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Construction and classification of p-ring class fields modulo p-admissible conductors

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Abstract: Each p-ring class field K_f modulo a p-admissible conductor f over a quadratic base field K with p-ring class rank ϱ_f mod f is classified according to Galois cohomology and differential principal factorization type of all members of its associated heterogeneous multiplet $\mathbf{M}(K_f) = [(N_{c,i})_{1\le i\le m(c)}]_{c|f}$ of dihedral fields $N_{c,i}$ with various conductors $c \mid f$ having p-multiplicities $m(c)$ over K such that $\sum_{c|f} m(c) = \frac{p^{\varrho_f}-1}{p-1}$. The advanced viewpoint of classifying the entire collection $\mathbf{M}(K_f)$, instead of its individual members separately, admits considerably deeper insight into the class field theoretic structure of ring class fields. The actual construction of the multiplet $\mathbf{M}(K_f)$ is enabled by exploiting the routines for abelian extensions in the computational algebra system Magma.

Keywords: p-ring class fields; p-admissible conductors; Quadratic base fields; Non-Galois cubic fields; S_3-fields; Dihedral fields; Multiplicity of discriminants; p-ring spaces; Heterogeneous multiplets; Galois cohomology; Differential principal factorizations; Capitulation of p-class groups; Statistics.

MSC: 11R37; 11R11; 11R16; 11R20; 11R27; 11R29; 11Y40.

1. Introduction

The aim of this article is to present an entirely new technique for the construction and classification of non-Galois fields L of odd prime degree p as subfields $L < K_f$ of a p-ring class field K_f modulo a p-admissible conductor f over a quadratic base field K. The innovative idea underlying this new method is the fact that, if the Galois closure N of such a field L is absolutely dihedral of degree $2p$ with automorphism group $\mathrm{Gal}(N/\mathbb{Q}) \simeq D_p = \langle \sigma, \tau \mid \sigma^p = \tau^2 = 1, \tau\sigma = \sigma^{-1}\tau\rangle$, then N is relatively cyclic of degree p with group $G = \mathrm{Gal}(N/K) \simeq C_p = \langle\sigma\rangle$ over its unique quadratic subfield $K = \mathrm{Fix}(\sigma)$ and can be viewed as an *abelian extension* modulo some conductor f over K within the scope of class field theory [1–4].

The construction process for the fields L is implemented as a program script for the computational algebra system Magma [5–7] using the *class field theoretic routines* by Fieker [3], and the normal fields N/L are classified according to the cohomology $\hat{H}^0(G, U_N)$ and $H^1(G, U_N)$ of their unit group U_N as a Galois module over G [8–10].

For $p \ge 5$, the results are completely new, whereas for $p = 3$, they admit an independent verification and a class field theoretic illumination of classical tables of cubic fields by Angell 1972 [11,12] and 1975 [13,14], Ennola and Turunen 1983 [15,16], Llorente and Quer 1988 [17], Fung and Williams 1990 [18,19], and Belabas 1997 [20]. However, in contrast to these well-known tables, where the focus was on the computation of fundamental systems of units and the structure of ideal class groups [11–16,18], or even only of generating polynomials and prime decompositions [17,20], our innovative database establishes an arrangement according to conductors with an increasing number of prime factors, pays attention to the phenomenon of *multiplicities of discriminants* [21–25], and constitutes the *first classification into* 9, respectively 3, *differential principal factorization types* of totally real, respectively simply real, cubic number fields [8–10,26,27]. This is a progressive new kind of structural information which has never been provided for algebraic number fields before, except for pure cubic fields [28–31] and pure quintic fields [8], but the present paper emphasizes the advanced viewpoint of *classifying an entire ring class field* K_f by its associated *heterogeneous multiplet* $\mathbf{M}(K_f)$ of dihedral fields with various conductors $c \mid f$.

2. Heterogeneous multiplets of objects and invariants

Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic base field with positive or negative fundamental discriminant $d = d_K \equiv 0,1 \pmod 4$, essentially squarefree except possibly for the 2-contribution $v_2(d)$. Suppose that p is an odd prime number and $f \geq 1$ is a p-admissible conductor over K [21,25]. Then the p-ring class field $K_{p,f}$ mod f of K contains all cyclic relative extensions N/K with some conductor $c \mid f$ which are absolutely dihedral with automorphism group $\mathrm{Gal}(N/\mathbb{Q}) \simeq D_p$ over the rational number field $\mathbb{Q}$. The crucial concept underlying this entire paper is the collection of all these dihedral fields in a *heterogeneous multiplet* $\mathbf{M}(K_{p,f}) = [(N_{c,i})_{1\leq i\leq m_p(K,c)}]_{c|f}$ according to the p-multiplicities $m_p(K,c)$ [21,25], which satisfy the relation $\sum_{c|f} m_p(K,c) = \frac{p^{\varrho_{p,f}}-1}{p-1}$ in terms of the p-ring class rank $\varrho_{p,f}$ modulo f of K. Since our principal aim is the classification of p-ring class fields $K_{p,f}$, it is essential to distinguish between a multiplet of *objects* (expressing the multiplicity of the discriminants d_N) and a corresponding multiplet of *invariants* (expressing the Galois cohomology of the unit groups U_N and differential principal factorizations of the fields N).

