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A new version of Ostrowski type integral inequalities for different differentiable mapping

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Abstract: In this paper, improved and generalized version of Ostrowski's type inequalities is established. The parameters used in the peano kernels help us to obtain previous results. The obtained bounds are then applied to numerical integration.

Keywords: Ostrowski inequality; Numerical integration; Kernel.

MSC: 26A51; 26A33; 33E12; 26D15; 26A51; 26B25.

1. Introduction

Since Ostrowski first time proved his inequality in 1938, after that many researchers did a lot of work on it. Some monographs presented by Barnet *et al.*, [1] and Dragomir *et al.* [2] on Ostrowski's type inequalities. In the past years, many researchers [3–7] did efforts to obtain tighter error bounds of Ostrowski type inequalities. Inspired and motivated by the work of above famous Mathematician [8,9] and [2,10,11], we started our work to extend and produce new and generalized Ostrowski's integral inequalities.

In this paper, we introduced some new generalized different types of Kernels, development of new identities and new error bounds of Ostrowski's type inequalities for first and second derivable mappings. By utilizing our obtained results, previous famous results are recaptured as special cases.

2. Results for Quadratic mapping

Theorem 1. *Let $J \subseteq \mathbb{R}$ such that $c, d \in J$, and $c < d$. If $s : J \to \mathbb{R}$ is a derivable function such that $\gamma \leq s'(t) \leq \Gamma$, and $\varphi, \psi, \gamma, \Gamma \in \mathbb{R}$, then we get*

$$\left| \frac{1}{\varphi+\psi} \left[\frac{1}{2} \left(\varphi (x-c)^2 - \psi (x-d)^2 \right) s'(x) - \left(\varphi (x-c) + \psi (x-d) \right) s(x) + \left(\varphi \int_c^x s(t)\,dt + \psi \int_x^d s(t)\,dt \right) \right.\right.$$
$$\left.\left. - \frac{\Gamma+\gamma}{12} \left(\varphi (x-c)^3 - \psi (x-d)^3 \right) \right] \right| \leq \frac{1}{\varphi+\psi} \frac{\Gamma-\gamma}{12} \left(\varphi (x-c)^3 - \psi (x-d)^3 \right), \tag{1}$$

for all $t \in [c,d]$.

Proof. Define a new peano Kernel $L(x,t) : [c,d] \to \mathbb{R}$ by

$$L(x,t) = \begin{cases} \frac{\varphi}{\varphi+\psi} \frac{(t-c)^2}{2}, & t \in [c,x] \\ \frac{\psi}{\varphi+\psi} \frac{(t-d)^2}{2}, & t \in (x,d] \end{cases} \tag{2}$$

for all $x \in [c,d]$. By using (2), we get

$$\int_c^d L(x,t)\, s''(t)\, dt = \frac{1}{\varphi+\psi}\left[\frac{\varphi}{2}(x-c)^2 s'(x) - \frac{\psi}{2}(x-d)^2 s'(x) - \varphi(x-c)\right.$$
$$\left. \times s(x) + \psi(x-d)\, s(x) + \left(\varphi\int_c^x s(t)\, dt + \psi\int_x^d s(t)\, dt\right)\right]. \tag{3}$$

Again, by using (2), we get

$$\int_c^d L(x,t)\, dt = \frac{1}{6(\varphi+\psi)}\left(\varphi(x-c)^3 - \psi(x-d)^3\right). \tag{4}$$

Using (3) and (4), we get

$$\int_c^d L(x,t)\left(s''(t) - C\right) dt = \frac{1}{\varphi+\psi}\left[\frac{\varphi}{2}(x-c)^2 s'(x) - \frac{\psi}{2}(x-d)^2 s'(x) - \varphi(x-c)\, s(x) + \psi(x-d)\, s(x)\right.$$
$$\left. + \left(\varphi\int_c^x s(t)\, dt + \psi\int_x^d s(t)\, dt\right) - \frac{C}{6}\left(\varphi(x-c)^3 - \psi(x-d)^3\right)\right]. \tag{5}$$

