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Asymptotic approximation of central binomial coefficients with rigorous error bounds

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Abstract: We show that a well-known asymptotic series for the logarithm of the central binomial coefficient is strictly enveloping in the sense of Pólya and Szegö, so the error incurred in truncating the series is of the same sign as the next term, and is bounded in magnitude by that term. We consider closely related asymptotic series for Binet's function, for $\ln\Gamma(z+\frac{1}{2})$, and for the Riemann-Siegel theta function, and make some historical remarks.

Keywords: Asymptotic series; Binet function; Binomial coefficient; Central binomial coefficient; Gamma function; Riemann-Siegel theta function; Stirling's approximation; strictly enveloping series.

MSC: 05A10; 11B65; 33B15; 41A60.

1. Introduction

Let $z \in \mathbb{C}$ and assume that $\Re z > 0$. It is well-known that

$$\ln\Gamma(z) = (z-\tfrac{1}{2})\ln z - z + \tfrac{1}{2}\ln(2\pi) + J(z), \tag{1}$$

where $J(z)$ can be written as

$$J(z) = \frac{1}{\pi}\int_0^\infty \frac{z}{\eta^2+z^2}\ln\left(\frac{1}{1-e^{-2\pi\eta}}\right)d\eta\,. \tag{2}$$

The analytic function $J(z)$ is known as *Binet's function* and has several equivalent expressions; see for example, Henrici [1, (8.5-7)].

Binet's function has an asymptotic expansion

$$J(z) \sim \frac{\beta_0}{z} - \frac{\beta_1}{z^3} + \frac{\beta_2}{z^5} - \cdots, \tag{3}$$

or more precisely, for non-negative integers k,

$$J(z) = \sum_{j=0}^{k-1}(-1)^j\frac{\beta_j}{z^{2j+1}} + r_k(z), \tag{4}$$

where

$$\beta_k = \frac{1}{\pi}\int_0^\infty \eta^{2k}\ln\left(\frac{1}{1-e^{-2\pi\eta}}\right)d\eta \tag{5}$$

and

$$r_k(z) = \frac{(-1)^k}{\pi z^{2k-1}}\int_0^\infty \frac{\eta^{2k}}{z^2+\eta^2}\ln\left(\frac{1}{1-e^{-2\pi\eta}}\right)d\eta\,. \tag{6}$$

It may be shown that

$$\beta_k = \frac{2(2k)!}{(2\pi)^{2k+2}}\,\zeta(2k+2) = \frac{(-1)^k}{(2k+1)(2k+2)}\,B_{2k+2}\,, \tag{7}$$

where B_{2k+2} is a Bernoulli number ($B_2 = 1/6, B_4 = -1/30$, etc.). Proofs of these results are given in Henrici's book [1, §11.1].[1] As far as possible, we have followed Henrici's notation.

Substituting (4) into (1) gives an asymptotic expansion for $\ln\Gamma(z)$ that is usually named after James Stirling, although some credit is due to Abraham de Moivre. For the history and early references, see Dutka [2]. It is interesting to note that de Moivre started (about 1721) by trying to approximate the central binomial coefficient $\binom{2n}{n}$, not the factorial (or Gamma) function – see Dutka [2, pg. 227].

It is easy to see from (5) and (6) that

$$r_k(z) = \theta_k(z)\,(-1)^k \frac{\beta_k}{z^{2k+1}}\,, \tag{8}$$

where

$$\theta_k(z) = \int_0^\infty \frac{z^2\,\eta^{2k}}{z^2+\eta^2} \ln\left(\frac{1}{1-e^{-2\pi\eta}}\right) d\eta \Bigg/ \int_0^\infty \eta^{2k} \ln\left(\frac{1}{1-e^{-2\pi\eta}}\right) d\eta\,. \tag{9}$$

Suppose now that z is real and positive. Since $z^2/(z^2+\eta^2) \in (0,1)$ and the logarithmic factors in (9) are positive for all $\eta \in (0,\infty)$, we see that

$$\theta_k(z) \in (0,1). \tag{10}$$

Thus, the remainder $r_k(z)$ given by (8) has the same sign as the next term $(-1)^k\beta_k/z^{2k+1}$ in the asymptotic series, and is smaller in absolute value. In the terminology used by Pólya and Szegö [3, Ch. 4], the asymptotic series for $\ln\Gamma(z)$ *strictly envelops*[2] the function $\ln\Gamma(z)$.[3]

§2 shows that we can deduce a strictly enveloping asymptotic series for $\ln(\Gamma(2z+1)/\Gamma(z+1)^2)$ or equivalently, if $z = n$ is a positive integer, for the logarithm of the central binomial coefficient $\binom{2n}{n}$. The series itself is well known, but we have not found the enveloping property or the resulting error bound mentioned in the literature. Henrici was aware of it, since in his book [1, §11.2, Problem 6] he gave the special case $k = 3$ as an exercise, along with a hint for the solution. Hence, we do not claim any particular originality. Our purpose is primarily to make some useful asymptotic series and their associated error bounds readily accessible. Related results and additional references may be found, for example, in [4–6].