Definition 1. By the **type of the p-ring class field** $K_{p,f}$ modulo f of K we understand the pair $(\mathrm{Obj}(K_{p,f}), \mathrm{Inv}(K_{p,f}))$ of heterogeneous multiplets

$$\begin{cases} \mathrm{Obj}(K_{p,f}) & = [(N_{c,i})_{1\leq i\leq m_p(K,c)}]_{c|f} \\ \mathrm{Inv}(K_{p,f}) & = [(\tau(N_{c,i}))_{1\leq i\leq m_p(K,c)}]_{c|f} \end{cases} \tag{1}$$

consisting of all absolutely dihedral fields $N_{c,i}$ with conductors c dividing f as *objects* and their differential principal factorization types (DPF types) $\tau(N_{c,i})$ as *invariants* [8,9].

3. Homogeneous multiplets of unramified extensions

The unique situation where the heterogeneous multiplets degenerate to *homogeneous multiplets* occurs for *unramified* relative extensions N/K with conductor $f = 1$ which has only itself as a divisor $c \mid f$. In this unramified case, which implies positive p-class rank $\varrho_p = \varrho_{p,1} \geq 1$ of the quadratic base field K, there occur *at most two* possible differential principal factorization types.

Theorem 1. *An **unramified** cyclic extension N with odd prime degree p of K possesses the conductor $f = 1$ without any prime divisors. For a **totally real** field N, there are two cases:*

1. *If the p-class rank of K is $\varrho_p = 1$, then N is of **type** δ_1.*
2. *If the p-class rank of K is $\varrho_p \geq 2$, then **two types** α_1 **and** δ_1 are possible for N.*

*If N is **totally complex**, then N is of **type** α_1, independently of the p-class rank of K.*

Proof. Since the conductor $f = q_1 \cdots q_t$ is essentially the square free product of all prime numbers $q_i \in \mathbb{P}$, whose overlying prime ideals $\mathfrak{q}_i \in \mathbb{P}_K$ are ramified in N, the following chain of equivalent statements is true: N/K is unramified $\Longleftrightarrow$ None of the prime ideals of K ramifies in N $\Longleftrightarrow$ The conductor $f = 1$ has no prime divisors, i.e., $t = 0$.

Now we use the fundamental equation in [9, Corollary 5.1] and the estimates in [9, Corollary 5.2] for the decision about possible types of principal factorizations. If $f = 1$, then there neither exist absolute principal factorizations in $L/\mathbb{Q}$, since $0 \leq A \leq \min(t,2) = 0$, nor relative principal factorizations in N/K, since $0 \leq R \leq \min(s,2) = 0$, where $s \leq t$ denotes the number of prime divisors q_i of f which split in K. Consequently, the fundamental equation degenerates to $U + 1 = C$ with $1 \leq U + 1 \leq 2$, which implies $1 \leq C \leq \min(\varrho_p, 2)$. Thus, only type δ_1 with $C = 1$ is possible for $\varrho_p = 1$, whereas type α_1 with $C = 2$ can arise additionally for $\varrho_p \geq 2$. $\square$

4. Conductors with a single prime divisor

For a *regular prime* conductor f, only two cases are possible.

Theorem 2. *Let K be a quadratic base field with p-class rank $\varrho = \varrho_p$. Suppose $f = q$ is a **regular** p-admissible **prime conductor** for K. Then the heterogeneous multiplet $\mathbf{M}(K_{p,f})$ associated with the p-ring class field $K_{p,f}$ mod f of K consists of two homogeneous multiplets with multiplicities $m_p(K,1)$ and $m_p(K,q)$. In this order, and in dependence on the p-ring space $V_p(q)$, these two multiplicities are given by*

1. $(1+p+\ldots+p^{\varrho-1},\ p^{\varrho})$, *if* $V_p(q) = V$ (***free*** *situation),*
2. $(1+p+\ldots+p^{\varrho-1},\ 0)$, *if* $V_p(q) < V$ (***restrictive*** *situation).*

Proof. See [25, Theorem 3.2, p. 2215, and Theorem 3.3, p. 2217]. □

In the special case $p = 3$, there also exists the possibility of an *irregular prime power* conductor $f = 3^2$, provided the discriminant of the quadratic field satisfies the congruence $d \equiv -3 \pmod 9$.

Theorem 3. *Assume that* $p = 3$*. Let* K *be a quadratic base field with* 3*-class rank* $\varrho = \varrho_3$ *and discriminant* $d \equiv -3 \pmod 9$*. Consider the* ***irregular*** 3*-admissible* ***prime power conductor*** $f = 3^2$ *for* K*. Then the heterogeneous multiplet* $\mathbf{M}(K_{p,f})$ *associated with the* 3*-ring class field* $K_{3,f}$ *mod* f *of* K *consists of three homogeneous multiplets with multiplicities* $m_3(K,1)$*,* $m_3(K,3)$ *and* $m_3(K,9)$*. In this order, and in dependence on the* 3*-ring spaces* $V_3(3)$ *and* $V_3(9)$*, these three multiplicities are given by*

1. $(1+3+\ldots+3^{\varrho-1},\ 3^{\varrho},\ 3^{\varrho+1})$, *if* $V_3(9) = V_3(3) = V$ (***free*** *situation),*
2. $(1+3+\ldots+3^{\varrho-1},\ 3^{\varrho},\ 0)$, *if* $V_3(9) < V_3(3) = V$,
3. $(1+3+\ldots+3^{\varrho-1},\ 0,\ 3^{\varrho})$, *if* $V_3(9) = V_3(3) < V$,
4. $(1+3+\ldots+3^{\varrho-1},\ 0,\ 0)$, *if* $V_3(9) < V_3(3) < V$ (***maximal restriction****).*

Proof. See [25, Theorem 3.4, p. 2217]. □

5. Conductors with two prime divisors

For *regular* conductors f divisible by *two primes*, more distinct situations may arise.

