, B) > 0\) such that

\[
\Phi(t, \tau, \theta_{-\omega} B(\tau, \theta_{-\omega})) \subseteq K(\tau, \omega), \forall t \geq T.
\]

**Definition 6.** Let \(\mathcal{D}\) be a collection of random subset of \(X\), the \(\Phi\) is said to be \(\mathcal{D}\)-pullback asymptotically compact in \(X\) if for \(p\)-a.e \(\omega \in \Omega\), \(\Phi(t_{n}, \theta_{-t_{n}} \omega, x_{n}))_{n=1}^{\infty}\) has a convergent subsequence in \(X\) when \(t_{n} \to \infty\) and \(x_{n} \in B(\theta_{-t_{n}} \omega)\) with \(\{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}\).

**Definition 7.** Let \(\mathcal{D}\) be a collection of random subset of \(X\) and \(\mathcal{A} = \{A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}\), then \(\mathcal{A}\) is called a \(\mathcal{D}\)-random attractor (or \(\mathcal{D}\)-pullback attractor) for \(\Phi\), if the following conditions are satisfied for all \(t \in \mathbb{R}^{+}, \tau \in \mathbb{R}\) and \(\omega \in \Omega\),

i) \(A(\tau, \omega)\) is compact, and \(\omega \mapsto d(x, A(\omega))\) is measurable for every \(x \in X\),

ii) \(A(\tau, \omega)\) is invariant, that is \(\Phi(t, \tau, \omega, A(\tau, \omega)) = A(t + \tau, \theta_{\tau}(\omega)), \forall t \geq \tau\),

iii) \(A(\tau, \omega)\) attracts every set in \(\mathcal{D}\), that is for every \(B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}\),

\[
\lim_{t \to \infty} d_{X}(\Phi(t, \tau, \theta_{-\omega} B(\tau, \theta_{-\omega})), A(\tau, \omega)) = 0,
\]

where \(d_{X}\) is the Hausdorff semi-distance given by \(d_{X}(Y, Z) = \sup_{y \in Y} \inf_{z \in Z} \|y - z\|_{X}\) for any \(Y \in X\) and \(Z \in X\).

**Lemma 1.** Let \(\mathcal{D}\) be a neighborhood-closed collection of \((\tau, \omega)\)-parameterized families of nonempty subsets of \(X\) and \(\Phi\) be a continuous cocycle on \(X\) over \(\mathbb{R}\) and \((\Omega, \mathcal{F}, \mathbb{P}, \{\theta_{t}\}_{t \in \mathbb{R}})\). Then \(\Phi\) has a pullback \(\mathcal{D}\)-attractor \(A \in \mathcal{D}\) if and only if \(\Phi\) is pullback \(\mathcal{D}\)-asymptotically compact in \(X\) and \(\Phi\) has a closed, \(\mathcal{F}\)-measurable pullback \(\mathcal{D}\)-absorbing set \(K \in \mathcal{D}\), the unique pullback \(\mathcal{D}\)-attractor \(\mathcal{A} = A(\tau, \omega)\) is given

\[
A(\tau, \omega) = \bigcap_{\mathcal{D}} \bigcup_{t \geq 0} \Phi(t, \tau - t, \theta_{-t}(\omega), K(\tau - t, \theta_{-t}(\omega))) \tau \in \mathbb{R}, \omega \in \Omega.
\]
3. Existence and uniqueness of solution

In this Section, first, we collect some important results that will help to achieve our goal. Let $A = \Delta^2$, $A^1 = -\Delta$ and $D(A) = \{u \in H^4 : \Delta u \in H^2_0\}$. We can define the powers $A^\nu$ is Hilbert space and a norm hold $D(A^\nu) = V_\nu = \|A^\nu u\|_2, v \in \mathbb{R}$. Especially, $V_0 \hookrightarrow L^2$ and $V_1 \hookrightarrow H^2 \cap H^1_0$. We denote that the injection $V_{v_1} \hookrightarrow V_{v_2}$ is compact embeddings, if $v_1 > v_2$ in conjunction with the generalized Poincaré inequality;

$$\|u\|_{v+1}^4 \geq \lambda_1 \|u\|_{v}^4,$$

where $\lambda_1$ is the first eigenvalue of $A$. Additionally we outline the subsequent

$$\begin{cases}
(u, v) = \int_{\Omega} uv \, dx = \|u\| \|v\|, \\
(u, u) = \|u\|^2, \\
((u, v)) = \int_{\Omega} \Delta u \Delta v \, dx = \|\Delta u\| \|\Delta v\|, \\
((u, u)) = \int_{\Omega} \Delta u \Delta u \, dx = \|\Delta u\|^2.
\end{cases}$$

(9)

Much like [18], for the memory kernel hypotheses $\mu(\cdot)$, we suppose $L^2_{\mu}(\mathbb{R}^+; V_\nu)$ the Hilbert space of function $\eta : \mathbb{R}^+ \rightarrow V_\nu$ endowed with the inner product and norm respectively,

$$\begin{cases}
(u, v)_{\mu, \nu} = \int_0^\infty \mu(s)(A^\nu u(s), A^\nu v(s)) \, ds, \\
\langle \eta_1, \eta_2 \rangle_{\mu, \nu} = \int_0^\infty \mu(s)(A^\nu \eta_1(s), A^\nu \eta_2(s)) \, ds, \\
\|\eta\|_{\mu, \nu}^2 = (A^\nu \eta, A^\nu \eta)_{\mu} = \int_0^\infty \mu(s)\|\eta\|^2 \, ds,
\end{cases}$$

(10)

specially, $\|u\|_{\mu, \nu}^2 = \|u\|_{\mu, \nu}^2$. Let, we define the product Hilbert space $E = V_0 \times V_1 \times L^2_{\mu}(\mathbb{R}^+; V_1)$.

To convert the version of Problem (6) with a random perturbation term right into a deterministic one with a random parameter $\omega$, we introduce an Ornstein-Uhlenbeck process driven by means of the Brownian motion, which satisfies the subsequent differential equation

$$dz_j + \delta z_j dt = dW_j(t),$$

(11)

Its unique stationary solution is given by

$$z_j(\theta_1 \omega_j) = -\delta \int_0^z e^{\delta s}(\theta_1 \omega_j)(s) \, ds, \quad s \in \mathbb{R}, \ t \in \mathbb{R}, \ \omega_j \in \Omega.$$  

(12)

From [6,16], it is recognized that the random variable $|z_j(\omega_j)|$ is tempered and there is an invariant set $\Omega \subseteq \Omega$ of full $P$ measure such that $z_j(\theta_1 \omega_j) = z_j(t, \omega)$ is continuous in $t$, for each $\omega \in \Omega$. For comfort, we shall write $\Omega$ as $\Omega$. It follows from Proposition 3.4 in [16], that for any $\epsilon > 0$, there exists a tempered characteristic $\gamma(\omega) > 0$ such that

$$\sum_{j=1}^m (|z_j(\omega_j)|^2 + |z_j(\omega_j)|^{\gamma + 2}) \leq \gamma(\omega),$$

(13)

where $\gamma(\omega)$ satisfies for, p-a.e. $\omega \in \Omega$,

$$\gamma(\theta_1 \omega) \leq e^{\epsilon |t|} \gamma(\omega), \ t \in \mathbb{R}.$$  

(14)

Then, it follows from the above inequality, for p-a.e. $\omega \in \Omega$,

$$\sum_{j=1}^m (|z_j(\theta_1 \omega_j)|^2 + |z_j(\theta_1 \omega_j)|^{\gamma + 2}) \leq e^{\epsilon |t|} \gamma(\omega), \ t \in \mathbb{R}.$$  

(15)

Put $\kappa h(x)z(\theta_1 \omega) = \kappa \sum_{j=1}^m h_j z_j(\theta_1 \omega_j)$, which solves $dz + \delta z \, dt = \sum_{j=1}^m h_j W_j(t)$.

Let $v(t, \tau, x, \omega) = u_t + eu - \kappa h(x)z(\theta_1 \omega)$, we handy to reduce (6) to an evolution equation of the first-order in time random partial differential equation (RPDE):
\[
\begin{aligned}
&\left\{
\begin{array}{ll}
\eta_t + h(x)z(\theta_t \omega), \\
\eta_s = -\nu u + v + kh(x)z(\theta_t \omega), \\
u_t = -\nu u + v + Au + \int_0^\infty \mu(s)A_i^2 \eta(s) ds = (1 + k(0)) \int_\Omega |\nabla u|^2 dx A_i^2 u - a(x)g(u_t) - f(u) + g(x, t) + \varepsilon kh(x)z(\theta_t \omega),
\end{array}
\right.
\end{aligned}
\]

Consequently the stochastic system for the system (16) becomes

\[
\begin{aligned}
&\left\{
\begin{array}{ll}
\psi' + H(\psi) = Q(\psi, t, \omega), \\
\psi(\tau, \omega) = (u_0(x), u_1(x) + \varepsilon u_0(x) - kh(x)z(\theta_t \omega), \eta_0)^T, \\
\end{array}
\right.
\end{aligned}
\]

in which \(\psi = (u \quad \nu \quad \eta \quad \varepsilon u - v)

\[
H(\psi) = \begin{bmatrix}
h(x)z(\theta_t \omega) \\
-\nu u + v + Au + \int_0^\infty \mu(s)A_i^2 \eta(s) ds
\end{bmatrix}
\]

and \(Q(\psi, \omega, t) = \begin{bmatrix}
kh(x)z(\theta_t \omega) \\
-\nu u + v + \eta_s
\end{bmatrix}
\]

By [21], we have the fact that \(H
\]
is the infinitesimal generator of \(C^0\)-semigroup \(e^{ht}\) on \(E(\Gamma)\). It is not difficult to check that the function \(Q(\psi, \omega, t) : E \to E\) is locally Lipschitz continuous with respect to \(\psi\) and bounded for each \(\omega \in \Omega\).

