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# Trinomial equation: the Hypergeometric way

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**Abstract:** This paper is devoted to the analytical treatment of trinomial equations of the form $y^n + y = x$, where $y$ is the unknown and $x \in \mathbb{C}$ is a free parameter. It is well-known that, for degree $n \geq 5$, algebraic equations cannot be solved by radicals; nevertheless, roots are described in terms of univariate hypergeometric or elliptic functions. This classical piece of research was founded by Hermite, Kronecker, Birkeland, Mellin and Brioschi, and continued by many other Authors. The approach mostly adopted in recent and less recent papers on this subject (see [1,2] for example) requires the use of power series, following the seminal work of Lagrange [3]. Our intent is to revisit the trinomial equation solvers proposed by the Italian mathematician Davide Besso in the late nineteenth century, in consideration of the fact that, by exploiting computer algebra, these methods take on an applicative and not purely theoretical relevance.

**Keywords:** Algebraic equations; Hypergeometric functions; Differential equations; Symbolic and numerical modelling.

**MSC:** 26C10, 65H05, 34A30, 65L80, 33C90, 33C20.

## 1. Introduction

Trinomial equation has always driven the attention of researchers. The first contributions beyond the purely algebraic approach dates back to [4,5], went through the works [6,7], to arrive to more recent fundamental contributions [8–12]. This problem also interested Ramanujan, who solved it via his famous *Master Theorem* in 1913 (see [13] pp.194–195 and [14] pp.306–307). From a theoretical, rather than computational, point of view, this problem is still currently studied [2,15]. Trinomial equations appear in several applications, among which some of the most recent are in financial mathematics [16] and motion analysis of aircraft planar trajectories [17].

Our contribution is based on the work in [18] and [19–21], where a theory is developed to treat an algebraic equation of the form:

$$f(y) = \phi(y) + x\,\psi(y) = 0\,, \tag{1}$$

being $\phi(y)\,,\psi(y)$ polynomials and $x \in \mathbb{R}$ a parameter.

Let $n$ be the $y$–degree of Equation (1) and denote $\alpha_1\,,\dots,\alpha_n$ the roots of $f(y)$ for $k = 1\,,\dots,n$; these roots are obviously functions of $x$. Consider the sum of a prescribed power, say $r \in \mathbb{N}$, of the roots of (1):

$$s_r = \sum_{k=1}^{n} \alpha_k^r\,, \tag{2}$$

and construct the $n \times n$ determinants:

$$D = \begin{Vmatrix} s_0 & s_1 & \dots & s_{n-1} \\ s_1 & s_2 & \dots & s_n \\ s_2 & s_3 & \dots & s_{n+1} \\ \dots & \dots & \dots & \dots \\ s_{n-1} & s_n & \dots & s_{2n-2} \end{Vmatrix}\,, \qquad Q = \begin{Vmatrix} s_0 & s_1 & \dots & s_{n-2} & 1 \\ s_1 & s_2 & \dots & s_{n-1} & y \\ s_2 & s_3 & \dots & s_n & y^2 \\ \dots & \dots & \dots & \dots & \dots \\ s_{n-2} & s_{n-1} & \dots & s_{2n-4} & y^{n-2} \\ 1 & y & \dots & y^{n-2} & 1 \end{Vmatrix}\,. \tag{3}$$

$D = D(x)$ is indeed the $y$–discriminant of the polynomial in (1). We refer to [18] where the following identity is obtained:

$$D\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = Q\left(f'(y)\right)^2. \tag{4}$$

To put into practice Equation (4), the sums $s_r$ need to be computed according to (2); to this aim, recall that, given an $n$–th degree polynomial:

$$p(y) = a_n\,y^n + a_{n-1}\,y^{n-1} + \ldots + a_1\,y + a_0 = a_n\,(y-\alpha_1)\,(y-\alpha_2)\ldots(y-\alpha_n)\,, \tag{5}$$

its logarithmic derivative is:

$$\frac{p'(y)}{p(y)} = \frac{1}{y-\alpha_1} + \frac{1}{y-\alpha_2} + \cdots + \frac{1}{y-\alpha_n}\,. \tag{6}$$

Factoring out $1/y$ and expanding in geometric series for $y \to \infty$ yields:

$$\begin{aligned}
\frac{p'(y)}{p(y)} &= \frac{1}{y}\left(\frac{1}{1-\dfrac{\alpha_1}{y}} + \frac{1}{1-\dfrac{\alpha_2}{y}} + \cdots + \frac{1}{1-\dfrac{\alpha_n}{y}}\right)\\
&= \frac{1}{y}\left(\left(1+\frac{\alpha_1}{y}+\frac{\alpha_1^2}{y^2}+\cdots\right)+\left(1+\frac{\alpha_2}{y}+\frac{\alpha_2^2}{y^2}+\cdots\right)+\left(1+\frac{\alpha_n}{y}+\frac{\alpha_n^2}{y^2}+\cdots\right)\right)\\
&= \frac{1}{y}\left(n+\frac{1}{y}\,(\alpha_1+\cdots+\alpha_n)+\frac{1}{y^2}\,(\alpha_1^2+\cdots+\alpha_n^2)+\cdots\right)\,.
\end{aligned} \tag{7}$$

Sums $s_r$ in (2) can then be evaluated expanding around infinity the logarithmic derivative of $p(y)$.

