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Article

Approximate solution of nonlinear ordinary differential equation using ZZ decomposition method

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Abstract: Nonlinear initial value problems are somewhat difficult to solve analytically as well as numerically related to linear initial value problems as their variety of natures. Because of this, so many scientists still searching for new methods to solve such nonlinear initial value problems. However there are many methods to solve it. In this article we have discussed about the approximate solution of nonlinear first order ordinary differential equation using ZZ decomposition method. This method is a combination of the natural transform method and Adomian decomposition method.

Keywords: ZZ transform, Adomain decomposition, Adomain polynomial, nonlinear differential equation.

MSC: 65R20, 45K05.

1. Introduction

In the literature there are numerous integral transforms [1] that are widely used in physics, astronomy as well as in engineering. In order to solve the differential equations, the integral transform was extensively used and thus there are several works on the theory and application of integral transform such as the Laplace, Fourier, Mellin, Hankel, Fourier Transform, Sumudu Transform, Elzaki Transform and Aboodh Transform. Aboodh Transform [2,3] was introduced by Khalid Aboodh in 2013, to facilitate the process of solving ordinary and partial differential equations in the time domain. This transformation has deeper connection with the Laplace and Elzaki Transform [4–6]. New integral transform, named as ZZ Transformation [7–10] introduce by Zain Ul Abadin Zafar. ZZ transform was successfully applied to integral equations and ordinary differential equations. The main objective of this article is to solve nonlinear ordinary differential equation using ZZ transform.

2. ZZ transform

Let $f(t)$ be a function defined for all $t \geq 0$. The ZZ transform of $f(t)$ is the function $Z(u, s)$ defined by

$$Z\left(u,\ s\right) = H\left\{f\left(t\right)\right\} = s\int_0^\infty f\left(ut\right)e^{-st}dt, \tag{1}$$

or Equation (1) equivalent to

$$Z\left(u,\ s\right) = H\left\{f\left(t\right)\right\} = \frac{s}{u}\int_0^\infty f\left(t\right)e^{\frac{-s}{u}t}dt. \tag{2}$$

2.1. The ZZ decomposition method

Consider the general nonlinear ordinary differential equation of the form:

$$Lv + Rv + Nv = g\left(t\right), \tag{3}$$

with initial condition

$$v\left(0\right) = f\left(t\right), \tag{4}$$

Table 1. ZZ transform of some functions

$f(t)$	$H\{f(t)\} = Z(u, s)$
1	1
t	$\frac{u}{s}$
t^2	$\frac{2!u^2}{s^2}$
t^n	$\frac{n!u^n}{s^n}$
e^{at}	$\frac{s}{s-au}$
$\cos(at)$	$\frac{s^2}{s^2+\alpha^2u^2}$
$\sin(at)$	$\frac{aus}{s^2+(au)^2}$

where v is the unknown function, L is the linear differential operator of highest derivative, R is the reminder of the differential operator, $g(t)$ is nonhomogeneous term and $N(v)$ is the nonlinear term.

Suppose L is a differential operator of the first order, then by taking the ZZ transform of Equation (3), we have

$$\frac{s}{u}V(u, s) - \frac{s}{u}V(0) + H[Rv] + H[Nv] = H[g(t)]. \tag{5}$$

Substituting the given initial condition from Equation (4), we get

$$\frac{s}{u}V(u, s) - \frac{s}{u}f(t) + H[Rv] + H[Nv] = H[g(t)],$$

or equivalent to

$$V(u, s) = f(t) + \frac{u}{s}H[Rv] - \frac{u}{s}H[Rv + Nv]. \tag{6}$$

Since, the solution can be written in the form of $v(t)$ and also $V(u, s)$ is the ZZ transform of $v(t)$. Therefore, Taking the inverse ZZ transform of Equation (6) to obtain the solution in the form of $v(t)$.