In order to obtain the random attractor of the Problem (17) has a unique solution in the mild sense, by the classical semigroup theory of existence and uniqueness of solutions of evolution differential equations [21], we get the following result.

**Theorem 1.** Let (2)-(5) and (7)-(8) hold. Then, for every \(t \in \mathbb{R}, \omega \in \Omega\) and \(\chi_t \in E(\Gamma)\), the Problem (17) has a unique solution \(\chi(t, \tau, \omega, \chi_t)\) which is continuous with respect to \((u_0, v_0, \eta_0)^T \in E(\Gamma)\) such that \(\chi_t\) and \(\chi(t, \tau, \omega, \chi_t)\) satisfies the integral equation

\[
\chi(t, \tau, \omega, \chi_t) = e^{\mathcal{H}(t)} \chi(t, \tau, \omega, \chi_t) + \int_0^t e^{\mathcal{H}(t-s)} Q(\chi, \tau, \omega, \chi_t) ds.
\]

Moreover, \(\chi(t, \tau, \omega, \chi_t)\) is continuous in \(\chi_t\) and measurable in \(\omega\).

**Theorem 2.** Let (2)-(4) and (7)-(8) hold. Then, for any \(t \in \mathbb{R}, \omega \in \Omega\) and \(\chi_t \in E(\Gamma)\), such that \(\chi(t, \tau, \omega, \chi_t) \in E(\Gamma)\) is a solution of the Problem (17) satisfy the properties of continuous random dynamical system over \(\mathbb{R}\) and \((\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})\). We can show that for P-a.s. every \(\omega \in \Omega\), for all \(T > 0\)

1. \(\chi(t, \omega) \in E\), then \(\chi(T, \omega, \chi_t) = \chi(T, \omega, \chi_t) \in C([T, T + T]; E)\),
2. \(\chi(t, \tau, \omega, \chi_t)\) is jointly continuous into \(t\) and measurable in \(\chi_t(\omega)\),
3. the solution mapping of (18) holds the properties of continuous cocycle.

From the Theorem 1, we can define a continuous random dynamical system over \(\mathbb{R}\) and \((\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})\), that is, \(\Phi(t, \tau, \omega, \chi_t) : \mathbb{R} \times \mathbb{R}^+ \times \Omega \times E \to E\), \(t \geq \tau\), such that

\[
\begin{aligned}
&\Phi(t, \tau, \theta^{-\tau} \omega, \chi_t(\theta^{-\tau} \omega)) = \chi(t, \tau, \theta^{-\tau} \omega, \chi_t(\theta^{-\tau} \omega)), \\
&\Phi(0, \tau, \theta^{-\tau} \omega, \chi_t(\theta^{-\tau} \omega)) = \chi(t, \tau, \theta^{-\tau} \omega, \chi_t(\theta^{-\tau} \omega)), \\
&\Phi(t, \tau, \theta^{-\tau} \omega, \chi_t(\theta^{-\tau} \omega)) = \chi(t, \tau, \theta^{-\tau} \omega, \chi_t(\theta^{-\tau} \omega)),
\end{aligned}
\]

It generates a random dynamical system. Moreover

\[
\begin{aligned}
&\Phi(t, \tau, \theta^{-\tau} \omega, \chi_t(\theta^{-\tau} \omega)) = \chi(t, \tau, \theta^{-\tau} \omega, \chi_t(\theta^{-\tau} \omega)) + (0, kh(x)z(\theta_t \omega), 0)^T, \\
&\Phi(t, \tau, \theta^{-\tau} \omega, \chi_t(\theta^{-\tau} \omega)) = \phi(t, \omega) + (0, kh(x)z(\theta_t \omega), 0)^T.
\end{aligned}
\]
To show the conjugation of the solution for the stochastic partial differential Equation (17) and the random partial differential Equation (19), introducing the homeomorphism $P(\theta t\omega)(y, w, \zeta(s)) = (y, w - ey + kh(x)z(\theta t\omega), \zeta(s)) \in E(\Gamma)$ with an inverse homeomorphism $P^{-1}(\theta t\omega)(y, w, \zeta(s)) = (y, w + ey - k\zeta(\theta t\omega), \zeta(s))$, then we have the transformation

$$\hat{\Phi}(\tau, t, \omega) = P(\theta t\omega)\Phi(t, \omega)P^{-1}(\theta t\omega), E \mapsto E, t \geq \tau. \quad (21)$$

Consider the equivalent RDS and introduce the isomorphism and has the inverse isomorphism:

$$\begin{cases}
\Phi(t, \omega) = T_{\tau}\Phi(t, \omega)T_{-\tau} : \chi_{\tau} \mapsto \varphi(t + \tau, \tau - \tau\omega, \chi_{\tau}(\theta t\omega)), \\
q' + H(q) = \tilde{Q}(q, t, \omega), \\
q(t, \omega) = q_{\tau} = (u_0(x), y_1(x) - \varepsilon y_0(x), \eta_0)^\top,
\end{cases} \quad (22)$$

where

$$q = (y, w, \eta)^\top = (y, y_1 + ey, \eta)^\top,$$

$$T_{\tau}q = (y, w, \eta)^\top = (y, y_1 + ey, \eta)^\top,$$

$$T_{-\tau}q = (y, w, \eta)^\top = (y, w + ey, \eta)^\top,$$

$$H(q) = \begin{pmatrix}
-\varepsilon w + \varepsilon^2 y + Ay + \int_0^\infty \mu(s)\mathcal{A}^2 \eta(s)ds \\
\varepsilon y - w + \eta s
\end{pmatrix},$$

and

$$\tilde{Q}(q, \omega, t) = \begin{pmatrix}
(1 + k(0) \int_\Omega |\nabla y|^2 dx)\Delta y - a(y_1)g(y_1) - f(y) + g(x, t) + kh(x)z(\theta t\omega) \\
0
\end{pmatrix}$$

is also a random dynamical systems corresponding to the Equation (17). Therefore, $\Phi, \hat{\Phi}$ and $\tilde{\Phi}$ are equivalent to each other in dynamics.

4. Random absorbing set

In this section, we will show boundedness of the solutions for Equation (17). The existence of a pullback absorbing set $\Phi \in \mathcal{D}$ and the asymptotic compactness of the random dynamical system associated with the Equation (17). We always assume that $\mathcal{D}$ is the collection of all tempered subsets of $E(\Gamma)$ from now on.

**Lemma 2.** Let (2)-(4) and (7)-(8) hold. Then, for any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\chi_{T-\tau} \in E(\Gamma)$, there exists a random ball $\{K(\omega)\}_{\omega \in \Omega} \subset \mathcal{D}$ centered at $0$ with random radius $M(\omega) \geq 0$ such that $\{K(\omega)\}$ is a random absorbing set for $\Phi$ in $\mathcal{D}$, that is, for any $B = \{B(\omega)\}_{\omega \in \Omega} \subset \mathcal{D}$, P-almost surely, there exists a $T = T(\tau, \omega, B) > 0$ and $\chi_{T-\tau}(\omega) \in B(\omega)$ such that

$$\|\chi(r, \tau - t, \theta^{-\tau}\omega, \chi_{T-\tau})\|_{E}^2 \leq M_0(\omega), \quad (23)$$

where $M_0(\omega)$ is a positive random function, that is

$$\Phi(t, \tau, \theta^{-\tau}\omega, B(\tau, \theta^{-\tau}\omega)) \subseteq K(\tau, \omega) \text{ for all } t > T. \quad (24)$$

**Proof.** Taking the inner product of the first term of (23) with $\chi = (u, v, \eta) \in E$, $v = \frac{du}{dt} + \varepsilon u - k\chi(x)z(\theta t\omega)$, we find that

$$(\chi', \chi) + (H(\chi), \chi) = (\mathcal{F}(t, x, \chi), \chi). \quad (25)$$

Using Hölder, Young and Poincaré inequalities and after simple computation, we gain

$$H(\chi) = \begin{pmatrix}
\varepsilon u - v \\
-\varepsilon v + \varepsilon^2 u + Au + \int_0^\infty \mu(s)\mathcal{A}^2 \eta(s)ds \\
\varepsilon u - v + \eta s
\end{pmatrix} = \varepsilon \|Au\|^2 + \varepsilon^2 (u, v) - \varepsilon \|v\|^2 + (\varepsilon u + \eta s, \eta)$$
and from (2) and (4), it is easy to show that

\[
\frac{\partial}{\partial \tau}t^2(u, v) - \epsilon ||v||^2 - \frac{\delta}{4} ||\nabla \eta||^2 - \frac{m_0\epsilon^2}{2\lambda} ||\Delta u||^2 + \frac{\delta}{2} ||\nabla \eta||^2.
\]