Betti's identity (4) is the starting point to develop an algorithm, following the contribution of Davide Besso [22] who worked at the quintic equation:

$$y^5 + y - x = 0. \tag{8}$$

We will adapt his procedure, also considered in [23] for the trinomial of degree $n$:

$$y^n + y - x = 0\,. \tag{9}$$

The importance of Besso's contribution is high, because (8) is in the Bring–Jerrard form, to which every quintic can be traced back through Tschirnhaus–Bring transformations; for details, we refer the reader to [24], page 165. Moreover, the fact that $1$ is the $y$–coefficient does not respresent a restriction, also in the general case (9): any equation of the form $y^n + a\,y - x = 0$ can indeed be transformed into $u^n + a\,\lambda^{n-1}\,u - \lambda^n\,x = 0$ with the change of variable $y = u/\lambda$, being $u$ the new variable and $\lambda$ a complex parameter; at this point, selecting $\lambda$ such that $a\,\lambda^{n-1} = 1$ originates an equation of the form (9).

The solution $y = y(x)$ to (9) such that $y(0) = 0$ is denominated *principal solution*, a translation of the original German term *Hauptlösung* used in [10]. We remark that for low degrees, say $n = 2, 3$, Equation (4) provides an alternative way to detect the well known solution formulas; see for instance the recent contribution [25].

## 2. Besso's approach to the quintic

### 2.1. Hypergeometric preliminary

Before presenting Besso's method for solving the fifth degree equation and its extension to degree $n$, to make our article easier to read, let us briefly recall the definition and main properties of the *generalised hypergeometric* function used in the following. The latter is defined as:

$${}_p\mathrm{F}_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array}\middle|\, x\right) = \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\,\frac{x^k}{k!}\,, \tag{10}$$

where $(a)_k$ stands for the Pochhammer symbol, a generalization of the factorial given in terms of the Gamma function $\Gamma$ :

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a\,(a+1)\,\cdots\,(a+k-1)\,.$$

Here we deal with the case $p = q+1$, where the radius of convergence of the power series (10) is $1$.

Introducing the differential operator $\delta := x\,(\mathrm{d}/\mathrm{d}x)$, it can be shown that ${}_{q+1}\mathrm{F}_q$ solves the linear differential equation:

$$\delta\,\left(\delta + b_1 - 1\right)\cdots\left(\delta + b_q - 1\right)\,y \;=\; x\,\left(\delta + a_1\right)\cdots\left(\delta + a_p\right)\,y\,. \tag{11}$$

A fundamental system of solutions of (11) is given by:

$$x^{1-b_\nu}\quad {}_{q+1}\mathrm{F}_q\left(\left.\begin{array}{c} 1+a_1-b_\nu\,,\;\ldots\ldots\,,\;1+a_{q+1}-b_\nu \\ 1+b_1-b_\nu\,,\;\ldots\,*\,\ldots\,,\;1+b_q-b_\nu\,,\;2-b_\nu \end{array}\right|\;x\right)\,,\qquad \nu = 1\,,\ldots\,,q\,. \tag{12}$$

The asterix $*$ means that $1+b_\nu-b_\nu$ is omitted from the sequence of "denominators". For further details and an exaustive treatment of the subject, refer to [26].

### 2.2. Besso's algorithm

Starting from (4) Besso derives a linear (hypergeometric) differential equation of fourth order, which yields the solution of the algebraic Equation (8). Here, we provide the details of his procedure, important for extending Besso's algorithm to the solution of any algebraic equation of the form (9). Indeed, the various solution steps involved in the worked–out example, presented in this Section 2.2 for the case $n = 5$, are to be followed carefully, as it will make easier to understand the implementation of our generalized algorithm in Section 5.