$$v(t) = G(t) - H^{-1}\left[\frac{u}{s}H[Rv + Nv]\right]. \tag{7}$$

We now assume an infinite series solution of the unknown function $v(t)$ of the form

$$v(t) = \sum_{n=0}^{\infty} v_n(t). \tag{8}$$

The nonlinear operator $Nv = \Psi(v)$ is decomposed as

$$Nv = \sum_{n=0}^{\infty} A_n(t),$$

where, A_n is called Adomian's polynomials. This can be calculated for various classes of nonlinearity according to

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[\Psi\left(\sum_{i=0}^{n}\lambda^i v_i\right)]_{\lambda=0}.$$

By using Equation (8), the Equation (7), can be rewritten as;

$$\sum_{n=0}^{\infty} v_n(t) = G(t) - H^{-1}\left[\frac{u}{s}H\left[R\sum_{n=0}^{\infty} v_n(t) + \sum_{n=0}^{\infty} A_n(t)\right]\right]. \tag{9}$$

Now, if we compare both sides of Equation (9), we can get the following recurrence relation

$$v_0 = G(t),$$
$$v_1 = -H^{-1}\left[\frac{u}{s}H[Rv_0(t) + A_0(t)]\right],$$
$$v_2 = -H^{-1}\left[\frac{u}{s}H[Rv_1(t) + A_1(t)]\right],$$

$$v_3 = -H^{-1}\left[\frac{u}{s}H\left[Rv_2(t) + A_2(t)\right]\right].$$

Finally, we have the following general recurrence relation;

$$v_{n+1} = -H^{-1}\left[\frac{u}{s}H\left[Rv_n(t) + A_n(t)\right]\right],\ n \geq 0. \tag{10}$$

Therefore, the exact or approximate solution is given by

$$v\left(t\right) = \sum_{n=0}^{\infty} v_n\left(t\right). \tag{11}$$

Example 1. Consider the non-linear system of initial value problems given by

$$y' = [y(t)]^2, \quad y\left(0\right) = 1, \tag{12}$$

with exact solution $y\left(t\right) = \frac{1}{1-t}$. Applying ZZ Transform of Equation (12), we have

$$\frac{s}{u}Y\left(u,\ s\right) - \frac{s}{u}y\left(0\right) = H\left[y^2\right]. \tag{13}$$

Substitute the given initial from Equation (13)

$$\frac{s}{u}Y\left(u,\ s\right) - \frac{s}{u} = H\left[y^2\right]. \tag{14}$$

After simple calculation from Equation (14), we have

$$Y\left(u,\ s\right) = 1 + \frac{u}{s}H\left[y^2\right]. \tag{15}$$

By taking the inverse ZZ transform of Equation (15), we have

$$y\left(t\right) = 1 + H^{-1}\left[\frac{u}{s}H\left[y^2\right]\right]. \tag{16}$$

We now assume an infinite series solution of the unknown function $y\left(t\right)$ of the form

$$y\left(t\right) = \sum_{n=0}^{\infty} y_n(t). \tag{17}$$

By using Equation (17), we can write Equation (16) in the form

$$\sum_{n=0}^{\infty} y_n\left(t\right) = 1 + H^{-1}\left[\frac{u}{s}\left[H\sum_{n=0}^{\infty} A_n(t)\right]\right], \tag{18}$$

where, A_n is called Adomian's polynomials of the nonlinear term $y^2(t)$. Now, by comparing both sides of Equation (18), we can get the following recurrence relation;

$$\begin{aligned}
y_0\left(t\right) &= 1,\\
y_1(t) &= H^{-1}\left[\frac{u}{s}H\left[A_0(t)\right]\right],\\
y_2(t) &= H^{-1}\left[\frac{u}{s}H\left[A_1(t)\right]\right],\\
y_3(t) &= H^{-1}\left[\frac{u}{s}H\left[A_2(t)\right]\right].
\end{aligned}$$

Finally, we have the following general recurrence relation;

$$y_{n+1}\left(t\right) = H^{-1}\left[\frac{u}{s}H\left[A_n\left(t\right)\right]\right],\ n \geq 0. \tag{19}$$