Using Cauchy-Schwartz inequality and Young inequality, we obtain

\[
(F(t, x, \chi), \chi) = \left(1 + k(0) \int_{\Omega} |\nabla g|^2 dx \right) A \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tau} t^2(u, v) - a(x)g(\frac{\partial}{\partial \tau} t^2(u, v) + f(u) + g(x, t) + kh(x)z(\theta_\omega)) \right) \left( \begin{array}{c}
u \\ \eta \end{array} \right).
\]

From (3) and (4), we obtain

\[
- \left(1 + k(0) ||\nabla u||^2 \right) \Delta u, v = - \left(1 + k(0) ||\nabla u||^2 \right) \nabla, \nabla \left( \frac{d}{dt} + \epsilon u - \alpha \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tau} t^2(u, v) \right)
\]

and from (2) and (4), it is easy to show that

\[
(a(x)g(u_1), v) = (a_0 g(\theta) \cdot v - \epsilon u - k h(x)z(\theta_\omega)) - g(0), \nu \leq a_0 a_1 ||v||^2 + a_0 \left(-a_2 \epsilon u + a' \cdot k h(x)z(\theta_\omega) \right),
\]

where \( \theta \) is between 0 and \( v = \epsilon u + k h(x)z(\theta_\omega) \).

\[
(q(x, t), v) = ||q(x, t)|| ||v|| \leq \frac{||q(x, t)||^2}{2(a_0, a_1 - \epsilon)} + \frac{a_0 a_1 - \epsilon}{2} ||v||^2,
\]

\[
((k h(x)z(\theta_\omega), u)) \leq ||\Delta u|| ||\Delta h(x)|| ||z(\theta_\omega)|| \leq \frac{\epsilon}{4} ||\Delta u||^2 + \frac{k^2}{\epsilon} ||\Delta h(x)||^2 ||z(\theta_\omega)||^2,
\]

\[
(k h(x)z(\theta_\omega), \eta) \leq \frac{m_0 k_0^2}{\epsilon} ||\nabla h(x)||^2 ||z(\theta_\omega)||^2 + \frac{\delta}{4} ||\nabla \eta||^2,
\]

\[
\left(a_0 g' + 2\epsilon k h(x)z(\theta_\omega), v \right) \leq \frac{2a_0^2 k_0^2}{\epsilon} ||h(x)||^2 ||z(\theta_\omega)||^2 + \frac{a_0 a_1 - \epsilon}{8} ||v||^2.
\]

By second term for right hand side of (26) and (29), we can get

\[
\epsilon (\epsilon - a_2 a_0) (u, v) \geq \frac{a_0 a_2 \epsilon}{\lambda} ||\nabla u|| ||v|| \geq \frac{2a_0^2 a_2^2 \epsilon^2}{(a_0 a_1 - \epsilon)^2} ||\nabla u||^2 + \frac{a_0 a_1 - \epsilon}{8} ||v||^2.
\]

About the nonlinearity, by (4), \(H\)older inequality and the Sobolev embedding theorem, we estimate that

\[
(f(u), \chi) = (f(u) + \epsilon \frac{d}{dt} F(u) + \epsilon C_3 (F(u) - ||\nu||) + (f(u), kh(x)z(\theta_\omega))) \geq \frac{d}{dt} F(u) + \epsilon C_3 (F(u) - ||\nu||).
\]

From (3) and (4), we have

\[
(f(u), kh(x)z(\theta_\omega)) \leq C_1 \int_M (1 + ||\nu||^2) kh(x)z(\theta_\omega) dx
\]

\[
\leq C_1 k ||h(x)|| ||z(\theta_\omega)|| + C_1 k \int_M (1 + ||\nu||^2) ||h(x)||_{L^{\gamma+1}(M)} ||z(\theta_\omega)||_{L^{\gamma+1}(M)}
\]

\[
\leq C_1 k ||h(x)|| ||z(\theta_\omega)|| + C_1 k \int_M \frac{1}{C_2} \frac{d}{dt} F(u) + \frac{1}{C_2} \frac{d}{dt} F(u) + \frac{1}{C_2} \frac{d}{dt} ||h(x)||_{L^{\gamma+1}(M)} ||z(\theta_\omega)||_{L^{\gamma+1}(M)}
\]

\[
\leq C_1 k ||h(x)|| ||z(\theta_\omega)|| + C_1 k \frac{1}{C_2} ||\nu|| + \frac{C_1}{2C_2} ||\nu||^2 + \frac{C_1}{2C_2} ||\nu||^2 + \frac{C_1}{2C_2} ||\nu||^2 + \frac{C_1}{2C_2} ||\nu||^2.
\]
Inserting the above two inequalities together, it yields that
\[
(f(u), v) = \frac{d}{dt} F(u) + \frac{\varepsilon}{2} \left( 2C_3 - C_1 C_2^{-1} \right) (F(u) - |\Gamma|) + C_1 \kappa \| h(x) \| |z(\theta_{r-\tau} \omega) + C_1 \tau^{\gamma+1} H_{h_{\tau}(\Gamma)} \|z(\theta_{r-\tau} \omega)\|^{\gamma+1},
\]
(37)
Collecting all inequalities (25)–(37), it leads to
\[
\frac{d}{dt} \left( \| v \|^2 + \| \nabla u \|^2 + (1 + k(0) \| \nabla u \|^2) \| \nabla u \|^2 + \frac{\delta}{4} \| \nabla w \|^2 - \frac{\varepsilon}{2} \left( 2C_3 - C_1 C_2^{-1} \right) \int_{\Gamma} F(u) dx \right)
\]
\[
+ \frac{\alpha_0 a_1 - \varepsilon}{2} \| v \|^2 + 2(1 + k(0) \| \nabla u \|^2) \| \nabla u \|^2 + 2\varepsilon (\frac{\alpha_0 a_1 - \varepsilon}{\alpha_0 a_1 - \varepsilon})^2 \| \nabla u \|^2 + \frac{\delta}{4} \| \nabla \eta \| \|^2
\]
\[
\leq \frac{\| q(x, t) \|^2}{(\alpha_0 a_1 - \varepsilon)} + C_1 \kappa \| h(x) \| |z(\theta_{r-\tau} \omega) + C_1 \tau^{\gamma+1} H_{h_{\tau}(\Gamma)} \|z(\theta_{r-\tau} \omega)\|^{\gamma+1}
\]
\[+ \kappa^2 \| \nabla h(x) \|^2 |z(\theta_{r-\tau} \omega)\| + \frac{\varepsilon}{2} \left( 2C_3 - C_1 C_2^{-1} \right) |\Gamma|.
\]
(38)
Thus
\[
\| \varphi \|^2_{E(\Gamma)} = \| v \|^2 + \| \nabla u \|^2 + (1 + k(0) \| \nabla u \|^2) \| \nabla u \|^2 + \frac{\delta}{4} \| \nabla \eta \|^2 - \frac{\varepsilon}{2} \left( 2C_3 - C_1 C_2^{-1} \right) \int_{\Gamma} F(u) dx,
\]
(39)
and
\[
\varphi(\theta_{r-\tau} \omega) = \frac{\| q(x, t) \|^2}{(\alpha_0 a_1 - \varepsilon)} + C_1 \kappa \| h(x) \| |z(\theta_{r-\tau} \omega) + C_1 \tau^{\gamma+1} H_{h_{\tau}(\Gamma)} \|z(\theta_{r-\tau} \omega)\|^{\gamma+1}
\]
\[+ \kappa^2 \| \nabla h(x) \|^2 |z(\theta_{r-\tau} \omega)\| + \frac{\varepsilon}{2} \left( 2C_3 - C_1 C_2^{-1} \right) |\Gamma|.
\]
(40)
Since \( \varepsilon \in (0, 1) \) be small enough such that \( \varepsilon^2 \left( m - \frac{\alpha_0 a_1 - \varepsilon}{\alpha_0 a_1 - \varepsilon} \right)^2 > 0, \frac{\alpha_0 a_1 - \varepsilon}{\alpha_0 a_1 - \varepsilon} > 0 \), we will choose \( \sigma = \left( \frac{\alpha_0 a_1 - \varepsilon}{\alpha_0 a_1 - \varepsilon}, 2\varepsilon (m - \frac{\alpha_0 a_1 - \varepsilon}{\alpha_0 a_1 - \varepsilon} \right)^2 \) and \( \bar{\sigma} = \min \{ \sigma, \frac{\varepsilon}{2} \left( 2C_3 - C_1 C_2^{-1} \right) \}, \)
(41)
Applying Gronwall’s Lemma over \( [\tau - t, r] \), we find that for \( r \geq \tau - t \),
\[
\| \varphi(r, \tau - t, \omega, \varphi_{\tau-1}(\omega)) \|^2_{E} \leq e^{-\bar{\sigma} (t - \tau)} \| \varphi_{\tau-1} \|^2_{E} + \int_{\tau - t}^{\tau} \varphi(\theta_{\tau-\tau}(\omega)) e^{-\bar{\sigma} (t - \tau)} d\xi.
\]
(42)
By replacing \( \omega \) by \( \theta_{\tau-\omega} \), we get from (42) such that for all \( t \geq 0 \)
\[
\| \varphi(r, \tau - t, \theta_{\tau-\omega}, \varphi_{\tau-1}(\theta_{\tau-\omega})) \|^2_{E} \leq \| \chi(\tau - t, \theta_{\tau-\omega}, \varphi_{\tau-1}(\theta_{\tau-\omega})) \|^2_{E}
\]
\[
\leq e^{-\bar{\sigma} t} \| \chi(\tau - t, \theta_{\tau-\omega}, \varphi_{\tau-1}(\theta_{\tau-\omega})) \|^2_{E} + \int_{\tau - t}^{\tau} \varphi(\theta_{\tau-\tau}(\omega)) e^{\bar{\sigma} (t - \tau)} d\xi.
\]
(43)
Since \( z(\theta_{\tau} \omega) \) is a tempered random variable and \( \lim_{t \to \pm \infty} \frac{z(\theta_{\tau} \omega)}{t} = 0, \int_{-\infty}^{0} \frac{1}{t} z(\theta_{\tau} \omega) dt = 0 \). Thus, there exists \( M_0(\omega) \) and \( T = T(\tau, \omega, B) > 0 \) such that
\[
\limsup_{t \to -\infty} e^{-\bar{\sigma} t} \| \chi(\tau - t, \theta_{\tau-\omega}) \|^2_{E} = 0,
\]
\[
\int_{-\infty}^{0} \varphi(\theta_{\tau-\tau}(\omega)) e^{\bar{\sigma} (t - \tau)} d\xi < +\infty = M_0(\omega),
\]
\[ \|\chi(r, \tau - t, \theta - r \omega, \chi_{\tau - t}(\theta - r \omega))\|_{L}^{2} \leq M_{0}^{2}(\omega). \] (44)