In §2 we consider the central binomial coefficient and its generalisation to a complex argument. Then, in §3, we consider some closely related asymptotic series that we can prove to be strictly enveloping. In §4 we make some remarks on asymptotic series that are *not* enveloping. An Appendix gives numerical values of the coefficients appearing in three of the asymptotic series.

Finally, we remark that it is possible to give asymptotic series related to $\Gamma(z+\frac{1}{2})/\Gamma(z)$ and $\binom{2n}{n}$, but in general these series do not alternate in sign. See, for example, [7], [8], [9, ex. 9.60 and pg. 602], [10], and [11].

2. Asymptotic series for central binomial coefficients

Define

$$\begin{aligned}
\widetilde{\Gamma}(z) &:= \frac{\Gamma(2z+1)}{\Gamma(z+1)^2}\,,\\
\widetilde{J}(z) &:= J(2z) - 2J(z),\\
\widetilde{r}_k(z) &:= r_k(2z) - 2r_k(z),
\end{aligned}$$

and

$$\widetilde{\beta}_k := (2-2^{-2k-1})\beta_k = (-1)^k \frac{(1-4^{-k-1})}{(k+1)(2k+1)} B_{2k+2}\,. \tag{11}$$

As noted above, the central binomial coefficient $\binom{2n}{n}$ is simply $\widetilde{\Gamma}(n)$.

Using elementary properties of the Gamma function, we have

$$\widetilde{\Gamma}(z) = \frac{2}{z}\frac{\Gamma(2z)}{\Gamma(z)^2}\,. \tag{12}$$

[1] There is an error in Henrici's equation (11.1-13): $2^{-2\pi\eta}$ should be replaced by $e^{-2\pi\eta}$.

[2] We refer to the English translation. In the original it is "in engerem Sinne umhüllen".

[3] When testing the enveloping property, we only consider the nonzero terms in the asymptotic series. See [3, Problem 142, footnote 1].

Thus, from (1) and the same equation with $z \mapsto 2z$, we have

$$\ln \widetilde{\Gamma}(z) = \ln\left(\frac{4^z}{\sqrt{\pi z}}\right) + \widetilde{J}(z). \tag{13}$$

Also, from (4) and the definition of $\widetilde{J}(z)$, we have an asymptotic series for $\widetilde{J}(z)$, namely:

$$\widetilde{J}(z) = -\sum_{j=0}^{k-1} (-1)^j \frac{\widetilde{\beta}_j}{z^{2j+1}} + \widetilde{r}_k(z). \tag{14}$$

Since $\binom{2n}{n} = \widetilde{\Gamma}(n)$, equations (13)–(14) give an asymptotic series for $\ln \binom{2n}{n}$. Lemma 1 shows that the remainder $\widetilde{r}_k(z)$ can be expressed as an integral analogous to the integral (6) for $r_k(z)$.

Lemma 1. *For $z \in \mathbb{C}, \Re z > 0$, and k a non-negative integer,*

$$\widetilde{\beta}_k = -\frac{1}{\pi}\int_0^\infty \eta^{2k} \ln \tanh(\pi\eta) d\eta\,, \tag{15}$$

$$\widetilde{r}_k(z) = \frac{(-1)^k}{\pi z^{2k-1}} \int_0^\infty \frac{\eta^{2k}}{z^2+\eta^2} \ln \tanh(\pi\eta) d\eta\,, \tag{16}$$

and

$$\widetilde{J}(z) = \widetilde{r}_0(z). \tag{17}$$

Proof. Making the change of variables $z \mapsto 2z$ and $\eta \mapsto 2\eta$ in (6), we obtain

$$r_k(2z) = \frac{(-1)^k}{\pi z^{2k-1}} \int_0^\infty \frac{\eta^{2k}}{z^2+\eta^2} \ln\left(\frac{1}{1-e^{-4\pi\eta}}\right) d\eta\,.$$

Now

$$\ln\left(\frac{1}{1-e^{-4\pi\eta}}\right) - 2\ln\left(\frac{1}{1-e^{-2\pi\eta}}\right) = \ln\left(\frac{1-e^{-2\pi\eta}}{1+e^{-2\pi\eta}}\right) = \ln\tanh(\pi\eta),$$

so (16)–(17) follow from the definitions of $\widetilde{r}_k(z)$ and $\widetilde{J}(z)$. The proof of (15) is similar. □

Corollary 1 gives a result analogous to Equations (8)–(9).