The first step consists of eliminating $x$ from the left–hand side of (4): this is done using the algebraic Equation (8) rewritten as $x = y + y^5$, so that:

$$D\,{y'}^{\,2} = 125\,y^{12} + 450\,y^{8} + 565\,y^{4} + 256\,. \tag{13}$$

The second step is to remove the exponent $2$ from $y'$ to derive an equivalent linear ordinary differential equation. To do this we differentiate (13) twice, with respect to $x$; notice that at the beginning of the computation it is not needed the explicit form of $D$ which, as we mention, is a function of the variable $x$. The first differentiation of (13) yields:

$$2\,D\,y'\,y'' + D^{(1)}\,\left(y'\right)^{2} = y'\,\left(1500\,y^{11} + 3600\,y^{7} + 2260\,y^{3}\right)\,. \tag{14}$$

Before performing the second differentiation, $y'$ is eliminated from both sides of (14): this goes in the desired direction of exponent removal; then, we lower the degree of the powers of $y$ using again (8) in the form $y^5 = x - y$, thus obtaining the following identity, equivalent to (14):

$$\begin{aligned} 2\,D\,y'' + D^{(1)}\,y' &= 1500\,y^{10}\,y + 3600\,y^{5}\,y^{2} + 2260\,y^{3} \\ &= 1500\,(x-y)^{2}\,y + 3600\,(x-y)\,y^{2} + 2260\,y^{3} \\ &= 1500\,x^{2}\,y + 600\,x\,y^{2} + 160\,y^{3}\,. \end{aligned}$$

The second derivative of both sides of (2.2) is now computed:

$$D^{(2)}\,y' + 3\,D^{(1)}\,y'' + 2\,D\,y''' = 1500\,x^{2}\,y' + 1200\,x\,y\,y' + 3000\,x\,y + 600\,y^{2} + 480\,y^{2}\,y'\,. \tag{15}$$

In (15), non–linearities in the right–hand side need to be removed. To eliminate terms containing $y\,y'$, Equation (8) is once more employed, differentiating which yields (after multiplication by $y$):

$$5\,y^{4}\,y' + y' - 1 = 0 \qquad\Longrightarrow\qquad y'\,(1+5\,y^{4}) = 1 \qquad\Longrightarrow\qquad y'\,(y+5\,y^{5}) = y\,.$$

A further use of (8) leads to:

$$y'\,(y + 5\,(x - y)) = y\,, \qquad \text{i.e.,} \quad y'\,(5\,x - 4\,y) = y\,.$$

In other words, the following identity is obtained as a consequence of (8):

$$y'\,y = \frac{5}{4}\,x\,y' - \frac{1}{4}\,y\,. \tag{16}$$

Inserting (16) in (15) yields:

$$D^{(2)}\,y' + 3\,D^{(1)}\,y'' + 2\,D\,y''' = 3750\,x^2\,y' + 2550\,x\,y + 480\,y^2\,. \tag{17}$$

To eliminate the non–linear term $y^2$ both sides of (17) are differentiated:

$$D^{(3)}\,y' + 4\,D^{(2)}\,y'' + 5\,D^{(1)}\,y''' + 2\,D\,y^{(4)} = 3750\,x^2\,y'' + 10050\,x\,y' + 960\,y\,y' + 2550\,y\,, \tag{18}$$

and terms containing $y\,y'$ can again be removed using (16).

At this point, recalling the definition of $D$, we arrive at the differential resolvent of Equation (8):

$$(256 + 3125\,x^4)\,y^{(4)} + 31250\,x^3\,y''' + 73125\,x^2\,y'' + 31875\,x\,y' - 1155\,y = 0\,. \tag{19}$$

To highlight the hypegeometric nature of the resolvent, the following independent variable transformation:

$$\xi = -\frac{3125}{256}\;x^4\,,$$

is applied to (19), which becomes:

$$\xi^3\,(\xi - 1)\,z^{(4)} + \xi^2\left(7\,\xi - \frac{9}{2}\right)z^{(3)} + \xi\left(\frac{411}{40}\,\xi - \frac{51}{16}\right)z^{(2)} + \left(\frac{183}{80}\,\xi - \frac{3}{32}\right)z' - \frac{231}{160000}\,z = 0\,, \tag{20}$$

where $z(\xi) = y(x)$.

Equation (20) has the form:

$$\alpha_0(\xi)\;z^{(4)} + \alpha_1(\xi)\;z^{(3)} + \alpha_2(\xi)\;z^{(2)} + \alpha_3(\xi)\;z' + \alpha_4(\xi)\;z = 0\,,$$