The proof is completed. \( \square \)

Now we decompose the Equation (6) into two parts and also decompose the nonlinear growth term \( f \in C^{1} \) in Equation (3) into two parts \( f = f_{1} + f_{2} \), where \( f_{1}, f_{2} \) satisfy the following respectively

\[
\begin{align*}
(A_{1}): & \ u f_{1}(u) \geq 0, \\
(A_{2}): & \ |f'_{1}(u)| \leq \mu_{1}(1 + |u|^{\gamma}), \ \forall \ u \in \mathbb{R}, \ n \geq 3 \\
(A_{3}): & \ |f_{2}(u)| \leq \mu_{2}(1 + |u|^{\gamma}), \ \forall \ u \in \mathbb{R}, \\
(A_{4}): & \ F(u) = \int_{0}^{u} f_{1}(r)dr, \\
(A_{5}): & \ u f_{1}(u) \geq \mu_{i}(F(u) - 1), \\
(A_{6}): & \ k_{0}(|u|^{\gamma+1} - 1) \leq F_{1}(u) \leq k_{1} u f_{1}(u) + C_{\mu}.
\end{align*}
\] (45)

where \( \mu_{i}, C_{\mu}, k_{0}, k_{1}, i = 1, 2 \) are positive constants. Let for any \( \tau \in \mathbb{R}, \omega \in \Omega \), there is a time \( T_{1} = T_{1}(B_{0}, \omega) \) satisfies

\[ B(\omega) = \bigcup_{i \geq 1} \chi(\tau, t - \theta \omega; \chi_{\tau - t}(\theta \omega)) = \chi_{\tau - t}(\omega) \in \hat{B}(\tau, \theta \omega), \ \forall \ t \geq \hat{T}, \] (46)

for any \( \omega \in \Omega \), where \( \hat{T} = \hat{T}(B_{0}, \omega) \geq \tau \) is the pullback absorbing time in Lemma 2, then it holds \( \hat{B}(\omega) \subseteq B_{0}(\omega) \)

In order to obtain the regularity estimates, we decompose the solution \( \chi(t, \tau, \omega) = (u(t, \tau, \omega), v(t, \tau, \omega), \eta^{1}(t, \tau, s, \omega))^{T} \) of system (6) with initial data \( \chi(\tau, \omega) = (u_{0}, v_{0}, \eta_{0})^{T} \) into two parts

\[
\begin{cases}
\chi(t, \tau, \omega) = \hat{\chi}(t, \tau, \omega) + \check{\chi}(t, \tau, \omega), \\
u = y + w, \\
\eta^{1} = \hat{\eta} + \check{\eta}.
\end{cases}
\] (48)

Then, we can rewrite the Equation (6) into the following systems

\[
\begin{align*}
y_{tt} + \Delta_{t} y - k(0)(1 + \int_{\Omega} |\nabla y|^{2} dx) \Delta y - \int_{0}^{\tau} \mu_{i} \hat{\eta} u(t - s) ds + a(x)g(y_{t}) + f_{1}(y) &= \hat{q}(x, t), \\
\eta_{t} &= -\eta_{ts} + y_{t}, \\
y(\tau, x) &= y_{0}(x), \ y_{t}(\tau, x) = y_{1}(x), \ x \in \Gamma, \ \tau \in \mathbb{R}, \\
\eta_{t}(\tau, x, s) &= y_{0}(x) - y_{0}(x, -s), \ x \in \Gamma, \ \tau \in \mathbb{R}, s \in \mathbb{R}^{+},
\end{align*}
\] (49)

Let \( \hat{\chi}(t, \omega) = (\hat{y}, \hat{\eta}^{1}(t, s))^{T}, \ \check{y} = y \) and \( \check{y} = \check{y}_{t} + e \hat{y} \), which are equivalent with

\[
\begin{align*}
\hat{\chi}' + H(\hat{\chi}) &= \hat{F}(\hat{\chi}, t, \omega), \\
\check{\chi}(t, \omega) &= (\check{y}_{0}(x), \check{y}_{1}(x) + e \check{y}_{0}(x), \check{\eta}_{0})^{T}, \ \hat{\chi} = (\hat{y}, \check{y}, \check{\eta})^{T},
\end{align*}
\] (50)

where

\[
H(\hat{\chi}) = \begin{pmatrix} \varepsilon \hat{g} - \hat{y} \\ -e \hat{g} + e^{2} \hat{g} + A \hat{g} + \int_{0}^{\tau} \mu(s) A^{\frac{1}{2}} \hat{\eta}(s) ds \\ \varepsilon \hat{g} - \hat{y} + \check{\eta}_{s} \end{pmatrix},
\]

\[ F(\hat{\chi}, \omega, t) = \begin{pmatrix} 0 \\ (1 + k(0) \int_{\Omega} |\nabla \hat{g}|^{2} dx) A^{\frac{1}{2}} y_{1} - a(x)g(\hat{y}_{t}) - f_{1}(\hat{y}) + \hat{q}(x, t) \\ 0 \end{pmatrix}, \]
and

\[
\begin{aligned}
&\begin{cases}
  w_{tt} + \Delta^2 w - k(0)(1 + \int_0^\infty |\nabla w|^2 dx)\Delta w - \int_0^\infty \mu \Delta \theta ds + a(x)g(w_t) + f(u) - f_1(y) = \hat{q}(x,t) + \kappa \sum_{j=1}^{m} h_j W_j, \\
  \hat{\eta}_t = -\hat{\eta}_s + \omega_t, \\
  \hat{w}(\tau,x) = w_0(\tau,x), w_t(\tau,x) = w_1(\tau,x), x \in \Gamma, \tau \in \mathbb{R}, \\
  \hat{\eta}_t(x,\tau,s) = w_0(x,\tau) - w_0(x,\tau-s), x \in \Gamma, \tau \in \mathbb{R}, s \in \mathbb{R}^+, \\
\end{cases}
\end{aligned}
\]

(51)

Since \( \hat{\chi} = (\hat{\omega}, \hat{\omega}, \hat{\eta})^\top \), \((\hat{\omega}, \hat{\omega}, \hat{\eta})^\top = \hat{\omega} = \hat{\omega}_t + \delta \hat{\omega} - \kappa z(\theta_\omega) \).

(52)

The above equations leads to

\[
\begin{aligned}
&\begin{cases}
  \hat{\chi}'' + H(\hat{\chi}) = \hat{F}(\hat{\chi}, t, \omega), \\
  \hat{\chi}(\tau, \omega) = (\hat{w}_0(x), \hat{w}_1(x) + \epsilon \hat{w}_0(x) - \kappa z(\theta_\omega), \hat{\eta}_0)^\top, \hat{\chi} = (\hat{\omega}, \hat{\omega}, \hat{\eta})^\top,
\end{cases}
\end{aligned}
\]

(53)

in which

\[
H(\hat{\chi}) = \begin{pmatrix}
\epsilon \hat{w} - \delta \hat{\omega} \\
-\epsilon \hat{w} + \epsilon^2 \hat{\omega} - A \hat{\omega} + \int_0^\infty \mu(s) A \hat{\omega} \eta(s) ds \\
\epsilon \hat{w} - \delta \hat{\omega} + \hat{\eta}
\end{pmatrix},
\]

and

\[
\hat{F}(\hat{\chi}, t, \omega) = \begin{pmatrix}
\kappa z(\theta_\omega) \\
\kappa z(\theta_\omega)
\end{pmatrix}.
\]

Now we need to establish some priori estimates for the solutions of Equation (50) and Equation (53), which are the basis of our later analysis.