Corollary 1. *For $z \in \mathbb{C}, \Re z > 0$, and k a non-negative integer,*

$$\widetilde{r}_k(z) = \widetilde{\theta}_k(z)(-1)^{k+1}\frac{\widetilde{\beta}_k}{z^{2k+1}}, \tag{18}$$

where

$$\widetilde{\theta}_k(z) = \int_0^\infty \frac{z^2\eta^{2k}}{z^2+\eta^2} \ln\tanh(\pi\eta) d\eta \Big/ \int_0^\infty \eta^{2k} \ln\tanh(\pi\eta) d\eta\,. \tag{19}$$

Proof. This is straightforward from Equations (15)–(16) of Lemma 1. □

Corollary 2 gives a result analogous to the bound (10).

Corollary 2. *If z is real and positive, then $\widetilde{\theta}_k(z) \in (0,1)$.*

Proof. We write (19) as

$$\widetilde{\theta}_k(z) = \frac{\displaystyle\int_0^\infty \frac{z^2\eta^{2k}}{z^2+\eta^2} \ln\coth(\pi\eta) d\eta}{\displaystyle\int_0^\infty \eta^{2k} \ln\coth(\pi\eta) d\eta}. \tag{20}$$

Observe that $\coth(y) = \cosh(y)/\sinh(y) > 1$ for $y \in (0,\infty)$, so $\ln\coth(y) > 0$ for $y = \pi\eta > 0$. Since $z^2/(z^2+\eta^2) \in (0,1)$ for real positive z and η, the result follows. □

Corollary 3. *If z is real and positive, then the asymptotic series (14) for $\widetilde{J}(z)$ is strictly enveloping.*

Proof. This is immediate from Corollary 2. □

Remark 1. We may compare Corollary 2 with (the proof of) Lemma 2.7 of [12]. The latter, after allowing for different notation, gives the bound

$$\frac{-1}{4^{k+1}-1} < \widetilde{\theta}_k(z) < \frac{4^{k+1}}{4^{k+1}-1}.$$

This is clearly weaker than the bound of Corollary 2, and not sufficient to prove Corollary 3.

3. Some related asymptotic series

Lemma 2. *If $z \in \mathbb{C}, \Re z > 0$, then*

$$\widetilde{J}(z) = \ln\left(\frac{\Gamma(z+\frac{1}{2})}{z^{1/2}\Gamma(z)}\right).$$

Proof. This follows from Equations (12)–(13) and the duplication formula $\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\pi^{1/2}\Gamma(2z)$. □

From Lemma 2 and (14) we immediately obtain an asymptotic expansion

$$\ln\left(\frac{\Gamma(z+\frac{1}{2})}{\Gamma(z)}\right) \sim \frac{\ln z}{2} + \sum_{j\geq 0}(-1)^{j+1}\frac{\widetilde{\beta}_j}{z^{2j+1}} \tag{21}$$

which is strictly enveloping if z is real and positive.

Define

$$\widehat{\beta}_j = \widetilde{\beta}_j - \beta_j = (1-2^{-2j-1})\beta_j. \tag{22}$$

Using the asymptotic expansion for $\ln\Gamma(z)$ given by Equations (1) and (3), we see from (21) that $\ln\Gamma(z+\frac{1}{2})$ has an asymptotic expansion

$$\ln\Gamma(z+\tfrac{1}{2}) \sim z\ln z - z + \tfrac{1}{2}\ln(2\pi) + \sum_{j\geq 0}(-1)^{j+1}\frac{\widehat{\beta}_j}{z^{2j+1}}. \tag{23}$$

In fact, the expansion (23) was already obtained by Gauss [13, Eqn. [59] of Art. 29] in 1812. However, Gauss did not explicitly bound the truncation error. Writing (23) as

$$\ln\Gamma(z+\tfrac{1}{2}) = z\ln z - z + \tfrac{1}{2}\ln(2\pi) + \sum_{j=0}^{k-1}(-1)^{j+1}\frac{\widehat{\beta}_j}{z^{2j+1}} + \widehat{r}_k(z), \tag{24}$$

we have an unsurprising result for the truncation error $\widehat{r}_k(z)$: the error is of the same sign as the first neglected term $(-1)^{k+1}\widehat{\beta}_k/z^{2k+1}$, and is bounded in magnitude by this term. This is shown in Lemma 3 and Corollaries 4–5 below.