where

$$\begin{cases} \alpha_0(\xi) = & \xi^3\;(1-\xi)\;\;, \\ \alpha_1(\xi) = & \xi^2\,\Big(3 + b_1 + b_2 + b_3\; - \;\xi\;(6 + a_1 + a_2 + a_3 + a_4)\Big)\;\;, \\ \alpha_2(\xi) = & \xi\,\Bigg(1 + b_1 + b_2 + b_3 + b_1\,b_2 + b_1\,b_3 + b_2\,b_3\; - \;\xi\,\Big(7\; + \;3\,(a_1 + a_2 + a_3 + a_4) \\ & \qquad\qquad + a_1\,a_2 + a_1\,a_3 + a_1\,a_4 + a_2\,a_3 + a_2\,a_4 + a_3\,a_4\Big)\Bigg)\;\;, \\ \alpha_3(\xi) = & b_1\,b_2\,b_3 - \xi\,\Big(1 + a_1 + a_2 + a_3 + a_4 + a_1\,a_2 + a_1\,a_3 + a_1\,a_4 + a_2\,a_3 + a_2\,a_4 + a_3\,a_4 \\ & \qquad + a_1\,a_2\,a_3 + a_1\,a_2\,a_4 + a_2\,a_3\,a_4\Big)\;\;, \\ \alpha_4(\xi) = & -a_1\,a_2\,a_3\,a_4\;\;. \end{cases} \tag{21}$$

Equation (20) is therefore hypergeometric of the form (11), and solved by the functions provided in (12).

In Section 2.3 we show how to generalise the solution of Equation (9) by exploiting the general solution of the differential Equation (11), the latter being of immediate determination and rapidly obtainable through symbolic calculus software.

### 2.3. Construction of the algebraic solutions

The final step to solve the quintic (8) consists in determining the most efficient way to choose, in its differential resolvent solutions space, those function that identify the solutions of the algebraic Equation (9). The procedure we present allows to find not only the main solution, but also the remaining ones, and is of general value. The method proposed, though naive in essence, is effective and easy to use, also thanks to the *Mathematica* capacity for symbolic calculus.

To illustrate it, we start from the general solution of Equation (19), which by its nature is the family of linear combinations of four sets of hypergeometric powers:

$$\begin{cases} Y_0^{(5)}(x) = \ {}_4F_3\left(\begin{array}{c} -\frac{1}{20}, \frac{3}{20}, \frac{7}{20}, \frac{11}{20} \\ \frac{1}{4}, \frac{2}{4}, \frac{3}{4} \end{array} \middle| -\frac{3125}{256}\, x^4\right), \quad Y_1^{(5)}(x) = x \ {}_4F_3\left(\begin{array}{c} \frac{4}{20}, \frac{8}{20}, \frac{12}{20}, \frac{16}{20} \\ \frac{2}{4}, \frac{3}{4}, \frac{5}{4} \end{array} \middle| -\frac{3125}{256}\, x^4\right), \\ Y_2^{(5)}(x) = x^2{}_4F_3\left(\begin{array}{c} \frac{9}{20}, \frac{13}{20}, \frac{17}{20}, \frac{21}{20} \\ \frac{3}{4}, \frac{5}{4}, \frac{6}{4} \end{array} \middle| -\frac{3125}{256}\, x^4\right), \quad Y_3^{(5)}(x) = x^3 \ {}_4F_3\left(\begin{array}{c} \frac{14}{20}, \frac{18}{20}, \frac{22}{20}, \frac{26}{20} \\ \frac{5}{4}, \frac{6}{4}, \frac{7}{4} \end{array} \middle| -\frac{3125}{256}\, x^4\right). \end{cases} \tag{22}$$

In other words, the fundamental system of solutions of (19) is obtained by linear combination of coefficients or weights $c_i$ with $i = 0, 1, \ldots, 3$ :

$$y_{\mathcal{G}^{(5)}}(x) = c_0\, Y_0^{(5)}(x) + c_1\, Y_1^{(5)}(x) + c_2\, Y_2^{(5)}(x) + c_3\, Y_3^{(5)}(x)\,. \tag{23}$$

Replacement of (23) in the left–hand side of the algebraic Equation (8):

$$y^5_{\mathcal{G}^{(5)}}(x) + y_{\mathcal{G}^{(5)}}(x) - x\,, \tag{24}$$

and expansion of (24) in third–degree McLaurin Series yields:

$$c_0 + c_0^5 + \left(5\, c_1\, c_0^4 + c_1 - 1\right)\, x + \left(5\, c_2\, c_0^4 + 10\, c_1^2\, c_0^3 + c_2\right)\, x^2 + \left(5\, c_3\, c_0^4 + 20\, c_1\, c_2\, c_0^3 + 10\, c_1^3\, c_0^2 + c_3\right)\, x^3\,. \tag{25}$$