Now, by using the recursive relation in Equation (19), we can easily compute the remaining components of the unknown function $y(t)$ in the following manner

$$y_1(t) = H^{-1}\left[\frac{u}{s}H\left[A_0(t)\right]\right] = H^{-1}\left[\frac{u}{s}H\left[{y_0}^2(t)\right]\right] = H^{-1}\left[\frac{u}{s+u}H(1)^2\right],$$
$$y_1(t) = H^{-1}\left[\frac{u}{s}\times 1\right] = H^{-1}\left[\frac{s}{s}\right] = t.$$
$$y_2(t) = H^{-1}\left[\frac{u}{s}H\left[A_1(t)\right]\right] = H^{-1}\left[\frac{u}{s}H\left[2y_0(t)\,y_1(t)\right]\right] = H^{-1}\left[\frac{u}{s}H(2t)\right] = 2H^{-1}\left[\frac{u^2}{s^2}\right] = t^2.$$

Similarly, we can find $y_3(t)$

$$y_3(t) = H^{-1}\left[\frac{u}{s}H\left[A_2(t)\right]\right] = H^{-1}\left[\frac{u}{s}H\left[2y_0(t)\,y_2(t)\right] + \left(y_1(t)\right)^2\right].$$

After some calculation, we obtain t^3 and so on. Hence, the approximate solution is given by;

$$y(t) = \sum_{n=0}^{\infty} y_n(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + \cdots = 1 + t + t^2 + t^3 + \cdots = \frac{1}{1-t}.$$

The Octave Code is;

$$\gg t = [0 : 0.05 : 0.9]\,;$$
$$\gg f = \frac{1}{1-t};$$
$$\gg g = 1 + t + t\wedge 2 + t\wedge 3 + t\wedge 5 + t\wedge 6 + t\wedge 7 + \ldots;$$
$$\gg plot(t, f,' r', t, g,' o');$$
$$\gg ylabel('y(t)');$$
$$\gg xlabel('t = 0 : 0.9');$$
$$\gg legend(‘Exact', ‘Approximate').$$

Hence, the exact solution is in closed agreement with the result obtained by ZZ decomposition

Example 2. Consider the non-linear system of initial value problems given by

$$x' = 1 - [x(t)]^2, \quad x(0) = 0. \tag{20}$$

Using the method of separation of variables, the exact solution is $x(t) = \frac{e^{2t}-1}{e^{2t}+1}$. Applying ZZ Transform of Equation (20), we have

$$\frac{s}{u}X(u,\, s) - \frac{s}{u}x(0) = 1 - H\left[x^2\right]. \tag{21}$$

Substitute the given initial condition from Equation (21), we have

$$\frac{s}{u}X(u,\, s) = 1 - H\left[x^2\right]. \tag{22}$$

After simple calculation from Equation (22), we have

$$X(u,\, s) = \frac{u}{s} - \frac{u}{s}H\left[x^2\right]. \tag{23}$$

By taking the inverse ZZ transform of Equation (23), we have

$$x(t) = t - H^{-1}\left(\frac{u}{s}H\left[x^2\right]\right). \tag{24}$$

We now assume an infinite series solution of the unknown function $x(t)$ of the form

$$x(t) = \sum_{n=0}^{\infty} x_n(t). \tag{25}$$

By using Equation (25), we can write Equation (24) in the form

$$\sum_{n=0}^{\infty} x_n\,(t) = t - H^{-1}\left[\frac{u}{s}\left[H\sum_{n=0}^{\infty} A_n(t)\right]\right], \tag{26}$$

where, A_n is called Adomian's polynomials of the nonlinear term $x^2(t)$. Now, by comparing both sides of Equation (26), we can get the following recurrence relation:

$$\begin{aligned}
x_0\,(t) &= t,\\
x_1\,(t) &= -H^{-1}\left[\frac{u}{s}H\left[A_0(t)\right]\right],\\
x_2\,(t) &= -H^{-1}\left[\frac{u}{s}H\left[A_1(t)\right]\right],\\
x_3(t) &= -H^{-1}\left[\frac{u}{s}H\left[A_2(t)\right]\right].
\end{aligned}$$