Theorem 4. *Let* K *be a quadratic base field with* p*-class rank* $\varrho = \varrho_p$*. Suppose* $f = q_1 \cdot q_2$ *is a* ***regular*** p*-admissible conductor for* K *with* ***two prime divisors*** q_1 *and* q_2*. Then the heterogeneous multiplet* $\mathbf{M}(K_{p,f})$ *associated with the* p*-ring class field* $K_{p,f}$ *mod* f *of* K *consists of four homogeneous multiplets with multiplicities* $m_p(K,1)$*,* $m_p(K,q_1)$*,* $m_p(K,q_2)$ *and* $m_p(K,f)$*. In this order, and in dependence on the* p*-ring spaces* $V_p(q_1)$*,* $V_p(q_2)$ *and* $V_p(f)$*, these four multiplicities are given by*

1. $(1+p+\ldots+p^{\varrho-1},\ p^{\varrho},\ p^{\varrho},\ p^{\varrho}(p-1))$, *if* $V_p(f) = V_p(q_1) = V_p(q_2) = V$ (***free*** *case),*
2. $(1+p+\ldots+p^{\varrho-1},\ p^{\varrho},\ 0,\ 0)$, *if* $V_p(f) = V_p(q_2) < V_p(q_1) = V$,
3. $(1+p+\ldots+p^{\varrho-1},\ 0,\ p^{\varrho},\ 0)$, *if* $V_p(f) = V_p(q_1) < V_p(q_2) = V$,
4. $(1+p+\ldots+p^{\varrho-1},\ 0,\ 0,\ p^{\varrho})$, *if* $V_p(f) = V_p(q_1) = V_p(q_2) < V$,
5. $(1+p+\ldots+p^{\varrho-1},\ 0,\ 0,\ 0)$, *if* $V_p(f) < V_p(q_1) \neq V_p(q_2) < V$ (***maximal restriction****).*

Proof. We use the terminology and notation in [25]. Generally, the p-ring class rank is given by $\varrho_{p,f} = \varrho + t + w - \delta_p(f)$. Here, we have either $t = 2$, $w = 0$ or $t = 1$, $w = 1$, and thus $\varrho_{p,f} = \varrho + 2 - \delta_p(f)$. Also, we know that generally $m_p(K,1) = \frac{p^{\varrho}-1}{p-1}$. Since $f = q_1 \cdot q_2$ is p-admissible, q_1 and q_2 must also be p-admissible, both.

1. In the free case with defect $\delta_p(f) = 0$, we have $V_p(f) = V_p(q_1) = V_p(q_2) = V$ and
$$\frac{p^{\varrho+2}-1}{p-1} - \frac{p^{\varrho}-1}{p-1} = \frac{p^{\varrho}(p^2-1)}{p-1} = p^{\varrho}(p+1) = p^{\varrho} + p^{\varrho} + p^{\varrho}(p-1),$$
which is exactly the desired partition
$$\frac{p^{\varrho_{p,f}}-1}{p-1} - m_p(K,1) = m_p(K,q_1) + m_p(K,q_2) + m_p(K,f).$$
2. If q_1 is free and q_2, f are restrictive, then $V_p(f) = V_p(q_2) < V_p(q_1) = V$ and the relation
$$\frac{p^{\varrho+1}-1}{p-1} - \frac{p^{\varrho}-1}{p-1} = \frac{p^{\varrho}(p-1)}{p-1} = p^{\varrho},$$
must be interpreted as $m_p(K,q_1) = p^{\varrho}$ and $m_p(K,q_2) = m_p(K,f) = 0$.
3. This case arises by interchanging the roles of q_1 and q_2 in the previous case.
4. Additionally to (2) and (3), there is another case of defect $\delta_p(f) = 1$ where neither q_1 nor q_2 is free but their p-ring spaces coincide $V_p(f) = V_p(q_1) = V_p(q_2) < V$. Then the formula in (2) has to be interpreted as $m_p(K,q_1) = m_p(K,q_2) = 0$ and $m_p(K,f) = p^{\varrho}$.

5. Finally, in the case of maximal restriction with defect $\delta_p(f) = 2$, which occurs for distinct p-ring spaces $V_p(f) < V_p(q_1) \neq V_p(q_2) < V$, there is no rank increment from ϱ to $\varrho_{p,f}$, and thus $m_p(K, q_1) = m_p(K, q_2) = m_p(K, f) = 0$. □