On the other hand

$$\left|\int_c^d L(x,t)\left(s''(t) - C\right)\right| \le \max_{t\in[c,d]} \left|s''(t) - C\right| \int_c^d L(x,t)\, dt. \tag{6}$$

$$\int_c^d \left|L(x,t)\right| dt = \frac{1}{6(\varphi+\psi)}\left(\varphi(x-c)^3 - \psi(x-d)^3\right). \tag{7}$$

Let $C = \frac{\Gamma+\gamma}{2}$, then, $\max_{t\in[c,d]} \left|s''(t) - C\right| \le \frac{\Gamma-\gamma}{2}$. Thus (6) becomes

$$\left|\int_c^d L(x,t)\left(s''(t) - C\right) dt\right| \le \frac{\Gamma-\gamma}{2}\left[\frac{1}{6(\varphi+\psi)}\left(\varphi(x-c)^3 - \psi(x-d)^3\right)\right]. \tag{8}$$

Using (5) in (8), we get our required result (1). □

Remark 1. By putting $\varphi = \psi$ in (1), we get

$$\left|(d-c)\left(x - \frac{c+d}{2}\right)s'(x) - (d-c)\, s(x) + \int_c^d s(t)\, dt - \frac{\Gamma+\gamma}{2}(d-c)\left(\frac{(d-c)^2}{24} + \frac{1}{2}\left(x - \frac{c+d}{2}\right)^2\right)\right|$$
$$\le \frac{\Gamma-\gamma}{2}(d-c)\left(\frac{(d-c)^2}{24} + \frac{1}{2}\left(x - \frac{c+d}{2}\right)^2\right). \tag{9}$$

Corollary 2. *By putting $x = \frac{c+d}{2}$ in (9), we get mid point inequality:*

$$\left|\int_c^d s(t)\, dt - (d-c)\, s\left(\frac{c+d}{2}\right) - \frac{1}{48}(\Gamma+\gamma)(d-c)^3\right| \le \frac{1}{48}(\Gamma-\gamma)(d-c)^3.$$

3. Applications in numerical integration

Using [5], we suppose that $J_n : c = x_0 < x_1 < x_2 < \ldots. < x_{n-1} < x_n = d$ a partition of $[c,d]$, $\xi_i \in \left[x_i + \delta\frac{\varrho_i}{2}, x_{i+1} - \delta\frac{\varrho_i}{2}\right]$, $(i = 0, 1, \ldots.., n-1)$ and $\varrho_i = x_{i+1} - x_i$, $(i = 0, 1, \ldots.., n-1)$, then following theorem exist:

Theorem 3. *Let $s : [c,d] \to \mathbb{R}$ be continuous on $[c,d]$ and derivable on (c,d), then following formula exist:*

$$\int_c^d s(t)\,dt = A(s,\xi,J_n) + R(s,\xi,J_n), \tag{10}$$

where

$$\begin{aligned} A(s,\xi,J_n) \leq & \sum_{i=0}^{n-1} \frac{1}{2(\varphi+\psi)} \left(\varphi(\xi_i - x_i)^2 - \psi(\xi_i - x_{i+1})^2\right) s'(\xi_i) \\ & - \sum_{i=0}^{n-1} \frac{1}{\varphi+\psi} \left(\varphi(\xi_i - x_i) + \psi(\xi_i - x_{i+1})\right) s(\xi_i), \end{aligned} \tag{11}$$

$$R(s,\xi,J_n) \leq \Gamma \sum_{i=0}^{n-1} \frac{\varrho_i}{6(\varphi+\psi)} \left(\varphi(\xi_i - x_i)^3 - \psi(\xi_i - x_{i+1})^3\right). \tag{12}$$

and remainder satisfies the estimation for all $\xi_i \in [x_i, x_{i+1}]$.