Lemma 3. *For $z \in \mathbb{C}, \Re z > 0$, and k a non-negative integer,*

$$\widehat{\beta}_k = \frac{1}{\pi}\int_0^\infty \eta^{2k}\ln\left(1+e^{-2\pi\eta}\right)d\eta \tag{25}$$

and

$$\widehat{r}_k(z) = \frac{(-1)^{k+1}}{\pi z^{2k-1}}\int_0^\infty \frac{\eta^{2k}}{z^2+\eta^2}\ln\left(1+e^{-2\pi\eta}\right)d\eta. \tag{26}$$

Proof. This is similar to the proof of Lemma 1. □

Corollary 4. *For $z \in \mathbb{C}, \Re z > 0$, and k a non-negative integer,*

$$\widehat{r}_k(z) = \widehat{\theta}_k(z)(-1)^{k+1}\frac{\widehat{\beta}_k}{z^{2k+1}}, \tag{27}$$

where

$$\widehat{\theta}_k(z) = \int_0^\infty \frac{z^2\,\eta^{2k}}{z^2+\eta^2} \ln\left(1+e^{-2\pi\eta}\right) d\eta \bigg/ \int_0^\infty \eta^{2k} \ln\left(1+e^{-2\pi\eta}\right) d\eta\,. \tag{28}$$

Proof. This is a straightforward consequence of Lemma 3. □

Corollary 5. *If z is real and positive, then the asymptotic expansion for $\ln\Gamma(z+\frac{1}{2})$ given in (24) is strictly enveloping.*

Proof. From (28) we have $\widehat{\theta}_k(z) \in (0,1)$. □

Remark 2. If we make the change of variables $z \mapsto n + \frac{1}{2}$ in (23), and assume that n is a positive integer, we obtain an asymptotic series for $n!$ in negative powers of $(n+\frac{1}{2})$:

$$\ln n! \sim (n+\tfrac{1}{2})\ln(n+\tfrac{1}{2}) - (n+\tfrac{1}{2}) + \tfrac{1}{2}\ln(2\pi) + \sum_{j\ge 0}(-1)^{j+1}\frac{\widehat{\beta}_j}{(n+\frac{1}{2})^{2j+1}}\,. \tag{29}$$

In fact, (29) was stated (without proof) by de Moivre [14,15] as early as 1730, see Dutka [2, (5), pg. 233].

4. Non-enveloping asymptotic series

Lest the reader has gained the impression that all "naturally occurring" asymptotic series are enveloping (for real positive arguments), we give two classes of examples to show that this is not the case. In fact, enveloping series are the exception, not the rule. Our first class of examples is given by the following Lemma.

Lemma 4. *Let $x \in (0,+\infty)$ and $f(x) := J(x) + \exp(-bx)$ for some constant $b \in (0, 2\pi)$. Then $f(x)$ has an asymptotic series*

$$f(x) \sim \sum_{j=0}^{\infty}(-1)^j \frac{\beta_j}{x^{2j+1}}\,. \tag{30}$$

However, the series (30) does not envelop f.

Proof. For all $k \ge 0$, $\exp(-bx) = O(x^{-2k-1})$ as $x \to +\infty$. Thus, it follows from (4) that $f(x)$ has the claimed asymptotic series (in fact the same series as the Binet function J.) This proves the first claim.

To prove the final claim, suppose, by way of contradiction, that the series (30) envelops f. For each integer $k > 0$, define $x_k := k/\pi$. From (7), the β_k grow like $(2k)!/(2\pi)^{2k}$, and from Stirling's approximation we see that

$$\beta_k / x_k^{2k+1} = O(\exp(-2\pi x_k)) \ \text{ as } \ k \to \infty. \tag{31}$$

Since the same series envelops both f and J, (31) implies that

$$|f(x_k) - J(x_k)| = O(\exp(-2\pi x_k)) \ \text{ as } \ k \to \infty.$$

Since $\exp(-2\pi x) = o(\exp(-bx))$, it follows that, for sufficiently large k,

$$|f(x_k) - J(x_k)| < \exp(-bx_k).$$

This contradicts the definition of f, so the assumption that the series (30) envelops f must be false. □

Remark 3. Lemma 4 can be generalised. For example, the conclusion holds if $f(x) = J(x) + g(x)$, where $g(x) = O(x^{-k})$ for all positive integers k, but $g(x) \ne O(\exp(-2\pi x))$. Also, we can replace the function $J(x)$ by a different function that has an enveloping asymptotic series whose terms grow at the same rate as those of $J(x)$.