6. Construction of p-ring class fields

This section describes how the classification of non-trivial p-ring class fields is prepared by their *construction* and *rigorous count*. The intended class field theoretic illumination of the structure of heterogeneous multiplets $\mathbf{M}(K_{p,f}) = [(N_{c,1}, \dots, N_{c,m(c)})]_{c|f}$ associated with p-ring class fields $K_{p,f}$ modulo p-admissible conductors f over quadratic fields K must pay *primary attention* to the *p-class rank* ϱ_p of the quadratic base fields $K = \mathbb{Q}(\sqrt{d})$, since ϱ_p enters the formula for the multiplicities $m(c)$. More precisely, since the existence of a torsion free fundamental unit $\varepsilon > 1$ in real quadratic fields K with $d > 0$, and the occurrence of the 3-torsion unit ζ_3 in the particular imaginary quadratic field K with $d = -3$ in the case $p = 3$, exerts a crucial impact on the codimension of p-ring spaces $V_p(c)$, the invariant ϱ_p must rather be replaced by the *p-Selmer rank* σ_p of K which describes all p-virtual units of K, those which arise from non-trivial p-classes and the units in the usual sense:

$$\sigma_p = \begin{cases} \varrho_p & \text{if } p \geq 5,\ d < 0 \text{ or } p = 3,\ d < -3, \\ \varrho_p + 1 & \text{if } d > 0 \text{ or } p = 3,\ d = -3. \end{cases} \tag{2}$$

The *secondary attention* is devoted to various p-admissible conductors $f = q_1 \cdots q_t$ with an increasing number $t \geq 0$ of prime divisors, starting with unramified extensions having $t = 0$, $f = 1$, and continuing with ramified extensions, beginning with prime or prime power conductors having $t = 1$, $f = q_1$ with a prime $q_1 \in \mathbb{P}$ or the critical prime power $q_1 = p^2$.

7. Multiplets over imaginary quadratic fields for $p = 3$

The focus of this section and most of the further sections is on $p = 3$, where the components $N_{c,i}$ of multiplets are cyclic cubic extensions of quadratic base fields K. Here, we begin with imaginary base fields K having the smallest possible 3-Selmer rank $\sigma_3 = \varrho_3$. The behavior of the particular imaginary quadratic field K with $d = -3$ where the extensions $N_{c,i}/K$ contain pure cubic fields is rather similar to real quadratic base fields K with $\sigma_3 = \varrho_3 + 1$, and thus the case $d = -3$ will be treated separately.

Theorem 5. *Let K be an imaginary quadratic field with fundamental discriminant $d < -3$ and trivial 3-class rank $\varrho_3 = 0$. Assume that $f = q_1 \cdots q_\tau$ is a 3-admissible conductor with $\tau \geq 1$ regular prime or prime power divisors q_i (that is, either $q_i \equiv \pm 1 \pmod 3$ or $q_\tau = 3$, $d \equiv \pm 3 \pmod 9$ or $q_\tau = 9$, $d \equiv \pm 1 \pmod 3$ but not $q_\tau = 9$, $d \equiv -3 \pmod 9$). Then the 3-ring class field $K_{3,f}$ modulo f of K contains a homogeneous multiplet $\mathbf{M}(K_{3,f}) = (N_{f,1}, \dots, N_{f,m})$ of dihedral fields with conductor f and multiplicity $m = 2^{\tau-1}$ (singlet, doublet, quartet, octet, hexadecuplet, etc.).*

Proof. All 3-ring spaces $V_3(q_i)$ coincide with 3-Selmer space $V = V_3$ [25, Theorem 3.2, p. 2215]. □

7.1. Classification of Angell's 3169 simply real cubic fields

In order to demonstrate the powerful performance of our innovative techniques, we construct all 3-ring class fields $K_{3,f}$ which contain the normal closures N of the simply real cubic fields L in Angell's table [11,12] as abelian extensions of the associated imaginary quadratic base fields $K < N$.

There arise four values of the *multiplicity* $m = 1, 2, 3, 4$, and accordingly simply real cubic fields are collected in singlets, doublets, triplets and quartets. *Nilets* with $m = 0$ complete the view.

The classification of the pure cubic fields, respectively non-pure simply real cubic fields, into **differential principal factorization types** was established in [28], respectively [9].

Although the types α and β of pure cubic fields are similar to the types α_2 and β of non-pure simply real cubic fields, we do not mix the classifications, since firstly the existence of radicals among the principal factors distinguishes pure cubic fields from non-pure simply real cubic fields, and secondly, type γ can only occur for the former, whereas type α_1 is only possible for the latter.

Results

According to Table 1, the number of all non-pure simply real cubic fields L having discriminants $-2 \cdot 10^4 < d_L < 0$ is given by **3134**. Together with 35 pure cubic fields in Table 2, the total number is **3169**, as announced correctly in [12].

Table 1. Cubic discriminants in the range $-2 \cdot 10^4 < d_L = f^2 \cdot d < 0$

			Multiplicity					DPF		
f	Condition	Total	0	1	2	3	4	α_1	α_2	β
q	$\equiv -1 \pmod 3$	454		454						454
3	$d \equiv +3 \pmod 9$	62		62						62
3	$d \equiv -3 \pmod 9$	58		58						58
9	$d \equiv -3 \pmod 9$	7				7				21
9	$d \equiv -1 \pmod 3$	23		23						23
9	$d \equiv +1 \pmod 3$	20		20					16	4
ℓ	$\equiv +1 \pmod 3$	64		64					49	15
q_1q_2	$\equiv -1 \pmod 3$	6			6					12
$3q$	$d \equiv +3 \pmod 9$	7			7					14
$3q$	$d \equiv -3 \pmod 9$	3			3					6
$9q$	$d \equiv -1 \pmod 3$	3			3					6
$9q$	$d \equiv +1 \pmod 3$	3			3					6
3ℓ	$d \equiv +3 \pmod 9$	1			1					2
$q\ell$	$\equiv \mp 1 \pmod 3$	1			1					2
1	$\varrho_3 = 1$	2143		2143				2143		
q	$\equiv -1 \pmod 3$	196	162			34		87		15
3	$d \equiv +3 \pmod 9$	24	22			2		4		2
3	$d \equiv -3 \pmod 9$	22	16			6		13		5
9	$d \equiv -1 \pmod 3$	5	5							
9	$d \equiv +1 \pmod 3$	9	8			1		2		1
ℓ	$\equiv +1 \pmod 3$	22	19			3		7		2
q_1q_2	$\equiv -1 \pmod 3$	2	1			1				3
$3q$	$d \equiv +3 \pmod 9$	3	1			2				6
$9q$	$d \equiv +1 \pmod 3$	1				1				3
$q\ell$	$\equiv \mp 1 \pmod 3$	2	1			1				3
1	$\varrho_3 = 2$	22					22	88		
	Summary	3163	235	2824	24	58	22	2344	65	725