Proof. By using Theorem 1 on $[x_i, x_{i+1}]$, $\xi_i \in [x_i, x_{i+1}]$, to get:

$$\begin{aligned} & \Bigg| \frac{1}{2(\varphi+\psi)} \left(\varphi(\xi_i - x_i)^2 - \psi(\xi_i - x_{i+1})^2\right) s'(\xi_i) - \frac{1}{\varphi+\psi} \left[\varphi(\xi_i - x_i) + \psi(\xi_i - x_{i+1})\right] s(\xi_i) \\ & + \frac{1}{\varphi+\psi} \left(\varphi \int_{x_i}^{\xi_i} s(t)\,dt + \psi \int_{\xi_i}^{x_{i+1}} s(t)\,dt\right) - \frac{\varrho_i}{12(\varphi+\psi)} (\Gamma+\gamma) \left(\varphi(\xi_i - x_i)^3 - \psi(\xi_i - x_{i+1})^3\right) \Bigg| \\ & \leq \frac{\varrho_i}{12(\varphi+\psi)} (\Gamma-\gamma) \left(\varphi(\xi_i - x_i)^3 - \psi(\xi_i - x_{i+1})^3\right), \end{aligned} \tag{13}$$

or

$$\begin{aligned} & \Bigg| \sum_{i=0}^{n-1} \frac{1}{2(\varphi+\psi)} \left(\varphi(\xi_i - x_i)^2 - \psi(\xi_i - x_{i+1})^2\right) s'(\xi_i) - \sum_{i=0}^{n-1} \frac{1}{\varphi+\psi} \left(\varphi(\xi_i - x_i) + \psi(\xi_i - x_{i+1})\right) s(\xi_i) \\ & + \sum_{i=0}^{n-1} \frac{1}{\varphi+\psi} \left(\varphi \int_c^x s(t)\,dt + \psi \int_x^d s(t)\,dt\right) - \frac{\Gamma+\gamma}{2} \sum_{i=0}^{n-1} \frac{\varrho_i}{6(\varphi+\psi)} \left(\varphi(\xi_i - x_i)^3 - \psi(\xi_i - x_{i+1})^3\right) \Bigg| \\ & \leq \frac{\Gamma-\gamma}{2} \sum_{i=0}^{n-1} \frac{\varrho_i}{6(\varphi+\psi)} \left(\varphi(\xi_i - x_i)^3 - \psi(\xi_i - x_{i+1})^3\right). \end{aligned}$$

With the help of generalized triangular inequality, we get the desired estimation. □

4. Results for generalized linear mapping

Theorem 4. *Let $r : I \to \mathbb{R}$ and $v, w \in I$, $v < w$. If $g' : I \to \mathbb{R}$, such that $\gamma \leq r'(t) \leq \Gamma$, $\forall\, t \in [v,w]$ and $\varphi, \psi, \gamma, \Gamma \in \mathbb{R}$. We have*

$$\begin{aligned} & \Bigg| \frac{1}{\varphi+\psi} \left[\varphi\left(x - v - \varrho\frac{w-v}{2}\right) - \psi\left(x - w + \varrho\frac{w-v}{2}\right)\right] r(x) + \frac{\varrho(w-v)}{2(\varphi+\psi)} \left(\varphi r(v) + \psi r(w)\right) \\ & - \frac{1}{\varphi+\psi} \left(\varphi \int_v^x r(t)\,wt + \psi \int_x^w r(t)\,wt\right) - \frac{\Gamma+\gamma}{4(\varphi+\psi)} \left(\varphi\left(x - v - \varrho\frac{w-v}{2}\right)^2 - \psi\left(x - w + \varrho\frac{w-v}{2}\right)^2\right) \\ & + \frac{\varrho^2(\varphi-\psi)}{16(\varphi+\psi)} (\Gamma+\gamma)(w-v)^2 \Bigg| \\ & \leq \frac{\Gamma-\gamma}{2} \left[\frac{\varrho^2}{8}(w-v)^2 + \frac{1}{2(\varphi+\psi)} \left(\varphi\left(x - v - \varrho\frac{w-v}{2}\right)^2 + \psi\left(x - w + \varrho\frac{w-v}{2}\right)^2\right)\right]. \end{aligned} \tag{14}$$