Our second class of examples involves asymptotic expansions where all (or all but a finite number) of the terms are of the same sign (assuming a positive real argument x). Such series can not be strictly enveloping [3, Ch. 4]. As examples, we mention the Bessel function $I_0(x)$ (see Olver and Maximon [16, §10.40.1]), the product of two Bessel functions $I_0(x)K_0(x)$ (see [16, §10.40.6] and [17, Lemma 3.1]), and the Riemann-Siegel theta function (see [18, §6.5]). In all these examples the terms have constant sign, so the remainder changes monotonically as the number of terms increases with the argument x fixed. Eventually the remainder changes sign and starts increasing in absolute value. Often the point where the remainder changes sign is close to where the terms are smallest in absolute value, but this is not always true – see for example [19, §§4–5].

5. Concluding remarks

We have considered three different but related asymptotic series that can all be proved to be strictly enveloping. Our proofs depend on the fact that the three relevant functions $-\ln(1-e^{-2\pi\eta})$, $\ln\coth(\pi\eta)$, and $\ln(1+e^{-2\pi\eta})$ are positive for all $\eta \in (0,\infty)$. We remark that these three functions are linearly dependent, since

$$\coth(\pi\eta) = \frac{1+e^{-2\pi\eta}}{1-e^{-2\pi\eta}}.$$

It follows that the sequences $(\beta_k)_{k\geq 0}$, $(\widetilde{\beta}_k)_{k\geq 0}$ and $(\widehat{\beta}_k)_{k\geq 0}$ are linearly dependent. In fact, $\widetilde{\beta}_k = \beta_k + \widehat{\beta}_k$ for all $k \geq 0$. A table of numerical values is given in the Appendix.

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Conflicts of Interest: "The author declares no conflict of interest".

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Appendix: Numerical values of the coefficients

The table below gives the exact values of the coefficients β_k, $\widetilde{\beta}_k$ and $\widehat{\beta}_k$ for $0 \le k \le 6$. The values have been computed from equations (7), (11) and (22). We recall from the discussion above that the coefficients occur in the asymptotic expansions

$$\ln\Gamma(z) \sim (z-\tfrac{1}{2})\ln z - z + \tfrac{1}{2}\ln(2\pi) + \frac{\beta_0}{z} - \frac{\beta_1}{z^3} + \frac{\beta_2}{z^5} - \cdots,$$
$$\ln\binom{2n}{n} \sim \ln\left(\frac{4^n}{\sqrt{\pi n}}\right) - \frac{\widetilde{\beta}_0}{n} + \frac{\widetilde{\beta}_1}{n^3} - \frac{\widetilde{\beta}_2}{n^5} + \cdots, \text{ and}$$
$$\ln\Gamma(z+\tfrac{1}{2}) \sim z\ln z - z + \tfrac{1}{2}\ln(2\pi) - \frac{\widehat{\beta}_0}{z} + \frac{\widehat{\beta}_1}{z^3} - \frac{\widehat{\beta}_2}{z^5} + \cdots,$$

the $\widehat{\beta}_k$ also occurring in de Moivre's series (29) and, with a different sign pattern, in a series related to the Riemann-Siegel theta function [19, eqn. (2)]: $2\vartheta(t) \sim t\ln(t/2\pi e) - \pi/4 + \widehat{\beta}_0/t + \widehat{\beta}_1/t^3 + \cdots$. In all but the Riemann-Siegel theta function case the asymptotic series are strictly enveloping, so the error incurred in truncating the series can be bounded by the first term omitted, provided that $z \in (0,\infty)$ is real and that n is a positive integer. For error bounds if z is complex, we refer to [19, §§2–3].

k	β_k	$\widetilde{\beta}_k$	$\widehat{\beta}_k$
0	1/12	1/8	1/24
1	1/360	1/192	7/2880
2	1/1260	1/640	31/40320
3	1/1680	17/14336	127/215040
4	1/1188	31/18432	511/608256
5	691/360360	691/180224	1414477/738017280
6	1/156	5461/425984	8191/1277952

Remark 4. We note that the sequence $((-1)^k\beta_k)_{k\ge 0}$ is in the Online Encyclopedia of Integer Sequences (OEIS) [20]. The (signed) numerators are sequence A046968, and the denominators are sequence A046969. The sequence $(\widehat{\beta}_k/2)_{k\ge 0}$ is also in the OEIS: the numerators are sequence A036282, and the denominators are sequence A114721. We have added the sequence $((-1)^k\widetilde{\beta}_k)_{k\ge 0}$ to the OEIS. The (signed) numerators are sequence A275994, and the denominators are sequence A275995. Other relevant sequences are A143503 and A061549.