We emphasize the difference between the *number of discriminants* (without multiplicities)

$$2824 + 24 + 58 + 22 = 2928,$$

and the *number of fields* (including multiplicities in a weighted sum)

$$1 \cdot 2824 + 2 \cdot 24 + 3 \cdot 58 + 4 \cdot 22 = 2824 + 48 + 174 + 88 = 3134,$$

which can be confirmed by adding the contributions to the 3 DPF types $\alpha_1, \alpha_2, \beta$

$$2344 + 65 + 725 = 3134.$$

In contrast, 235 is the number of *formal cubic discriminants* $d_L = f^2 \cdot d_K$ with fundamental discriminants d_K of imaginary quadratic fields and 3-admissible conductors f for each K, where the relevant multiplicity formula [25] yields the value zero. So the formal cubic discriminants belong to *nilets*, i.e., multiplets with multiplicity $m_3(K, f) = 0$. The total number of all (actual) cubic discriminants and formal cubic discriminants is the number of admissible cubic discriminants

$$2928 + 235 = 3163.$$

According to Theorem 5, *Nilets* can only arise for $\varrho_3 \geq 1$, but not for $\varrho_3 = 0$.

Table 2. Pure cubic discriminants in the range $-2 \cdot 10^4 < d_L = -3 \cdot f^2 < 0$

			Multiplicity					DPF		
f	Condition	Total	0	1	2	3	4	α	β	γ
q	$\equiv -1 \pmod 3$	11	8	3						3
9	$d = -3$	1		1						1
ℓ	$\equiv +1 \pmod 3$	10	7	3				3		
q_1q_2	$\equiv -1 \pmod 3$	6	1	5					5	
$3q$	$d = -3$	5	1	4					4	
$9q$	$d = -3$	2			2				4	
3ℓ	$d = -3$	3	1	2				2		
9ℓ	$d = -3$	1			1			2		
$q\ell$	$\equiv \mp 1 \pmod 3$	8	2	6				4	2	
$q_1q_2\ell$	$\equiv \mp 1 \pmod 3$	1		1					1	
$3q_1q_2$	$d = -3$	2		2					2	
$3q\ell$	$d = -3$	2		2					2	
	Summary	52	20	29	3			11	20	4

According to Table 2, the number of pure cubic fields L with discriminant $-2 \cdot 10^4 < d_L < 0$ is 35. Actually, triplets and quartets of pure cubic fields do not occur in this range.

There is a difference between the *number of discriminants* (without multiplicities)

$$29 + 3 = 32,$$

and the *number of fields* (including multiplicities in a weighted sum)

$$1 \cdot 29 + 2 \cdot 3 = 29 + 6 = 35,$$

which can be confirmed by adding the contributions to the 3 DPF types

$$11 + 20 + 4 = 35.$$

The total number of all (actual) cubic discriminants and formal cubic discriminants (of the 20 nilets) is the number of admissible pure cubic discriminants $d_L = -3 \cdot f^2$,

$$32 + 20 = 52.$$

8. Multiplets over real quadratic fields for $p = 3$

We continue with real quadratic base fields K having elevated 3-Selmer rank $\sigma_3 = \varrho_3 + 1$, due to the existence of a torsion free fundamental unit $\varepsilon > 1$.

8.1. Classification of Angell's 4804 totally real cubic fields

In order to demonstrate our progressive perspective of classification of heterogeneous multiplets $\mathbf{M}(K_{3,f})$ into an enigmatic variety of differential principal factorization types, we construct all 3-ring class fields $K_{3,f}$ which contain the normal closures N of the totally real cubic fields L in Angell's table [13,14] as abelian extensions of the associated real quadratic base fields $K < N$.

Again there arise four values of the *multiplicity* $m = 1, 2, 3, 4$, and accordingly totally real cubic fields are collected in singlets, doublets, triplets and quartets. Formal *nilets* complete the view.

The classification into **differential principal factorization types** for non-cyclic totally real cubic fields was developed in [9,26,27].

Results

According to Table 3, the number of non-cyclic totally real cubic fields L with discriminant $0 < d_L < 10^5$ is **4753**, in perfect accordance with the results by Llorente and Oneto [32,33], who discovered the ommission of ten fields in the table by Angell [13,14]. Together with 51 cyclic cubic fields in Table 4, the total number is **4804** (not 4794, as announced erroneously in [14]).