Proof. First we define the mapping $L(x,t):[v,w]\to\mathbb{R}$ by

$$L(x,t)=\begin{cases}\frac{\varphi}{\varphi+\psi}\left[t-\left(v+\varrho\frac{w-v}{2}\right)\right], & t\in[v,x]\\ \frac{\psi}{\varphi+\psi}\left[t-\left(w-\varrho\frac{w-v}{2}\right)\right], & t\in(x,w]\end{cases} \tag{15}$$

By using (15), we get

$$\int_v^w L(x,t)\,r'(t)\,dt=\frac{1}{\varphi+\psi}\left[\varphi\left(x-\left(v+\varrho\frac{w-v}{2}\right)\right)r(x)-\psi\left(x-\left(w-\varrho\frac{w-v}{2}\right)\right)\right.$$
$$\left.\times r(x)+\varrho\frac{w-v}{2}\left(\varphi r(v)+\psi r(w)\right)-\left(\varphi\int_v^x r(t)\,dt+\psi\int_x^w r(t)\,dt\right)\right], \tag{16}$$

and

$$\int_v^w L(x,t)\,dt=\frac{\varphi}{2(\varphi+\psi)}\left[\left(x-\left(v+\varrho\frac{w-v}{2}\right)\right)^2-\frac{\varrho^2}{4}(w-v)^2\right]$$
$$+\frac{\psi}{2(\varphi+\psi)}\left[\frac{\varrho^2}{4}(w-v)^2-\left(x-\left(w-\varrho\frac{w-v}{2}\right)\right)^2\right]. \tag{17}$$

We put $C=\frac{\Gamma+\gamma}{2}$ and using (16) and (17), we get

$$\int_v^w L(x,t)\left(r'(t)-C\right)dt=\frac{\varphi}{\varphi+\psi}\left[\left(x-\left(v+\varrho\frac{w-v}{2}\right)\right)r(x)+\frac{\varrho}{2}(w-v)\,r(v)-\int_v^x r(t)\,dt\right]$$
$$+\frac{\psi}{\varphi+\psi}\left[\frac{\varrho}{2}(w-v)\,r(w)-\left(x-\left(w-\varrho\frac{w-v}{2}\right)\right)r(x)-\int_x^w r(t)\,dt\right]$$
$$-\frac{C\varphi}{2(\varphi+\psi)}\left[\left(x-\left(v+\varrho\frac{w-v}{2}\right)\right)^2-\frac{\varrho^2}{4}(w-v)^2\right]$$
$$-\frac{C\psi}{2(\varphi+\psi)}\left[\frac{\varrho^2}{4}(w-v)^2-\left(x-\left(w-\varrho\frac{w-v}{2}\right)\right)^2\right]. \tag{18}$$

Let

$$C=\frac{\Gamma+\gamma}{2}.$$

Then

$$\left|\int_v^w L(x,t)\left(r'(t)-C\right)dt\right|\le\max_{t\in[v,w]}\left|r'(t)-C\right|\int_v^w |L(x,t)|\,dt. \tag{19}$$

Now

$$\int_v^w |L(x,t)|\,dt=\frac{\varrho^2}{8}(w-v)^2+\frac{1}{2(\varphi+\psi)}\left[\varphi\left(x-v-\varrho\frac{w-v}{2}\right)^2+\psi\left(x-w+\varrho\frac{w-v}{2}\right)^2\right], \tag{20}$$

$$\max_{t\in[v,w]}\left|r'(t)-C\right|\le\frac{\Gamma-\gamma}{2}\ \text{ for all }\ \gamma\le t\le\Gamma. \tag{21}$$

Using (19) and (21), we have

$$\left|\int_v^w L(x,t)\left(r'(t)-\frac{\Gamma+\gamma}{2}\right)dt\right|$$
$$\le\frac{\Gamma-\gamma}{2}\left[\frac{\varrho^2}{8}(w-v)^2+\frac{1}{2(\varphi+\psi)}\left(\varphi\left(x-v-\varrho\frac{w-v}{2}\right)^2+\psi\left(x-w+\varrho\frac{w-v}{2}\right)^2\right)\right]. \tag{22}$$