By imposing that (25) vanishes identically, the following five solutions of (8) are obtained:

$$\begin{cases} Y_1^{(5)}(x)\,, \\ -e^{\frac{3i\pi}{4}}\, Y_0^{(5)}(x) - \frac{1}{4}\, Y_1^{(5)}(x) - \frac{5}{32}\, e^{\frac{i\pi}{4}}\, Y_2^{(5)}(x) - \frac{5i}{32}\, Y_3^{(5)}(x)\,, \\ e^{\frac{3i\pi}{4}}\, Y_0^{(5)}(x) - \frac{1}{4}\, Y_1^{(5)}(x) + \frac{5}{32}\, e^{\frac{i\pi}{4}}\, Y_2^{(5)}(x) - \frac{5i}{32}\, Y_3^{(5)}(x)\,, \\ -e^{\frac{i\pi}{4}}\, Y_0^{(5)}(x) - \frac{1}{4}\, Y_1^{(5)}(x) - \frac{5}{32}\, e^{\frac{3i\pi}{4}}\, Y_2^{(5)}(x) + \frac{5i}{32}\, Y_3^{(5)}(x)\,, \\ e^{\frac{i\pi}{4}}\, Y_0^{(5)}(x) - \frac{1}{4}\, Y_1^{(5)}(x) + \frac{5}{32}\, e^{\frac{3i\pi}{4}}\, Y_2^{(5)}(x) + \frac{5i}{32}\, Y_3^{(5)}(x)\,. \end{cases} \tag{26}$$

In other words, we imposed that the $c_i$ cancel the first four terms of the power series, thus identifying, by uniqueness, the linear combinations of solutions of the differential Equation (19).

From (26) we notice that the principal solution to (8) is indeed $Y_1^{(5)}(x)$.

### 2.4. Hypergeometric summations

Even if, in general, a quintic is not solvable by radicals, there are special situations, widely investigated in the literature, in which this happens: these cases, seen from a hypergeometric point of view, produce interesting summation formulas, some of which we recall here.

- Equation $y^5 + y + 1 = 0$ admits only one real (negative) solution, given by:

$$r_1 = \frac{1}{3}\left(1 - \sqrt[3]{\frac{1}{2}\left(25 - 3\sqrt{69}\right)} - \sqrt[3]{\frac{1}{2}\left(25 + 3\sqrt{69}\right)}\right)\,.$$

This is indeed the principal solution, evaluated at $x = 1$, that is:

$$r_1 = {}_4\mathrm{F}_3 \left( \begin{matrix} \frac{4}{20}, \frac{8}{20}, \frac{12}{20}, \frac{16}{20} \\ \frac{2}{4}, \frac{3}{4}, \frac{5}{4} \end{matrix} \middle| -\frac{3125}{256} \right) .$$

- Equation $y^5 + 15\,y + 12 = 0$ is very popular. It was studied in [27–29] where it is found that its unique real root is:

$$r_2 = -\frac{1}{5^{3/5}} \left( \sqrt[5]{75 - 21\sqrt{10}} + \sqrt[5]{75 + 21\sqrt{10}} + \sqrt[5]{-225 + 72\sqrt{10}} - \sqrt[5]{225 + 72\sqrt{10}} \right) .$$

Evaluating the principal solution, we see that:

$$r_2 = -\frac{4}{5}\; {}_4F_3 \left( \begin{matrix} \frac{4}{20}, \frac{8}{20}, \frac{12}{20}, \frac{16}{20} \\ \frac{2}{4}, \frac{3}{4}, \frac{5}{4} \end{matrix} \middle| -\frac{1}{3} \right) .$$

- Equation $y^5 - \frac{11}{4}\,y + 1 = 0$ is treated in [30], where the following factorization is provided:

$$y^5 - \frac{11}{4}\,y + 1 = \left( y^2 + y - \frac{1}{2} \right) \left( y^3 - y^2 + \frac{3}{2}\,y - 2 \right) .$$

This allows to see that the principal solution argument leads to:

$$\frac{\sqrt{3} - 1}{2} = \frac{4}{11}\; {}_4F_3 \left( \begin{matrix} \frac{4}{20}, \frac{8}{20}, \frac{12}{20}, \frac{16}{20} \\ \frac{2}{4}, \frac{3}{4}, \frac{5}{4} \end{matrix} \middle| \frac{12500}{161051} \right) .$$

## 3. Elementary cases

### 3.1. Degree 2

In this simplest case, following Besso's method, we obtain the differential equation:

$$(1 + 4\,x)\,y'' + 2\,y' = 0\,,$$

which is of immediate integration, leading to the general solution:

$$y_{\mathcal{G}} = \frac{1}{2}\,c_1\,\sqrt{1 + 4\,x} + c_2\,.$$

It is thus easy to see that the algebraic Equation (9) is satisfied by the choice $c_1 = \pm 1$ and $c_2 = -1/2$. In view of the transition to equations of higher order, this solution can be seen hypergeometrically as:

$$Y_0^{(2)}(x) = -\frac{1}{2} - \frac{1}{2}\; {}_1\mathrm{F}_0 \left( \begin{matrix} -\frac{1}{2} \\ - \end{matrix} \middle| -4\,x \right) , \qquad Y_1^{(2)}(x) = -\frac{1}{2} + \frac{1}{2}\; {}_1\mathrm{F}_0 \left( \begin{matrix} -\frac{1}{2} \\ - \end{matrix} \middle| -4\,x \right) .$$