Again we emphasize the difference between the *number of discriminants* (without multiplicities)

$$4652 + 9 + 21 + 5 = 4687,$$

and the *number of fields* (including multiplicities in a weighted sum)

$$1 \cdot 4652 + 2 \cdot 9 + 3 \cdot 21 + 4 \cdot 5 = 4652 + 18 + 63 + 20 = 4753,$$

which can be confirmed by adding the contributions to the 7 DPF types (α_2, α_3 do not occur)

$$16 + 10 + 76 + 106 + 3349 + 79 + 1117 = 4753.$$

Table 3. Cubic discriminants in the range $0 < d_L = f^2 \cdot d < 10^5$

			Multiplicity					Differential Principal Factorization						
f	Condition	Total	0	1	2	3	4	α_1	β_1	β_2	γ	δ_1	δ_2	ε
q	$\equiv -1\,(\mathrm{mod}\,3)$	3025	2219	806										806
3	$d \equiv +3\,(\mathrm{mod}\,9)$	396	287	109										109
3	$d \equiv -3\,(\mathrm{mod}\,9)$	389	284	105										105
9	$d \equiv -3\,(\mathrm{mod}\,9)$	48	9	38		1								41
9	$d \equiv -1\,(\mathrm{mod}\,3)$	136	102	34										34
9	$d \equiv +1\,(\mathrm{mod}\,3)$	127	96	31						8			20	3
ℓ	$\equiv +1\,(\mathrm{mod}\,3)$	402	316	86						20			59	7
q_1q_2	$\equiv -1\,(\mathrm{mod}\,3)$	70	30	38	2						38			4
$3q$	$d \equiv +3\,(\mathrm{mod}\,9)$	46	23	23							23			
$3q$	$d \equiv -3\,(\mathrm{mod}\,9)$	45	19	25	1						25			2
$9q$	$d \equiv -3\,(\mathrm{mod}\,9)$	5			4	1					9			2
$9q$	$d \equiv -1\,(\mathrm{mod}\,3)$	14	6	8							8			
$9q$	$d \equiv +1\,(\mathrm{mod}\,3)$	15	5	10						10				
9ℓ	$d \equiv -1\,(\mathrm{mod}\,3)$	1		1						1				
3ℓ	$d \equiv +3\,(\mathrm{mod}\,9)$	6	1	5						5				
3ℓ	$d \equiv -3\,(\mathrm{mod}\,9)$	5	2	3						3				
$q\ell$	$\equiv \mp 1\,(\mathrm{mod}\,3)$	43	13	29	1					29				2
$3q_1q_2$	$d \equiv +3\,(\mathrm{mod}\,9)$	2		1	1						3			
1	$\varrho_3 = 1$	3300		3300								3300		
q	$\equiv -1\,(\mathrm{mod}\,3)$	275	261			14			4			36		2
3	$d \equiv -3\,(\mathrm{mod}\,9)$	35	34			1						3		
ℓ	$\equiv +1\,(\mathrm{mod}\,3)$	28	25			3			3			6		
$3q$	$d \equiv -3\,(\mathrm{mod}\,9)$	2	1			1			3					
1	$\varrho_3 = 2$	5					5	16				4		
	Summary	8420	3733	4652	9	21	5	16	10	76	106	3349	79	1117

In contrast, 3733 is the number of *formal cubic discriminants* $d_L = f^2 \cdot d_K$ with fundamental discriminants d_K of real quadratic fields and 3-admissible conductors f for each K, where the relevant multiplicity formula [25] yields the value zero. So the formal cubic discriminants belong to *nilets*, i.e., multiplets with multiplicity $m_3(K, f) = 0$. The total number of all (actual) cubic discriminants and formal cubic discriminants is the number of admissible cubic discriminants

$$4687 + 3733 = 8420.$$

Table 4. Cyclic cubic discriminants in the range $0 < d_L = f^2 < 10^5$

		M		DPF
f	Condition	1	2	ζ
9	$d = 1$	1		1
ℓ	$\equiv +1\,(\mathrm{mod}\,3)$	30		30
9ℓ	$d = 1$		4	8
$\ell_1\ell_2$	$\equiv +1\,(\mathrm{mod}\,3)$		6	12
	Summary	31	10	51

According to Table 4, the number of cyclic cubic fields L with discriminant $0 < d_L < 10^5$ is 51, with 31 arising from singlets having conductors f with a single prime divisor, and 20 from doublets having two prime divisors of the conductor f. (M denotes the multiplicity.)

We point out that cyclic cubic fields are rather contained in *ray class fields* over $\mathbb{Q}$ than in ring class fields over real quadratic base fields. The single possible DPF type ζ has nothing to do with the 9 DPF types $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \gamma, \delta_1, \delta_2, \varepsilon$ of non-abelian totally real cubic fields in [9].

9. Conclusion and outlook

In this paper, we have classified all multiplets $\mathrm{Obj}(K_{3,f})$ of *non-pure simply real* cubic fields L (more precisely of their normal closures N) according to the associated multiplets of invariants, namely the **differential principal factorization types**, $\mathrm{Inv}(K_{3,f})$, where $K_{3,f}$ denotes the 3-ring class field modulo a 3-admissible conductor f of the imaginary quadratic subfield $K < N$: (Recall that $\mathrm{Obj}(K_{3,f}) = (N_{f,i})_{1 \le i \le m}$ and $\mathrm{Inv}(K_{3,f}) = (\tau(N_{f,i}))_{1 \le i \le m}$, here *homogeneously*.)