**Lemma 3.** Let (2)-(5),(7)-(8) and (45) hold. Let \( \bar{B}(\tau, \omega) \subseteq B_0(\tau, \omega), \bar{B} = \{ \bar{B}(\tau, \omega) \}_{\omega \in \Omega} \subseteq D(E) \) and \( \bar{\chi}_0(\omega) \in \bar{B}(\tau, \omega) \). Then there exists \( T = T(\bar{B}, \omega) > 0 \) and \( M_0(\omega) \), such that the solution \( \tilde{\chi}(T, \omega, \bar{\chi}(\omega)) \) of (50) satisfies for \( P-a.e. \omega \in \Omega, \forall t \geq \hat{T} \):

\[
\|\tilde{\chi}(t, \tau, \omega, \bar{\chi}(\omega))\|_E^2 \leq \|\tilde{\chi}_0\|_E^2 e^{-2\alpha_1 t} + \int_0^T \hat{r}(\omega) dt \leq M(\omega).
\]

(54)

**Proof.** Taking inner product of (50) with \( \tilde{\chi} \) in \( E \), we have

\[
\langle \tilde{\chi}', \tilde{\chi} \rangle + \langle H(\tilde{\chi}), \tilde{\chi} \rangle = \langle \hat{F}(t, x, \tilde{\chi}), \tilde{\chi} \rangle.
\]

(55)

Using Hölder, Young and Poincaré inequalities, we get

\[
\langle H(\tilde{\chi}), \tilde{\chi} \rangle = \epsilon \|\Delta \tilde{\gamma}\|^2 + \epsilon^2 \|g(\tilde{\gamma}, \hat{\gamma}) - \epsilon \|\tilde{\gamma}\|^2 - \frac{m_0 \epsilon^2}{2} \|\nabla \tilde{\gamma}\|^2 + \frac{\delta}{4} \|\nabla \theta\|^2.
\]

(56)

Now, we estimate the terms on the right hand side of (55) one by one:

\[
\left( \left( 1 + k(0) \|\nabla g\|^2 \right) \Delta \tilde{\gamma} \hat{\gamma} \right) \leq \left( 1 + k(0) \|\nabla \tilde{\gamma}\|^2 \right) \nabla \tilde{\gamma}, \nabla \left( \frac{d \tilde{\gamma}}{dt} + \epsilon \hat{\gamma} \right)
\]

\[
\leq \left( 1 + k(0) \|\nabla \tilde{\gamma}\|^2 \right) \left( \frac{1}{2} \frac{d}{dt} \|\nabla \tilde{\gamma}\|^2 + \frac{\epsilon}{2} \|\nabla \tilde{\gamma}\|^2 \right),
\]

(57)

and from (2), it is easy to show that

\[
\left( a(x)g(\tilde{\gamma}, \hat{\gamma}) \right) \leq a_0 a_1 \|\tilde{\gamma}\|^2 - a_0 a_2 \epsilon (\hat{\gamma}, \hat{\gamma}),
\]

(58)
where \( \theta \) is between 0 and \( \tilde{y} - \varepsilon \tilde{y} \).

\[
(q(x,t), v) = \|q(x,t)\|\|y\| \leq \frac{\|\hat{q}(x,t)\|^2}{(\alpha_0\alpha_1 - \varepsilon)} + \frac{\alpha_0\alpha_1 - \varepsilon}{4} \|\tilde{y}\|^2,
\]

by second term for right hand side of (55) and (59), we can get

\[
\varepsilon(\varepsilon - \alpha_2\alpha_0)(\tilde{y}, \tilde{y}) \geq -\frac{\alpha_0\alpha_2\varepsilon}{\lambda} \|\nabla \tilde{y}\|\|\tilde{y}\| \geq \frac{\alpha_0\alpha_1 - \varepsilon}{4} \|\tilde{y}\|^2 - \frac{\alpha_0\alpha_2^2\varepsilon^2}{(\alpha_0\alpha_1 - \varepsilon)^2} \|\nabla \tilde{y}\|^2,
\]

Further, from (45)\(_{(A_2), (A_4)}\), we infer

\[
(f_1(y), \tilde{y}) = (f_1(y), \frac{d\hat{q}}{dt} + \varepsilon \tilde{y}) \geq d \frac{\hat{F}_1(y)}{\hat{F}_1(y)} + \frac{\varepsilon}{k_1}(F_1(y) - c_y|\Gamma|),
\]

Thus, applying together in (55) we conclude that

\[
\frac{1}{2} \frac{d}{dt} \left( \|y\|^2 + (1 + k(0)) \|\nabla y\|^2 \right) \|\nabla y\|^2 + \|\nabla y(r)^2\|^2 + \|\nabla \tilde{y}(r)^2\|^2 + \tilde{F}_1(y)
\]

\+
\[
\frac{\varepsilon}{2} \left( \|y\|^2 + (1 + k(0)) \|\nabla y\|^2 \right) \|\nabla y\|^2 + \varepsilon \left( \frac{1}{\lambda} - \frac{m\varepsilon}{2} - \frac{\alpha_0\alpha_2^2\varepsilon}{(\alpha_0\alpha_1 - \varepsilon)^2} \right) \|\nabla \tilde{y}(r)^2\|^2
\]

\+
\[
\frac{\alpha_0\varepsilon_1 - \varepsilon}{2} \|y\|^2 + \frac{\delta}{4} \|\tilde{y}(r)^2\|^2 + \frac{\varepsilon}{k_1} \tilde{F}_1(y(r))
\]

\leq \frac{\|q(x,t)\|^2}{(\alpha_0\alpha_1 - \varepsilon)} + \frac{\varepsilon}{k_1} c_y|\Gamma|,
\]

Since the inequalities above has nonnegative terms, we obtain \( \varepsilon \left( \frac{1}{\lambda} - \frac{m\varepsilon}{2} - \frac{\alpha_0\alpha_2^2\varepsilon}{(\alpha_0\alpha_1 - \varepsilon)^2} \right) > 0, \frac{\alpha_0\varepsilon_1 - \varepsilon}{2} > 0 \).

We will choose \( \sigma = \left( \frac{\varepsilon}{2} + \frac{\alpha_0^2\varepsilon}{2}, 2\varepsilon \left( \frac{1}{\lambda} - \frac{m\varepsilon}{2} - \frac{\alpha_0\alpha_2^2\varepsilon}{(\alpha_0\alpha_1 - \varepsilon)^2} \right) \right) \) and \( \delta = \min \{ \sigma, \frac{\varepsilon}{k_1} \} \), which obviously implies that

\[
\frac{d}{dt} \|\hat{x}(r)\|_E^2 + \delta \|\hat{x}(r)\|_E^2 \leq \|\hat{q}(x,t)\|^2 + \frac{\varepsilon}{k_1} c_y|\Gamma|.
\]

Note that \( \hat{x}(r, \tau - t, \omega, \hat{x}_{\tau-t}(\omega)) = \chi(r, \tau - t, \omega, \hat{x}_{\tau-t}(\omega)) - (0, z(\theta_t \omega), 0) \in B_0(\tau, \omega) \). By definition of \( B_0(\tau, \omega) \), it follows that \( \|\hat{x}(r, \tau - t, \omega, \hat{x}_{\tau-t}(\omega))\|_E^2 \leq \hat{r}(\omega) + |z(\theta_t \omega)| = \hat{M}(\omega) \). Now, by the Gronwall inequality to \( \tau - t, r \), we arrive to (54); \( \|\hat{x}(r, \tau - t, \omega, \hat{x}_{\tau-t}(\omega))\|_E^2 \leq \hat{M}(\omega) \). Hence, for every \( \tilde{y} \in H_0^1 \), by \( H_0^1 \subset L^{2\varepsilon} \) and (62), we have

\[
0 \leq \int_{\Omega} f_1(u)dx \leq \mu_1 \left( \|y\|^2 + \|y\| \frac{\alpha_0^2\varepsilon}{2} \right) \leq \hat{r}(\omega) \|\nabla y\|^2 \forall u \in \mathbb{R}, n \geq 3.
\]

The proof is completed. \( \Box \)

**Lemma 4.** Suppose (2)-(4) hold. Let \( B_1(\omega) \subseteq B_0(\omega) \), \( B = \{ B(\omega) \}_{\omega \in \Omega} \in \mathcal{D}(E) \) and \( \hat{x}(r) \in B(\omega) \), then there exists \( \hat{T} = \hat{T}(\hat{B}, \omega) > 0 \) and a random radius \( \hat{M}(\omega) \), such that the solution \( \hat{x}(t, \tau, \omega, \hat{x}_{\tau-t}(\omega)) \) of (53) satisfies for P-a.e \( \omega \in \Omega, \forall t \geq \hat{T} \)

\[
\|A^\tau \hat{x}(r, \tau - t, \omega, \hat{x}_{\tau-t}(\omega))\|_E^2 \leq \|A^\tau \hat{x}_{\tau-t}(\theta_{-\tau} \omega))\|_E^2 e^{-2\gamma(t-\tau)} + \hat{r}(\omega) \leq \hat{M}(\omega), t \geq \tau.
\]

We denote

\[
\nu \in \left( 0, \min \left\{ \frac{1}{4}, \frac{n + 2 - (n - 2)\gamma}{4} \right\} \right), \forall 1 \leq \gamma \leq \frac{n + 2}{n - 2}.
\]