### 3.2. Degree 3

The iterative nature of Besso's procedure allows the method to be adapted to equations of any degree. It is interesting to dwell on the elementary case $n = 3$ of (9), where Besso's differential equation reads as:

$$\left( 4 + 27\,x^2 \right) y'' + 27x\,y' - 3y = 0\,. \tag{27}$$

Equation (27) can be solved in terms of elementary functions but, before doing it, it is interesting to apply the change of independent variable, that reveals the hypergeometric nature of the equation. Indeed, from:

$$\xi = -\frac{27}{4}\,x^2\,,$$

it follows that (27) is transformed in a Gauss hypergeometric equation:

$$\xi\,(1-\xi)\,z''+\left(\frac{1}{2}-\xi\right)\,z'+\frac{1}{36}\,z=0\,. \tag{28}$$

The general solution of (28) is therefore (see [31] Section 7):

$$z=c_1\quad {}_2\mathrm{F}_1\left(\begin{array}{c}-\frac{1}{6},\frac{1}{6}\\ \frac{1}{2}\end{array}\middle|\,\xi\right)+c_2\;\xi^{1/2}\quad {}_2\mathrm{F}_1\left(\begin{array}{c}\frac{1}{3},\frac{2}{3}\\ \frac{3}{2}\end{array}\middle|\,\xi\right)\,.$$

Returning to (27), the general solution can be written as:

$$y_{\mathcal{G}}=c_0\quad {}_2\mathrm{F}_1\left(\begin{array}{c}-\frac{1}{6},\frac{1}{6}\\ \frac{1}{2}\end{array}\middle|\,-\frac{27}{4}\,x^2\right)+c_1\;x\quad {}_2\mathrm{F}_1\left(\begin{array}{c}\frac{1}{3},\frac{2}{3}\\ \frac{3}{2}\end{array}\middle|\,-\frac{27}{4}\,x^2\right):=c_0\;Y_0^{(3)}(x)+c_1\;Y_1^{(3)}(x)\,.$$

For $n=3$, therefore, solutions to (9) are:

$$\begin{aligned}
Y_0^{(3)}(x) &= i\;{}_2\mathrm{F}_1\left(\begin{array}{c}-\frac{1}{6},\frac{1}{6}\\ \frac{1}{2}\end{array}\middle|\,-\tfrac{27}{4}\,x^2\right)-\frac{1}{2}\,x\;{}_2\mathrm{F}_1\left(\begin{array}{c}\frac{1}{3},\frac{2}{3}\\ \frac{3}{2}\end{array}\middle|\,-\tfrac{27}{4}\,x^2\right)\,,\\
Y_1^{(3)}(x) &= x\;{}_2\mathrm{F}_1\left(\begin{array}{c}\frac{1}{3},\frac{2}{3}\\ \frac{3}{2}\end{array}\middle|\,-\tfrac{27}{4}\,x^2\right)\,,\\
Y_2^{(3)}(x) &= -i\;{}_2\mathrm{F}_1\left(\begin{array}{c}-\frac{1}{6},\frac{1}{6}\\ \frac{1}{2}\end{array}\middle|\,-\tfrac{27}{4}\,x^2\right)-\frac{1}{2}\,x\;{}_2\mathrm{F}_1\left(\begin{array}{c}\frac{1}{3},\frac{2}{3}\\ \frac{3}{2}\end{array}\middle|\,-\tfrac{27}{4}\,x^2\right)\,.
\end{aligned}$$

The elementary nature of solutions of (27) is due to known properties of ${}_2\mathrm{F}_1$ (see [32] entries 07.23.03.7392.01 and 07.23.03.7393.01) that imply the following equality, where only the expression for the principal solution is considered:

$$Y_1^{(3)}(x)=\frac{2}{\sqrt{3}}\,\sinh\left[\frac{1}{3}\,\sinh^{-1}\left(\frac{3\sqrt{3}}{2}\,x\right)\right]\,,$$

or in algebraic form:

$$Y_1^{(3)}(x)=\frac{1}{\sqrt{3}}\,\left(\Bbbk-\frac{1}{\Bbbk}\right)\,,\qquad\text{where}\qquad \Bbbk=\sqrt[3]{\frac{3\,\sqrt{3}}{2}\,x+\sqrt{\frac{27}{4}\,x^2+1}}\,.$$