- 2824 *singlets* of type either (α_1) or (α_2) or (β), according to Table 1;
- 24 *doublets* of exclusive type (β, β) (without 3 pure cubic doublets);
- 58 *triplets* with the following distribution of types:
  - 7 triplets of type (β, β, β) for $f = 9$ singular, $\varrho_3 = 0$,
  - 51 triplets sharing common 3-class rank $\varrho_3 = 1$ of K [34, Table 1, pp. 118–121], namely
    * 34 triplets of type $(\alpha_1, \alpha_1, \alpha_1)$ for $f = q, \ell, 3$,
    * 3 triplets of type $(\alpha_1, \alpha_1, \beta)$ for $f = \ell, 9$ split,
    * 5 triplets of type (α_1, β, β) for $f = q, 3$, and
    * 9 triplets of type (β, β, β) for $f = q, 3, 3q, q\ell, 9q$, finer than [34] since α_2 does not occur;
- 22 *quartets* of exclusive type $(\alpha_1, \alpha_1, \alpha_1, \alpha_1)$ (see [35] for details concerning the capitulation).

Similarly, we have classified all multiplets $\mathrm{Obj}(K_{3,f})$ of *non-cyclic totally real* cubic fields L (more precisely of their normal closures N) according to the associated multiplets of invariants, namely the **differential principal factorization types**, $\mathrm{Inv}(K_{3,f})$, where $K_{3,f}$ denotes the 3-ring class field modulo a 3-admissible conductor f of the real quadratic subfield $K < N$:

- 4652 *singlets* of type either (β_2) or (γ) or (δ_1) or (δ_2) or (ε), according to Table 3;
- 9 *doublets*, 4 of type (γ, γ) for $f = 9q, 3q_1q_2$ and 5 of type $(\varepsilon, \varepsilon)$ for $f = 3q, 9q, q_1q_2, q\ell$;
- 21 *triplets* with the following distribution of types:
  - 1 triplet of type $(\varepsilon, \varepsilon, \varepsilon)$ for $f = 9$ singular, $\varrho_3 = 0$,
  - 1 triplet of type (γ, γ, γ) for $f = 9q$ singular, $\varrho_3 = 0$, and
  - 19 triplets sharing common 3-class rank $\varrho_3 = 1$ of K (with considerable refinement of Schmithals' coarse distinction of only two alternatives [34, Table 2, pp. 122–123]), namely
    * 13 triplets of type $(\delta_1, \delta_1, \delta_1)$ for $f = 3, q$,
    * 1 triplet of type $(\beta_1, \beta_1, \beta_1)$ for $f = 3q$,
    * 2 triplets of type $(\beta_1, \beta_1, \varepsilon)$ for $f = q$ (conspicuously with symbol "$-$" in [34]), and
    * 3 triplets of type $(\beta_1, \delta_1, \delta_1)$ for $f = \ell$,
- 5 *quartets*, 1 of type $(\alpha_1, \alpha_1, \alpha_1, \alpha_1)$ and 4 of type $(\alpha_1, \alpha_1, \alpha_1, \delta_1)$ (more details in [36,37]).

In the same manner, we shall *refine more extensive tables* by Fung and Williams [18], Ennola and Turunen [15,16], Llorente and Quer [17] in the new year 2021.

Moreover, we shall provide extensive evidence of the *truth of Scholz' conjecture*, which we have proved for $p = 3$ in [9], also for $p = 5$ and $p = 7$, and probably for any odd prime p.

Data Availability: Implementations of our innovative algorithms in Magma [5–7] may be requested via email.

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References

[1] Artin, E. (1927, December). Beweis des allgemeinen Reziprozitätsgesetzes. In *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* (Vol. 5, No. 1, pp. 353-363). Springer-Verlag.

[2] Belabas, K. (2004). Topics in computational algebraic number theory. *Journal de théorie des nombres de Bordeaux, 16*(1), 19-63.

[3] Fieker, C. (2001). Computing class fields via the Artin map. *Mathematics of Computation, 70*(235), 1293-1303.

[4] Hasse, H. (1930). Arithmetische Theorie der kubischen Zahlkörper auf klassenkörpertheoretischer Grundlage. *Mathematische Zeitschrift, 31*(1), 565-582.

[5] Bosma, W., Cannon, J., & Playoust, C. (1997). The Magma algebra system I: The user language. *Journal of Symbolic Computation*, *24*(3-4), 235-265.

[6] Cannon, J., Bosma, W., Fieker, C., & Steel, A. (2020). *Handbook of Magma functions*, Edition 2.25, Sydney.

[7] MAGMA Developer Group, MAGMA *Computational Algebra System*, Version 2.25-8, Univ. Sydney, 2020, (`http://magma.maths.usyd.edu.au`).

[8] Mayer, D. C. (2019). Differential principal factors and Pólya property of pure metacyclic fields. *International Journal of Number Theory, 15*(10), 1983–2025.

[9] Mayer, D. C. (2019). Generalized Artin pattern of heterogeneous multiplets of dihedral fields and proof of Scholz's conjecture. *arXiv:1904.06148v1 [math.NT]*.

[10] Moses, N. P. (1979, April). Unités et nombre de classes d'une extension Galoisienne diédrale de $\mathbb{Q}$. In *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* (Vol. 48, No. 1, pp. 54-75). Springer-Verlag.