Using (18) and (22), we get our required result (14). □

Remark 2. By putting $\varrho = 0$ in (14), we get

$$\left|\frac{1}{\varphi+\psi}\left[(\varphi(x-v)-\psi(x-w))r(x)-\left(\varphi\int_v^x r(t)\,dt+\psi\int_x^w r(t)\,dt\right)\right.\right.$$
$$\left.\left.-\frac{1}{4}(\Gamma+\gamma)\left(\varphi(x-v)^2-\psi(x-w)^2\right)\right]\right|$$
$$\leq \frac{\Gamma-\gamma}{2}\left[\frac{\varrho^2}{8}(w-v)^2+\frac{1}{2(\varphi+\psi)}\left(\varphi(x-v)^2+\psi(x-w)^2\right)\right].$$

5. Results for generalized Quadratic mapping

Theorem 5. *Let $z : I \subseteq \mathbb{R}$, and $c, d \in I$, $c < d$. If $z : I \to \mathbb{R}$ is a derivable function such that $\gamma \leq z'(t) \leq \Gamma$, $\forall\, t \in [c,d]$, the constants $\varphi, \psi, \gamma, \Gamma \in \mathbb{R}$. Then, we get*

$$\left|\frac{1}{2(\varphi+\psi)}\left(\varphi\left(x-c-\varrho\frac{d-c}{2}\right)^2-\psi\left(x-d+\varrho\frac{d-c}{2}\right)^2\right)z'(x)+\frac{\varrho^2}{8(\varphi+\psi)}(d-c)^2(\psi z'(d)-\varphi z'(c))\right.$$
$$+\frac{1}{\varphi+\psi}\left(\psi\left(x-d+\varrho\frac{d-c}{2}\right)-\varphi\left(x-c-\varrho\frac{d-c}{2}\right)\right)z(x)-\frac{\varrho}{2(\varphi+\psi)}(d-c)(\varphi z(c)+\psi z(d))$$
$$+\frac{1}{\varphi+\psi}\left(\varphi\int_c^x z(t)\,dt+\psi\int_x^d z(t)\,dt\right)-\frac{\Gamma+\gamma}{2}\left[\frac{\varrho^3}{48}(d-c)^3+\frac{1}{6(\varphi+\psi)}\left(\varphi\left(x-c-\varrho\frac{d-c}{2}\right)^3\right.\right.$$
$$\left.\left.\left.-\psi\left(x-d+\varrho\frac{d-c}{2}\right)^3\right)\right]\right|$$
$$\leq \frac{\Gamma-\gamma}{2}\left[\frac{\varrho^3(\psi-\varphi)}{48(\varphi+\psi)}(d-c)^3+\frac{1}{6(\varphi+\psi)}\left(\varphi\left(x-d+\varrho\frac{d-c}{2}\right)^3+\psi\left(x-c-\varrho\frac{d-c}{2}\right)^3\right)\right]. \tag{23}$$

Proof. Let us define the mapping

$$L(x,t)=\begin{cases}\frac{\varphi}{2(\varphi+\psi)}\left[t-\left(c+\varrho\frac{d-c}{2}\right)\right]^2, & t\in[c,x]\\ \frac{\psi}{2(\varphi+\psi)}\left[t-\left(d-\varrho\frac{d-c}{2}\right)\right]^2, & t\in(x,d]\end{cases} \tag{24}$$