**Proof.** By (64), (23) and \( \hat{x} = \chi - \hat{x} \), there exists a random variable \( \hat{r}(\omega) > 0 \) such that

\[
\max \{ \|\chi(0, \omega, \hat{x}(0, \omega))\|_E, \|\hat{x}(0, \omega, \hat{x}(0, \omega))\|_E \} \leq \hat{r}(\omega).
\]
By the embedding relations, we have $V_{v_1} \subset V_{v_2}$, if $v_1 \geq v_2$ and $V_v \subset L^q$, where $\frac{1}{q_1} = \frac{1}{2} - \frac{\epsilon}{4}$, $\frac{1}{q_2} = \frac{1}{2} - \frac{\epsilon}{4}$ and $H^1_\theta = D(A^{\frac{1}{2}}) = V_v \subset L^q \subset L^{q} \subset V_{-\nu} = D(A^{\frac{1}{2}})$. Multiplying (53) with $A^r \tilde{\chi}(r)$ and integrating over $\Gamma$, we can get

$$
(H(\tilde{\chi}(r)), A^r \tilde{\chi}(r)) = \left( -\epsilon u - v + f_0 u + f_0^\infty(s)A^{\frac{1}{2}} \eta(z) s ds \right) \left( A^r u \atop A^r \eta \right) = \epsilon \| A^{\frac{1}{2}} \tilde{w} \|^2 + e^2 (\tilde{w}, A^{r} \tilde{w}) - \epsilon \| A^{\frac{1}{2}} \tilde{w} \|^2 - \frac{m_0^2}{2} \| A^{\frac{1}{2}} \tilde{w} \|^2 + \frac{\delta}{4} \| A^{\frac{1}{2}} \eta \|^2. \tag{66}
$$

Now using Cauchy-Schwartz inequality and Young inequality one by one as

$$
(F(t, x, \chi), A^r \chi) = \left( kh(x) z(\theta t) \atop (1 + k(0) \| A^{\frac{1}{2}} \tilde{w} \|^2) A^{\frac{1}{2}} \tilde{w} - a(x) g(\tilde{w}) + f_1(y) - f(u) + \eta(x, t) + kh(x) z(\theta t) \right) \left( A^r \tilde{w} \atop A^r \tilde{\eta} \right). \tag{67}
$$

From (3)_{(A_2), (A_4)} one has that

$$
- \left(1 + k(0) \| A^{\frac{1}{2}} \tilde{w} \|^2\right) A^{\frac{1}{2}} \tilde{w}, A^{r} \tilde{w} = \left( 1 + k(0) \| A^{\frac{1}{2}} \tilde{w} \|^2 \right) A^{\frac{1}{2}} \tilde{w}, A^{r} \tilde{w} \frac{d \tilde{w}}{dt} + \epsilon \tilde{w} - ah(x) z(\theta t) \omega
$$

$$
\leq - \left(1 + k(0) \| A^{\frac{1}{2}} \tilde{w} \|^2\right) \left( \frac{1}{2} \frac{d}{dt} \| A^{\frac{1}{2}} \tilde{w} \|^2 + \frac{\epsilon}{2} \| A^{\frac{1}{2}} \tilde{w} \|^2 \right) + \frac{\kappa^2}{2} \| A^{\frac{1}{2}} h(x) \|^2 \|z(\theta t)\|^2. \tag{68}
$$

Therefore from (2) and (4), it is straightforward to show that

$$
(a(x) g(\tilde{w}), A^{r} \tilde{w}) = -(a_0 g(x) A^{\frac{1}{2}} (\tilde{w} - \epsilon \tilde{w} + kh(x) z(\theta t) - g(0)), A^{\frac{1}{2}} \tilde{w})
$$

$$
\leq -a_0 a_2 \| A^{\frac{1}{2}} \tilde{w} \|^2 + a_0 a_2 \epsilon \left( A^{\frac{1}{2}} \tilde{w}, A^{\frac{1}{2}} \tilde{w} \right) - a_0 g'(x) \left( h(x) z(\theta t), A^{\frac{1}{2}} \tilde{w} \right), \tag{69}
$$

where $\tilde{\theta}$ is between 0 and $\tilde{\omega} - \epsilon \tilde{w} + kh(x) z(\theta t) \omega$.

$$
\left( k h(x) z(\theta t) \right) \left( k h(x) z(\theta t) \right) \leq \frac{\epsilon}{4} \left\| A^{\frac{1}{2}} \tilde{w} \right\|^2 + \epsilon k \left\| A^{\frac{1}{2}} \tilde{w} \right\|^2 \|z(\theta t)\|^2, \tag{70}
$$

$$
\left( k h(x) z(\theta t) \right) \left( k h(x) z(\theta t) \right) \leq \frac{2 m_0^2 k^2}{8} \left\| A^{\frac{1}{2}} \tilde{w} \right\|^2 \|z(\theta t)\|^2 + \frac{\delta}{8} \left\| A^{\frac{1}{2}} \tilde{w} \right\|^2, \tag{71}
$$

$$
\left( -a_0 g'(\tilde{\theta}) - 2 k h(x) z(\theta t), A^{r} \tilde{w} \right) \leq \frac{2 (a_0 a_2 \epsilon)^2}{a_0 a_1 - \epsilon} \left\| A^{\frac{1}{2}} \tilde{w} \right\|^2 \|z(\theta t)\|^2 + \frac{\delta}{8} \left\| A^{\frac{1}{2}} \tilde{w} \right\|^2. \tag{72}
$$

Through 2d term for right hand side of (26) and (29), we will get

$$
\epsilon \left( -a_2 a_0 \right) (\tilde{w}, A^{r} \tilde{w}) \geq -a_0 a_2 \epsilon \| A^{\frac{1}{2}} \tilde{w} \| \| A^{\frac{1}{2}} \tilde{w} \| \geq \frac{a_0 a_1 - \epsilon}{4} \| A^{\frac{1}{2}} \tilde{w} \|^2 - \frac{(a_0 a_2 \epsilon)^2}{(a_0 a_1 - \epsilon) k^2} \| A^{\frac{1}{2}} \tilde{w} \|^2. \tag{73}
$$

For the nonlinearity, with the aid of (4), Hölder inequality and the Sobolev embedding theorem, we estimate that

$$
(\tilde{f}(u) - f_1(\tilde{w}), A^{r} \tilde{w}) = (\tilde{f}(u) - f_1(\tilde{w}), A^{r} (\tilde{w} + \epsilon \tilde{w} - kh(x) z(\theta t) \omega))
$$

$$
\leq \frac{d}{dt} \int \tilde{f}(u) - f_1(\tilde{w}) A^{r} \tilde{w} dx + \epsilon \int \tilde{f}(u) - f_1(\tilde{w}) A^{r} \tilde{w} dx - \int (\tilde{f}(u) - f_1(\tilde{w})) A^{r} \tilde{w} - \kappa \int (\tilde{f}(u) - f_1(\tilde{w})) A^{r} \tilde{w} dx - \kappa \int (\tilde{f}(u) - f_1(\tilde{w})) A^{r} \tilde{w} dx.
$$

Infer to $A_{1, \nu}$, (45)-(46), use Cauchy-Schwartz, Young’s inequality and embedding theorem $V_{1+v} \subset L^{\frac{2}{2\nu-1}}$, $V_{1-v} \subset L^{\frac{2}{2\nu+1}}$ and $V_1 \hookrightarrow L^{\frac{2}{\nu}}$, we gain
\begin{align*}
\int_I (f(u) - f_1(\hat{\omega})) A^r h(x) \|z(\theta_1 \omega)) \| dx &\leq \int_I ((f_1(u) + f_2(u) - f_1(\hat{\omega})) A^r \kappa(h(x)) \|z(\theta_1 \omega)) \| dx \\
&\leq \mu_1 \kappa \left( \int_I \left( 1 + |\hat{\omega} \frac{\Gamma}{\nu} \right) \frac{2n}{n - 2(1 + \gamma)} \right) \frac{2 \eta}{n} \left( \int_I |\hat{\omega}| \frac{n - 2n}{n - 2(1 + \gamma)} \right) \frac{2 \eta}{n} \left( \int_I \|A^r h(x)\| \|z(\theta_1 \omega)) \| \right) \\
&+ \mu_2 \kappa \left( \int_I \left( 1 + |\hat{u}| \frac{\Gamma}{\nu} \right) \frac{2n}{n - 2(1 + \gamma)} \right) \frac{2 \eta}{n} \left( \int_I \|A^r h(x)\| \|z(\theta_1 \omega)) \| \right) \\
&\leq \mu_3 \left( 1 + \|u\| \frac{2n}{L^{n - 2(1 + \gamma)}} \right) \frac{2 \eta}{n} \left( \int_I \|\hat{\omega}\| \frac{n - 2n}{n - 2(1 + \gamma)} \right) \frac{2 \eta}{n} \left( \int_I \|A^r h(x)\| \|z(\theta_1 \omega)) \| \right) \\
&+ \mu_4 \left( 1 + \|u\| \frac{2n}{L^{n - 2(1 + \gamma)}} \right) \frac{2 \eta}{n} \left( \int_I \|A^r h(x)\| \|z(\theta_1 \omega)) \| \right) \\
&\leq \mu_5 \left( \int_I \left( f(u) - f_1(\hat{\omega}) \right) A^r h(x) \|z(\theta_1 \omega)) \| \right) + \mu_6 \left( \int_I \|A^r h(x)\|^2 \|z(\theta_1 \omega)) \|^2 \right). \tag{75}
\end{align*}

and therefore

\begin{align*}
\int_I (f(u)u_1 - f_1(\hat{\omega}) \hat{\omega}_1) A^r \hat{\omega} dx = \int_I ((f_1(u) - f_1(\hat{\omega})) u_1 + f_2(u) \hat{\omega}_1 + f_2(u) u_1) A^r \hat{\omega} dx.
\end{align*}