Finally, we have the following general recurrence relation

$$x_{n+1}\,(t) = -H^{-1}\left[\frac{u}{s}H\left[A_n\,(t)\right]\right],\ n \geq 0. \tag{27}$$

Then by using the recursive relation in Equation (27), we can easily compute the remaining components of the unknown function $x\,(t)$ in the following manner

$$\begin{aligned}
x_1(t) &= -H^{-1}\left[\frac{u}{s}H\left[A_0(t)\right]\right] = -H^{-1}\left[\frac{u}{s}H[x_0(t)]^2\right] = -H^{-1}\left[\frac{u}{s+u}H(t)^2\right] = -H^{-1}\left[\frac{u}{s}\times 2!\frac{u^2}{s^2}\right]\\
&= -2!H^{-1}\left[\frac{u^3}{s^3}\right] = -\frac{t^3}{3},\\
x_2\,(t) &= -H^{-1}\left[\frac{u}{s}H\left[A_1\,(t)\right]\right] = -H^{-1}\left[\frac{u}{s}H\left[2y_0\,(t)\,y_1\,(t)\right]\right] = -H^{-1}\left[\frac{u}{s}H\left(2t\times-\frac{t^3}{3}\right)\right]\\
&= -H^{-1}\left[\frac{u}{s}H\left(\frac{-2}{3}t^4\right)\right] = -H^{-1}\left[\frac{-2}{3}\frac{u}{s}H\left(t^4\right)\right] = \frac{2}{3}H^{-1}\left[4!\frac{u^5}{s^5}\right] = \frac{2}{3}\times 4!\ H^{-1}\left[\frac{u^5}{s^5}\right]\\
&= \frac{2}{3}\times 4!\frac{t^5}{5!} = \frac{2}{15}t^5.
\end{aligned}$$

Similarly, we can find $x_3\,(t)$

$$x_3\,(t) = -H^{-1}\left[\frac{u}{s}H\left[A_2\,(t)\right]\right] = -H^{-1}\left[\frac{u}{s}H\left[2x_0\,(t)\,x_2\,(t)\right] + \left(x_1\,(t)\right)^2\right].$$

After, some calculation step, we obtain $x_3\,(t) = \frac{-17}{315}t^7$ and so on. Hence, the approximate solution is given by;

$$x\,(t) = \sum_{n=0}^{\infty} x_n\,(t) = x_0\,(t) + x_1\,(t) + x_2\,(t) + x_3\,(t) + \cdots = t - \frac{t^3}{3} + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \ldots.$$

We obtain the following graph, that is the comparison of approximate and exact solution of the given differential equation depend on the order of expansion using Octave. The line (graph) in the red color indicates the actual solution, while the ring line (o) indicates the approximate solution.

Example 3. Solve

$$\frac{dv}{dt} + \left(\frac{dv}{dt}\right)^2 = 4v\,(t), \qquad v\,(0) = 1. \tag{28}$$

Applying ZZ Transform of Equation (28), we have

$$\frac{s}{u}V\,(u,\ s) - \frac{s}{u}v\,(0) + H\left[\left(\frac{dv}{dt}\right)^2\right] = 4V\,(u, s)\,. \tag{29}$$

Substitute the given initial condition from Equation (29), we have

$$\frac{s}{u}V\left(u,\ s\right)-\frac{s}{u}=4V\left(u,s\right)-H\left[\left(\frac{dv}{dt}\right)^{2}\right]. \tag{30}$$

After simple calculation from Equation (30), we have

$$V\left(u,\ s\right)=\frac{s}{s-4u}-\frac{u}{s-4u}H\left[\left(\frac{dv}{dt}\right)^{2}\right]. \tag{31}$$