By using (24), we get

$$\int_c^d L(x,t)z''(t)\,dt=\frac{1}{\varphi+\psi}\left[\frac{\varphi}{2}\left[x-\left(c+\varrho\frac{d-c}{2}\right)\right]^2z'(x)-\frac{\psi}{2}\left[x-\left(d-\varrho\frac{d-c}{2}\right)\right]^2z'(x)-\frac{\varphi}{8}\varrho^2(d-c)^2z'(c)\right.$$
$$+\frac{\psi}{8}\varrho^2(d-c)^2z'(d)-\varphi\left[x-\left(c+\varrho\frac{d-c}{2}\right)\right]z(x)+\psi\left[x-\left(d-\varrho\frac{d-c}{2}\right)\right]z(x)$$
$$\left.-\frac{\varphi}{2}\varrho(d-c)z(c)-\frac{\psi}{2}\varrho(d-c)z(d)+\varphi\int_c^x z(t)\,dt+\psi\int_x^d z(t)\,dt\right] \tag{25}$$

and

$$\int_c^d L(x,t)\,dt=\frac{1}{6(\varphi+\psi)}\left(\varphi\left(x-c-\varrho\frac{d-c}{2}\right)^3-\psi\left(x-d+\varrho\frac{d-c}{2}\right)^3\right)+\frac{\varrho^3}{48}(d-c)^3. \tag{26}$$

Using (25) and (26), we get

$$\int_c^d L(x,t)\left(z''(t)-C\right)dt = \frac{1}{\varphi+\psi}\left[\frac{\varphi}{2}\left[x-\left(c+\varrho\frac{d-c}{2}\right)\right]^2 z'(x)-\frac{\psi}{2}\left[x-\left(d-\varrho\frac{d-c}{2}\right)\right]^2 z'(x)\right.$$
$$-\frac{\varphi}{8}\varrho^2(d-c)^2 z'(c)+\frac{\psi}{8}\varrho^2(d-c)^2 z'(d)-\varphi\left[x-\left(c+\varrho\frac{d-c}{2}\right)\right]z(x)+\psi\left[x-\left(d-\varrho\frac{d-c}{2}\right)\right]z(x)$$
$$\left.-\frac{\varphi}{2}\varrho(d-c)z(c)-\frac{\psi}{2}\varrho(d-c)z(d)+\varphi\int_c^x z(t)\,dt+\psi\int_x^d z(t)\,dt\right]-C\left[\frac{\varrho^3}{48}(d-c)^3+\frac{1}{6(\varphi+\psi)}\right.$$
$$\left.\times\left(\varphi\left(x-c-\varrho\frac{d-c}{2}\right)^3-\psi\left(x-d+\varrho\frac{d-c}{2}\right)^3\right)\right]. \tag{27}$$

But on the other side,

$$\left|\int_c^d L(x,t)\left(z''(t)-C\right)dt\right| \le \max_{t\in[c,d]}\left|z''(t)-C\right|\int_c^d |L(x,t)|\,dt. \tag{28}$$

Now, again by using (24), we get

$$\int_c^d |L(x,t)|\,dt = -\frac{\varrho^3(\varphi-\psi)}{48(\varphi+\psi)}(d-c)^3+\frac{1}{6(\varphi+\psi)}\left(\varphi\left[x-\left(c+\varrho\frac{d-c}{2}\right)\right]^3+\psi\left[x-\left(d-\varrho\frac{d-c}{2}\right)\right]^3\right) \tag{29}$$

and

$$C=\frac{\Gamma+\gamma}{2}. \tag{30}$$

Also

$$\max_{t\in[c,d]}\left|z''(t)-C\right| \le \frac{\Gamma-\gamma}{2}. \tag{31}$$

Using (28) and (29), we get

$$\left|\int_c^d L(x,t)\left(z''(t)-C\right)dt\right|$$
$$= -\frac{\varrho^3(\varphi-\psi)}{48(\varphi+\psi)}(d-c)^3+\frac{1}{6(\varphi+\psi)}\left(\varphi\left[x-\left(c+\varrho\frac{d-c}{2}\right)\right]^3+\psi\left[x-\left(d-\varrho\frac{d-c}{2}\right)\right]^3\right). \tag{32}$$

Using (27) and (32), we get our required result (23). □

6. Conclusion

In this paper, we proved the results by using quadratic mapping, generalized linear mapping and generalized quadratic mapping. We developed application for numerical integration also.

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Conflicts of Interest: "The authors declare no conflict of interest".

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