Estimate the above inequality, we get

\begin{align*}
\int_I (f_1(u) - f_1(\hat{\omega})) u_1 A^r \hat{\omega} dx \leq \mu_7 \int_I \left( 1 + |\hat{\omega} \frac{\Gamma}{\nu} \right) \frac{2n}{n - 2(1 + \gamma)} \|A^r \hat{\omega}\| \|u_1\| \|dx dx \\
&\leq \mu_8 \left( \int_I \left( 1 + |\hat{\omega} \frac{\Gamma}{\nu} \right) \frac{2n}{n - 2(1 + \gamma)} \right) \frac{2 \eta}{n} \left( \int_I |u_1|^2 \right) \frac{2 \eta}{n} \left( \int_I \|A^r \hat{\omega}\| \|u_1\| \right) \frac{2 \eta}{n} \left( \int_I \|A^r \hat{\omega}\| \|u_1\| \right) \\
&\leq \mu_9 \left( 1 + \|\hat{\omega}\| \frac{2n}{L^{n - 2(1 + \gamma)}} \right) \frac{2 \eta}{n} \left( \int_I \|u_1\| \|A^r \hat{\omega}\| \|u_1\| \right) \frac{2 \eta}{n} \left( \int_I \|A^r \hat{\omega}\| \|u_1\| \right) \\
&\leq \mu_{10} \left( \|\hat{\omega}\| \frac{2n}{L^{n - 2(1 + \gamma)}} \|A^r \hat{\omega}\| \right) \left( \|\hat{\omega}\| \frac{2n}{L^{n - 2(1 + \gamma)}} \|A^r \hat{\omega}\| \right) \\
&\leq \mu_{12} \left( \|A^r \hat{\omega}\|^2 + \|A^r \hat{\omega}\|^2 \right). \tag{76}
\end{align*}

Similarly, by (45) \(_A \) and (65), we get

\begin{align*}
\int_I f_1(\hat{\omega}) \hat{\omega}_1 A^r \hat{\omega} dx \leq \mu_{13} \left( \int_I \left( 1 + |\hat{\omega} \frac{\Gamma}{\nu} \right) \frac{2n}{n - 2(1 + \gamma)} \right) \frac{2 \eta}{n} \left( \int_I |\hat{\omega}| \frac{n - 2n}{n - 2(1 + \gamma)} \right) \frac{2 \eta}{n} \left( \int_I \|A^r \hat{\omega}\| \^2 \|z(\theta_1 \omega)) \|^2 \right) \\
&\leq \mu_{14} \left( 1 + \|\hat{\omega}\| \frac{2n}{L^{n - 2(1 + \gamma)}} \right) \frac{2 \eta}{n} \left( \int_I \|A^r \hat{\omega}\| \|z(\theta_1 \omega)) \|^2 \right) \\
&\leq \mu_{15} \left( \|A^r \hat{\omega}\| \right) \left( \|A^r \hat{\omega}\| \right) \left( \|A^r \hat{\omega}\| \right) \\
&\leq \mu_6 \left( \|A^r \hat{\omega}\| \right) \left( \|A^r \hat{\omega}\| \right) \left( \|A^r \hat{\omega}\| \right). \tag{77}
\end{align*}

Furthermore, by (45) \(_A \) and (65), note \( \nu \leq \frac{n + 2 - (n - 2)\gamma}{4} \)

\begin{align*}
\int_I f_1(\hat{\omega}) \hat{\omega}_1 A^r \hat{\omega} dx \leq \mu_{17} \int_I \left( 1 + |u|^\gamma \right) \|u_1\| \|A^r \hat{\omega}\| \|dx dx \\
&\leq \mu_{18} \left( \int_I \left( 1 + |u|^\gamma \right) \frac{2n}{n - 2(1 + \gamma)} \right) \frac{2 \eta}{n} \left( \int_I |u_1|^2 \|A^r \hat{\omega}\| \|dx dx \\
&\leq \mu_{19} \left( 1 + \|u\| |u|^\gamma \frac{2n}{n - 2(1 + \gamma)} \right) \|u_1\| \|A^r \hat{\omega}\| \|dx dx \\
&\leq \mu_{20} \left( \|A^r \hat{\omega}\| \right) \left( \|A^r \hat{\omega}\| \right) \left( \|A^r \hat{\omega}\| \right). \tag{78}
\end{align*}
Including above inequalities together (66)-(78), we achieve
\[
\frac{1}{2} \frac{d}{dt} \left[ \| A^{\frac{1}{2}} \tilde{X} \|^2 \right] + 2(f(u) - f_1(\tilde{w})) + \frac{\epsilon}{4} \| A^{\frac{1}{2}} \tilde{X} \|^2 + \frac{\epsilon}{2} (f(u) - f_1(\tilde{w})) \\
\leq |z(\theta \omega)| |A^{\frac{1}{2}} \tilde{X}|^2 + \mu_C |1 + r_1^2(\omega) + r_2^2(\omega) + r_3^2(\omega) + r_4^2(\omega) \\
+ |z(\theta \omega)|^2 + \| A^{\frac{1}{2}} h(x) \|^2 + \| A^{\frac{1}{2}} \tilde{q}(x) \|^2].
\]

By Gronwall’s inequality in (79) on \([0, r]\) and changing \(\omega\) to \(\theta_{-1} \omega\), we deduce that
\[
\| A^{\frac{1}{2}} \tilde{q}(t, \theta_{-1} \omega; \varrho_0) \|_E^2 \leq \left( \| A^{\frac{1}{2}} \tilde{q}(r, \theta_{-1} \omega; \varrho_0) \|_E^2 + 2(f(u(r, \theta_{-1} \omega; \chi_0)) - f_1(\tilde{w}(r, \theta_{-1} \omega; \chi_0))) \right) \\
\leq \left( \| A^{\frac{1}{2}} \tilde{X} \|_E^2 + (f(u) - f_1(\tilde{w})) \right) \exp^{2 \int_0^r \| \tilde{q} \| d\omega} + \int_0^r \epsilon_1(\theta \omega) \exp^{2 \int_0^r \| \tilde{q} \| d\omega} \| \tilde{q} \| d\omega} ds.
\]

We can choose \(\epsilon_1(\theta \omega)\) and \(\mu_C\) to depend on \([1, \delta, \gamma, \alpha_0, \alpha_1, \alpha_2, m_0, m_1]\) are positive constants, such that
\[
\epsilon_1(\theta \omega) = \mu_C |1 + r_1^2(\omega) + r_2^2(\omega) + r_3^2(\omega) + r_4^2(\omega) \\
+ \| A^{\frac{1}{2}} h(x) \|^2 |z(\theta \omega)|^2 + \| A^{\frac{1}{2}} \tilde{q}(x) \|^2 \| \tilde{q} \| \| z(\theta \omega) \| ^2 + \| A^{\frac{1}{2}} \tilde{q}(x) \|^2].
\]

Note that
\[
\int_0^r ((f(u) - f_1(\tilde{w})) A^{\frac{1}{2}} \tilde{w} dx \leq \int_0^r ((f(u) + f_2(u) - f_1(\tilde{w})) A^{\frac{1}{2}} \tilde{w} dx
\]
\[
\leq \mu_2 \left( \int_0^r (1 + |\tilde{w}|^\frac{1}{2}) |\tilde{w}| \left| A^{\frac{1}{2}} \tilde{w} \right| dx \right) + \mu_2 \left( \int_0^r (1 + |\tilde{w}|^\| \tilde{w} \| dx \right). \]

Thus, by the Sobolev embedding
\[
\left( \int_0^r (1 + |\tilde{w}|^\frac{1}{2}) |\tilde{w}| \left| A^{\frac{1}{2}} \tilde{w} \right| dx \right) + \left( \int_0^r (1 + |\tilde{w}|^\| \tilde{w} \| dx \right) \left| A^{\frac{1}{2}} \tilde{w} \right| dx
\]
\[
\leq \mu_23 \left( \int_0^r (1 + |\tilde{w}|^\frac{1}{2}) |\tilde{w}| \| A^{\frac{1}{2}} \tilde{w} \| dx \right) \int_0^r |\tilde{w}| \| A^{\frac{1}{2}} \tilde{w} \| dx
\]
\[
\leq \mu_24 \left( \int_0^r (1 + |\tilde{w}|^\| \tilde{w} \| dx \right) \| A^{\frac{1}{2}} \tilde{w} \| dx
\]
\[
\leq \mu_25 \left( \| A^{\frac{1}{2}} \tilde{w} \| \right)^2,
\]
where the constants \(\mu_i, i = 1, 2, \ldots, 25\), comes from the embedding \(D(A^{\frac{1}{2}}) \hookrightarrow L^2 \| A^{\frac{1}{2}} \| \), \(D(A^{\frac{1}{2}}) \hookrightarrow L^2 \| A^{\frac{1}{2}} \| \), and \(V_1 = H_0^1 \hookrightarrow L^2 \| A^{\frac{1}{2}} \| \).