By taking the inverse ZZ transform of Equation (31), we have

$$v\left(t\right)=e^{4t}-\frac{u}{s-4u}H\left[\left(\frac{dv}{dt}\right)^{2}\right]. \tag{32}$$

We now assume an infinite series solution of the unknown function $v\left(t\right)$ of the form

$$v\left(t\right)=\sum_{n=0}^{\infty}v_n(t). \tag{33}$$

By using Equation (33), we can write Equation (32) in the form

$$\sum_{n=0}^{\infty}x_n\left(t\right)=e^{4t}-H^{-1}\left[\frac{u}{s-4u}\left[H\sum_{n=0}^{\infty}A_n(t)\right]\right], \tag{34}$$

where, A_n is called Adomian's polynomials of the nonlinear term $\left(\frac{dv}{dt}\right)^2$. Now, by comparing both sides of Equation (34), we can get the following recurrence relation:

$$\begin{aligned}
v_0\left(t\right)&=e^{4t},\\
v_1\left(t\right)&=-H^{-1}\left[\frac{u}{s-4u}H\left[A_0(t)\right]\right],\\
v_2\left(t\right)&=-H^{-1}\left[\frac{u}{s-4u}H\left[A_1(t)\right]\right],\\
v_3(t)&=-H^{-1}\left[\frac{u}{s-4u}H\left[A_2(t)\right]\right].
\end{aligned}$$

Finally, we have the following general recurrence relation

$$v_{n+1}\left(t\right)=-H^{-1}\left[\frac{u}{s-4u}H\left[A_n\left(t\right)\right]\right],\ n\geq 0. \tag{35}$$

Then by using the recursive relation in Equation (35), we can easily compute the remaining components of the unknown function $v\left(t\right)$ in the following manner

$$\begin{aligned}
v_1(t)&=-H^{-1}\left[\frac{u}{s-4u}H\left[A_0(t)\right]\right]=-H^{-1}\left[\frac{u}{s-4u}H[v'_0]^2\right]=-H^{-1}\left[\frac{u}{s-4u}H\left[\left(e^{4t}\right)'\right]^2\right]\\
&=-4H^{-1}\left[\frac{u}{s-4u}\times\frac{4s}{s-8u}\right]=-4H^{-1}\left[\frac{s}{s-4u}-\frac{s}{s-8u}\right]=-4e^{4t}+4e^{8t},\\
v_2\left(t\right)&=-H^{-1}\left[\frac{u}{s-4u}H\left[A_1\left(t\right)\right]\right]=-H^{-1}\left[\frac{u}{s-4u}H\left[2v_0\left(t\right)y_1\left(t\right)\right]\right]=-H^{-1}\left[\frac{u}{s-4u}H\left(-8e^{8t}+8e^{12t}\right)\right]\\
&=-H^{-1}\left[\frac{-8us}{(s-4u)(s-8u)}+\frac{8us}{(s-4u)(s-12u)}\right]=-H^{-1}\left[-2\left(\frac{s}{s-8u}-\frac{s}{s-4u}\right)+\frac{s}{s-12u}-\frac{s}{s-4u}\right]\\
&=2e^{8t}-e^{4t}-e^{12t}.
\end{aligned}$$

Similarly, we can find $v_3(t)$ as;

$$v_3(t) = -H^{-1}\left[\frac{u}{s-4u}H\left[A_2(t)\right]\right] = -H^{-1}\left[\frac{u}{s-4u}H\left[2v_0(t)\,v_2(t)\right] + (v_1(t))^2\right].$$