[11] Angell, I. O. (1972). *Table of complex cubic fields*, Royal Holloway College, University of London, Surrey, England. (53 sheets of computer output, deposited in the UMT files of the Royal Society and of Math. Comp.)

[12] Angell, I. O. (1973). A table of complex cubic fields. *Bulletin of the London Mathematical Society, 5*(1), 37-38.

[13] Angell, I. O. (1975). *Table of totally real cubic fields*, Royal Holloway College, University of London, Surrey, England. (79 sheets of computer output, deposited in the UMT file of Math. Comp.)

[14] Angell, I. O. (1976). A table of totally real cubic fields. *Mathematics of Computation, 30*(133), 184–187.

[15] Ennola, V., & Turunen, R. Tables of totally real cubic fields, University of Turku, Finland, 1983. *Computer output, deposited in the UMT file of Math. Comp.*

[16] Ennola, V., & Turunen, R. (1985). On totally real cubic fields. *Mathematics of Computation, 44*(170), 495-518.

[17] Llorente, P., & Quer, J. (1988). On totally real cubic fields with discriminant $D < 10^7$. *Mathematics of Computation, 50*(182), 581-594.

[18] Fung, G. W., & Williams, H. C. (1990). On the computation of a table of complex cubic fields with discriminant $D > -10^6$. *Mathematics of Computation, 55*(191), 313-325.

[19] Fung, G. W., & Williams, H. C. (1994). Errata: "On the computation of a table of complex cubic fields with discriminant $D > -10^6$" [Math. Comp. 55 (1990), no. 191, 313–325; MR1023760 (90m: 11155)]. *Mathematics of Computation, 63*(207), 433.

[20] Belabas, K. (1997). A fast algorithm to compute cubic fields. *Mathematics of Computation, 66*(219), 1213-1237.

[21] Mayer, D. C. (1992). Multiplicities of dihedral discriminants. *Mathematics of Computation, 58*(198), 831-847.

[22] Mayer, D. C. (2001). *Multiplicities of discriminants of p-ring class fields over quadratic fields with modified p-class rank* $\sigma \geq 2$, 15. Austrian Math. Congress and 111. Annual Meeting of the DMV 2001, Univ. Vienna, Austria.

[23] Mayer, D. C. (2012). *Quadratic p-ring spaces for counting dihedral fields*, Workshop International NTCCCS 2012, Université Mohammed Premier, Faculté des Sciences d' Oujda, Oujda, Morocco.

[24] Mayer, D. C. (2012). *Number fields sharing a common discriminant*, 122. Annual Meeting of the DMV 2012, University of Saarland, Faculty of Mathematics, Saarbrücken, Germany.

[25] Mayer, D. C. (2014). Quadratic p-ring spaces for counting dihedral fields. *International Journal of Number Theory, 10*(08), 2205-2242.

[26] D. C. Mayer, *Classification of dihedral fields*, Preprint, Dept. of Computer Science, Univ. of Manitoba, 1991.

[27] D. C. Mayer, *List of discriminants* $d_L < 200\,000$ *of totally real cubic fields L, arranged according to their multiplicities m and conductors* f, Computer Centre, Department of Computer Science, University of Manitoba, Winnipeg, Canada, 1991, Austrian Science Fund, Project Nr. J0497-PHY.

[28] Aouissi, S., Mayer, D. C., Ismaili, M. C., Talbi, M., & Azizi, A. (2020). *3-rank of ambiguous class groups in cubic Kummer extensions*, Period. Math. Hung. 250–274.

[29] Barrucand, P., & Cohn, H. (1971). Remarks on principal factors in a relative cubic field. *Journal of Number Theory, 3*(2), 226-239.

[30] Mayer, D. C. (1989). *Differential principal factors and units in pure cubic number fields*. Preprint, Dept. of Math., Univ. Graz.

[31] Williams, H. C. (1982). Determination of principal factors in $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt[3]{D})$. *Mathematics of Computation, 38*(157), 261-274.

[32] Llorente, P., & Oneto, A. V. (1980). Cuerpos Cúbicos, Cursos, Seminarios y Tesis del PEAM, No. 5, Univ. *Zulia, Maracaibo, Venezuela*.

[33] Llorente, P., & Oneto, A. V. (1982). On the real cubic fields. *Mathematics of Computation, 39*(160), 689-692.

[34] Schmithals, B. (1982). Zur Kapitulation in zyklischen Zahlkörpererweiterungen und Einheitenstruktur in Diederkörpern von Grad 2ℓ.

[35] Mayer, D. C. (1991). Principalization in complex S_3-fields. *Congressus Numerantium,* 73-73.

[36] Mayer, D. C. (2012). The second p-class group of a number field. *International Journal of Number Theory, 8*(02), 471-505.

[37] Mayer, D. C. (2014). Principalization algorithm via class group structure. *Journal de Théorie des Nombres de Bordeaux, 26*(2), 415-464.

[38] Voronoi, G. F. E. (1896). Ob odnom obobshchenii algorifma nepreryvnykh drobei (Russian) (Doctoral dissertation, St. Petersburg.).

[39] Mayer, D. C. (1992). Final report on Erwin Schrödinger Project J0497-PHY, *Galois number fields with S_3-group*, Department of Computer Science, University of Manitoba, 1992, Austrian Science Fund.

[40] Mayer, D. C. (1993). Discriminants of metacyclic fields. *Canadian Mathematical Bulletin, 36*(1), 103-107.