Note that, \(|z(\theta \omega)| \) is tempered, and hence applying the inequalities (81) and (82) in (80), the integrand of the second term on the righthand side of (80) is convergent to zero exponentially as \(r \rightarrow -\infty\). Then, we can shows that the following result
\[
\| A^{\frac{1}{2}} \tilde{q}(t, \theta_{-1} \omega; \chi_0) \|_E^2 \leq M_2^2(\omega).
\]

The proof is complete. \(\square\)

Now we obtain our main result about the existence of a random attractor for random dynamical system \(\Phi\) as following Lemma. It follows from Lemma 2, that \(\Phi\) has a closed random absorbing set in \(D\), then apply Lemmas in Section 4, we prove the existence of a random attractor by using tail estimates and the decompose technique of solutions, which along with the \(D\)-pullback asymptotic compactness.
Lemma 5. (see [2,3,15]) Let $X_0, X, X_1$ be three Banach spaces such that $X_0 \hookrightarrow X \hookrightarrow X_1$ is projection operator $X_0 \hookrightarrow X$ is compact. Let $Y = \chi(t, B(t, \omega)) \subset L^2(\mathbb{R}^+, X)$ be a random bounded absorbing set from Lemma 4, $\psi(t)$ is the solution operators of (53) and by Lemma 4, there is a positive random radius $M_0(\omega)$ dependent on $t$, such that

\begin{align*}
1). & \quad Y \text{ is bounded in } L^2_\mu(\mathbb{R}^+, X_0) \bigcap H^{1}_\mu(\mathbb{R}^+, X_1), \\
2). & \quad \sup_{\eta \in Y, s \in \mathbb{R}^+} \|\nabla \eta(s)\|_{X}^2 \leq M_0(\omega). \tag{83}
\end{align*}

Then $Y$ is relatively compact in $L^2_\mu(\mathbb{R}^+, X)$. Further, for every $\tau \in \mathbb{R}, \omega \in \Omega$, $t \geq 0$, so that

\begin{align*}
\hat{\eta}(t, \tau, \theta\_\omega, \chi_0(\theta\_\omega)) = \left\{ \begin{array}{ll}
\hat{\omega}(t, \tau, \theta\_\omega, \chi_0(\theta\_\omega), s) & s \leq t, \\
\hat{\omega}(t, \tau, \theta\_\omega, \chi_0(\theta\_\omega), t) & s = t,
\end{array} \right.
\tag{84}
\end{align*}

\begin{align*}
\hat{\eta}_\tau(t, \tau, \theta\_\omega, \chi_0(\theta\_\omega)) = \left\{ \begin{array}{ll}
\hat{\omega}_\tau(t - s, \tau, \theta\_\omega, \chi_0(\theta\_\omega), s) & 0 \leq s \leq t,
0, & t \leq s.
\end{array} \right.
\tag{85}
\end{align*}

Denote by $\bar{B}$ the closed ball of $L^2_\mu(\mathbb{R}^+, X_0) \bigcap H^{1}_\mu(\mathbb{R}^+, X_1)$ of random variable radius $M_0(\omega)$, since we apply on a finite domain. $\bar{B}$ is compact subset of $X_0 \times X_1$. Let a set $\hat{B}(\tau, \omega)$

\begin{align*}
\hat{B}(\tau, \omega) = \bigcup_{\hat{\chi}_{\tau-1}(\theta\_\omega) \in \bar{B}(\theta\_\omega)} \left\{ \begin{array}{ll}
\hat{\eta}(\tau, t, \theta\_\omega, \chi_0(\theta\_\omega), s) & s \in \mathbb{R}^+ \tau \in \mathbb{R} \omega \in \Omega,
\end{array} \right.
\tag{86}
\end{align*}

where $\nu$ is as in (65). Thus, employing (3), Lemma 2, Lemma 4 and (84), we get that

\begin{align*}
\sup_{\eta \in \mathcal{B}, s \in \mathbb{R}^+} \|\nabla \eta(s)\|_{\mu}^2 = \sup_{t \geq 0} \sup_{\chi_{\tau-1}(\theta\_\omega) \in \bar{B}(\theta\_\omega), s \in \mathbb{R}^+} \|\nabla \eta(t, \tau - t, \theta\_\omega, \chi_{\tau-1}(\theta\_\omega), s)\|^2 \leq M_0(\omega), \tag{87}
\end{align*}

implying that

\begin{align*}
\|\nabla \eta(s)\|_{\mu}^2 = \int^{+\infty}_t \mu(s)\|\nabla \eta(s)\|^2ds \leq M_0(\omega) \int^{+\infty}_t e^{\sigma s}ds \leq \frac{M_0(\omega)}{\sigma}. \tag{88}
\end{align*}

We obtain our main result about the existence of a random attractor for random dynamical system $\Phi$ as following Theorem.

**Theorem 3.** Suppose (2)-(4) hold. Then the continuous cocycle $\Phi$ associated with Problem (16), has a unique $D$-pullback random attractor $A(\tau, \omega) \in \mathcal{D}$ in $\Gamma$.

**Proof.** For any $(\tau, \omega) \in (\mathbb{R} \times \Omega)$. Let $\hat{\chi}_{\tau-1}(\theta\_\omega) \in \bar{B}(\tau, \theta\_\omega)$, $B \subset \bar{B}(\theta\_\omega)$ is compact in $\mathcal{D}(E)$. It follows that $\hat{B}$ be the closed ball of $\mathcal{V}_{2\nu-1} \times \mathcal{V}_2 \hookrightarrow E$ is compact of radius $\hat{M}(\omega) \in \mathcal{D}(E)$, where $\nu$ satisfy (65). Therefore, $\Lambda(\tau, \omega)$ is compact in $E$ for any bounded non-random set $B$ of $E$. By Lemma 3 and $\hat{\chi}_{\tau-1}(\theta\_\omega) \in \hat{B}(\tau, \theta\_\omega)$, we have $\chi_{\tau-1} = \hat{\chi}_{\tau-1} - \hat{\chi}_{\tau-1} \in \Lambda(\tau, \omega)$, where $\chi_{\tau-1}$ is given by (50). Then, there exists a random set $\hat{M}(\omega) \in \hat{B} \subset B(\tau, \omega) \in \mathcal{D}(E)$, as follows

\begin{align*}
d_H(\Phi(t, \tau - t, \theta\_\omega, B(\tau, \theta\_\omega)), \Lambda(\tau, \omega)) \leq \hat{M}(\omega)e^{-\sigma t} \rightarrow 0, \text{ as } t \rightarrow +\infty. \tag{89}
\end{align*}

From Lemma 3, there exists $T = \hat{T}(\tau, \omega, B) \geq 0$, then we dedicate the following attraction property

\begin{align*}
\hat{\chi}(\tau, t, \tau - t, \theta\_\omega, B(\tau, \theta\_\omega), \Lambda(\tau, \omega))) \subseteq B_0(\tau, \omega), \forall t \geq \hat{T}.
\end{align*}

Let $t \geq \hat{T}$ and $\hat{T} = t - \hat{T} \geq T(\tau, \omega, B_0) \geq 0$ using cocycle property (iii) of $\Phi$, we show that

\begin{align*}
\hat{\chi}(\tau, t, \tau - t, \theta\_\omega, B(\tau - \hat{T} \theta\_\omega)) = \hat{\chi}(\tau, T - \hat{T} \theta\_\omega, B(\tau - \hat{T} \theta\_\omega)) \\
= \hat{\chi}(\tau, \tau - \hat{T} \theta\_\omega, B(\tau - \hat{T} \theta\_\omega)) \\
= \hat{\chi}(\tau, \tau - \hat{T} \theta\_\omega, B(\tau - \hat{T} \theta\_\omega)) \subseteq B(\tau, \theta\_\omega).
\tag{90}
\end{align*}
Take any \( \hat{x}(\tau, \tau - t, (\theta - \tau, \omega), \chi(\theta - \tau, \omega)) \) \( \in \hat{x}(\tau, \tau - t, \theta - \tau, B(\tau - t, \theta - \tau, \omega)) \), for \( t \geq \hat{T} + T(\tau, \omega, B_0) \), where \( \hat{x}(\theta - \tau, \omega) \) \( \in B(\tau - t, \theta - \tau, \omega) \). It follow to Lemmas 2, 3 and (90), such that

\[
\inf_{\eta \in \Lambda(\tau, \omega)} \| \chi(\tau, \tau - t, \theta - \tau, \omega), \chi_{\tau - t}(\theta - \tau, \omega) \) - \eta(\tau, \tau - t, \theta - \tau, \omega), \chi_{\tau - t}(\theta - \tau, \omega) \|_E^2 \\
\leq \| \eta(\tau, \tau - t, \theta - \tau, \omega), \chi_{\tau - t}(\theta - \tau, \omega) \|_E^2 \leq \hat{M}^2(\omega) e^{-\epsilon_0 t}, \forall t > \hat{T} + T(\tau, \omega, B_0).
\]

Thus from the relation (19) between \( \Phi, \Phi \), one could easily obtain that for any nonrandom bounded

\[
d_H(\Phi(t, \tau, \theta - \tau, \omega), B(\tau, \theta - \tau, \omega)), \Lambda(\tau, \omega) \leq \hat{M}(\omega) e^{-\epsilon_0 t} \rightarrow 0 \text{ as } t \rightarrow +\infty.
\]

Follows from Lemma 1, Lemma 2, Lemma 3 and Lemma 4, \( \Phi \) related to (16) possesses a \( \mathcal{D} \) pull-back random attractor \( \mathcal{A}(\tau, \omega) \subseteq \Lambda(\tau, \omega) \cap B_0(\omega) \). The proof is completed. \( \square \)

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**References**


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