After, some calculation step, we obtain

$$\begin{aligned}
v_3(t) &= -H^{-1}\left[\frac{u}{s-4u}H\left[14e^{8t} - 28e^{12t} + 14e^{16t}\right]\right]\\
&= -H^{-1}\left[\frac{u}{s-4u}H\left[\frac{14s}{s-8u} - \frac{28s}{s-12u} + \frac{14s}{s-16u}\right]\right]\\
&= -\frac{7}{2}H^{-1}\left[\frac{s}{s-8u} - \frac{s}{s-4u}\right] - \frac{7}{2}H^{-1}\left[\frac{s}{s-4u} - \frac{s}{s-12u}\right] - \frac{7}{6}H^{-1}\left[\frac{s}{s-16u} - \frac{s}{s-4u}\right]\\
&= -\frac{7}{2}e^{8t} + \frac{7}{2}e^{12t} - \frac{7}{6}e^{16t} + \frac{7}{6}e^{4t}.
\end{aligned}$$

Hence, the approximate solution is given by;

$$v(t) = \sum_{n=0}^{\infty} v_n(t) = v_0(t) + v_1(t) + v_2(t) + v_3(t) + \cdots = -\frac{17}{6}e^{4t} + \frac{5}{2}e^{8t} - \frac{5}{2}e^{12t} - \frac{7}{6}e^{16t} + \ldots.$$

3. Conclusion

In this paper, the ZZ decomposition method has been successfully applied to find approximate solution of the first order initial value problems of nonlinear ordinary differential equations. If the approximate solution of the given problems is compared with their analytical solutions, the ZZ decomposition is very effective and convergence are quite close. It may be concluded that ZZ decomposition method is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of nonlinear ordinary differential equations.

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References

[1] Abdelilah, K., & Hassan, S. (2017). The use of Kamal transform for solving partial differential equations. *Advances in Theoretical and Applied Mathematics, 12*(1), 7-13.

[2] Aggarwal, S., Gupta, A. R., Singh, D. P., Asthana, N., & Kumar, N. (2018). Application of Laplace transform for solving population growth and decay problems. *International Journal of Latest Technology in Engineering, Management & Applied Science, 7*(9), 141-145.

[3] Debnath, L., & Bhatta, D. (2014). *Integral transforms and their applications.* CRC press.

[4] Chauhan, R., & Aggarwal, S. (2018). Solution of linear partial integro-differential equations using Mahgoub transform. *Periodic Research, 7*(1), 28-31.

[5] Aggarwal, S., Sharma, N., Chauhan, R., Gupta, A. R., & Khandelwal, A. (2018). A new application of Mahgoub transform for solving linear ordinary differential equations with variable coefficients. *Journal of Computer and Mathematical Sciences, 9*(6), 520-525.

[6] Zill, D. G. (2016). *Advanced engineering mathematics*. Jones & Bartlett.

[7] Shaikh, S. L. (2018). Introducing a new integral transform: Sadik Transform. *American International Journal of Research in Science, Technology, Engineering & Mathematics, 22*(1), 100-102.

[8] Aggarwal, S., Sharma, N., & Chauhan, R. (2018). Applications of Kamal Transform for solving Volterra integral equation of first kind. *International Journal of Research in Advent Technology, 6*(8), 2081–2088.

[9] Song, Y., & Kim, H. (2014). The solution of Volterra integral equation of the second kind by using the Elzaki transform. *Applied Mathematical Sciences, 8*(11), 525-530.

[10] Mahgoub, M. M. A. (2019). The new integral transform"Sawi Transform". *Advances in Theoretical and Applied Mathematics, 14*(1), 81-87.

[11] Elzaki, T. M., & Elzaki, S. M. (2011). On the ELzaki Transform and System of Partial Differential Equations. *Advances in Theoretical and Applied Mathematics, 6*(1), 115-123.

[12] Osman, M., & Bashir, M. A. (2016). Solution of partial differential equations with variables coefficients using double Sumudu transform. *International Journal of Scientific and Research Publications, 6*, 37-46.

[13] Gore (Jagtap) Jyotsana, S., & Gore Shukracharya, S. (2015). Solution of partial integro-differential equations by using Laplace, Elzaki and double Elzaki transform methods. *International Research Journal of Engineering and Technology*, *2*(3), 1825–1830.