### 3.3. Degree 4

As already mentioned, the iterative nature of Besso's process allows to find a specific hypergeometric equation of order $n-1$ for each trinomial equation of degree $n$ of the form (9). In the case $n=4$, the hypergeometric equation is:

$$\left(27+256\,x^3\right)\,y'''+1152\,x^2\,y''+688\,x\,y'-40\,y=0\,. \tag{29}$$

The fundamental system of solutions of (29) is given by:

$$\begin{aligned}
Y_0^{(4)}(x) &= {}_3\mathrm{F}_2\left(\begin{array}{c}-\frac{1}{12},\frac{2}{12},\frac{5}{12}\\ \frac{1}{3},\frac{2}{3}\end{array}\middle|\,-\tfrac{256}{27}\,x^3\right)\,,\\
Y_1^{(4)}(x) &= x\;{}_3\mathrm{F}_2\left(\begin{array}{c}\frac{1}{4},\frac{2}{4},\frac{3}{4}\\ \frac{2}{3},\frac{4}{3}\end{array}\middle|\,-\tfrac{256}{27}\,x^3\right)\,,\\
Y_2^{(4)}(x) &= x^2\;{}_3\mathrm{F}_2\left(\begin{array}{c}\frac{7}{12},\frac{10}{12},\frac{13}{12}\\ \frac{4}{3},\frac{5}{3}\end{array}\middle|\,-\tfrac{256}{27}\,x^3\right)\,.
\end{aligned}$$

By the same procedure used to solve the quintic Equation (8), we arrive at the following solutions of (9), when $n = 4$ :

$$\begin{cases} Y_1^{(4)}(x) \ , \\ -e^{\frac{2i\pi}{3}}\, Y_0^{(4)}(x) - \frac{1}{3}\, Y_1^{(4)}(x) - \frac{2}{9}\, e^{\frac{i\pi}{3}}\, Y_2^{(4)}(x) \ , \\ -\, Y_0^{(4)}(x) - \frac{1}{3}\, Y_1^{(4)}(x) + \frac{2}{9}\, Y_2^{(4)}(x) \ . \end{cases} \tag{30}$$

Unlike the case of order $n = 3$ , here no reductions of the hypergeometric function (30) identified by the Besso's differential Equation (29) are known. Rather, the opposite is true: it is the Besso's procedure that detects new reduction relations for ${}_3\mathrm{F}_2$ , when the hypergeometric roots are equated with the roots of the 4–th degree equation, determined by classical methods [33]. In the case of the principal solution, the hypergeometric expression $Y_1^{(4)}(x)$ in (30) is equal to a sum of radicals; indeed, formula (31) is the principal solution, in algebraic form, to the quartic equation; the situation is similar for the remaining solutions, which we omit here for brevity:

$$-\frac{1}{2}\sqrt{\mathbb{k}} - \frac{1}{2}\sqrt{\mathbb{k} + \frac{2}{\sqrt{\mathbb{k}}}} \ , \tag{31}$$

where:

$$\mathbb{k} = 4\, x\, \sqrt[3]{\frac{2}{27 + 3\sqrt{3}\ \sqrt{27 + 256\, x^3}}} - \sqrt[3]{\frac{9 + \sqrt{3}\ \sqrt{27 + 256\, x^3}}{18}} \ .$$

If the quartic is reducible in $\mathbb{Q}[Y]$ , the equality between classical and hypergeometric representation of the solution is expressed in a simpler form; for example, equation $y^4 + 12\, y - 5 = 0$ can be solved in an elementary way by observing that the polynomial can be factored as $(y^2 - 2\, y + 5)\,(y^2 + 2\, y - 1)$ ; as a consequence, the principal solution can be computed following (30), which yields:

$$\sqrt{2} - 1 = -\frac{5}{12}\ {}_3\mathrm{F}_2\left( \left. \begin{matrix} \frac{1}{4}, \frac{2}{4}, \frac{3}{4} \\ \frac{2}{3}, \frac{4}{3} \end{matrix} \right| -\frac{125}{2187} \right) .$$

## 4. Degree n > 5

The extension is at this point straightforward. In the case $n = 6$ , we find the following solutions:

$$\begin{cases} Y_1^{(6)}(x) \ , \\ -Y_0^{(6)}(x) \ -\frac{1}{5}\, Y_1^{(6)}(x) \ +\frac{3}{25}\ Y_2^{(6)}(x) \ -\frac{14}{125}\ Y_3^{(6)}(x) \ +\frac{78}{625}\ Y_4^{(6)}(x) \ , \\ e^{\frac{i\pi}{5}}\, Y_0^{(6)}(x) \ -\frac{1}{5}\, Y_1^{(6)}(x) \ +\frac{3}{25}\, e^{\frac{4i\pi}{5}}\, Y_2^{(6)}(x) \ +\frac{14}{125}\, e^{\frac{3i\pi}{5}}\, Y_3^{(6)}(x) \ +\frac{78}{625}\, e^{\frac{2i\pi}{5}}\, Y_4^{(6)}(x) \ , \\ -e^{\frac{2i\pi}{5}}\, Y_0^{(6)}(x) \ -\frac{1}{5}\, Y_1^{(6)}(x) \ -\frac{3}{25}\, e^{\frac{3i\pi}{5}}\, Y_2^{(6)}(x) \ +\frac{14}{125}\, e^{\frac{i\pi}{5}}\, Y_3^{(6)}(x) \ +\frac{78}{625}\, e^{\frac{4i\pi}{5}}\, Y_4^{(6)}(x) \ , \\ e^{\frac{3i\pi}{5}}\, Y_0^{(6)}(x) \ -\frac{1}{5}\, Y_1^{(6)}(x) \ +\frac{3}{25}\, e^{\frac{2i\pi}{5}}\, Y_2^{(6)}(x) \ -\frac{14}{125}\, e^{\frac{4i\pi}{5}}\, Y_3^{(6)}(x) \ -\frac{78}{625}\, e^{\frac{i\pi}{5}}\, Y_4^{(6)}(x) \ , \\ -e^{\frac{4i\pi}{5}}\, Y_0^{(6)}(x) \ -\frac{1}{5}\, Y_1^{(6)}(x) \ -\frac{3}{25}\, e^{\frac{i\pi}{5}}\, Y_2^{(6)}(x) \ -\frac{14}{125}\, e^{\frac{2i\pi}{5}}\, Y_3^{(6)}(x) \ -\frac{78}{625}\, e^{\frac{3i\pi}{5}}\, Y_4^{(6)}(x) \ , \end{cases}$$

where functions $Y_i^{(6)}$ are the a fundamental system of solution of the resolvent equation of degree 6, namely:

$$\begin{aligned} Y_0^{(6)}(x) &= \ {}_5\mathrm{F}_4\left( \left. \begin{matrix} -\frac{1}{30}, \frac{4}{30}, \frac{9}{30}, \frac{14}{30}, \frac{19}{30} \\ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \end{matrix} \right| -\frac{46656\, x^5}{3125} \right) , \\ Y_1^{(6)}(x) &= x\ \ {}_5\mathrm{F}_4\left( \left. \begin{matrix} \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6} \\ \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{6}{5} \end{matrix} \right| -\frac{46656\, x^5}{3125} \right) , \\ Y_2^{(6)}(x) &= x^2\ {}_5\mathrm{F}_4\left( \left. \begin{matrix} \frac{11}{30}, \frac{16}{30}, \frac{21}{30}, \frac{26}{30}, \frac{31}{30} \\ \frac{3}{5}, \frac{4}{5}, \frac{6}{5}, \frac{7}{5} \end{matrix} \right| -\frac{46656\, x^5}{3125} \right) , \end{aligned}$$

$$Y_3^{(6)}(x) = x^3 \; {}_5F_4\left( \left. \begin{matrix} \frac{17}{30}, \frac{22}{30}, \frac{27}{30}, \frac{32}{30}, \frac{37}{30} \\ \frac{4}{5}, \frac{6}{5}, \frac{7}{5}, \frac{8}{5} \end{matrix} \right| -\frac{46656\,x^5}{3125} \right),$$

$$Y_4^{(6)}(x) = x^4 \; {}_5F_4\left( \left. \begin{matrix} \frac{23}{30}, \frac{28}{30}, \frac{33}{30}, \frac{38}{30}, \frac{43}{30} \\ \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5} \end{matrix} \right| -\frac{46656\,x^5}{3125} \right).$$

For the general case, the fundamental system of solutions consists of $n-1$ hypergeometric functions of the type ${}_{n-1}F_{n-2}$ :

$$\begin{aligned} Y_0^{(n)}(x) &= \; {}_{n-1}F_{n-2}\left( \left. \begin{matrix} \dots \\ \dots \end{matrix} \right| -\frac{n^n\; x^{n-1}}{(n-1)^{n-1}} \right), \\ Y_1^{(n)}(x) &= x \; {}_{n-1}F_{n-2}\left( \left. \begin{matrix} \dots \\ \dots \end{matrix} \right| -\frac{n^n\; x^{n-1}}{(n-1)^{n-1}} \right), \\ &\dots \\ Y_{n-2}^{(n)}(x) &= x^{n-2} \; {}_{n-1}F_{n-2}\left( \left. \begin{matrix} \dots \\ \dots \end{matrix} \right| -\frac{n^n\; x^{n-1}}{(n-1)^{n-1}} \right). \end{aligned}$$

The upper parameters are of the form:

$$\frac{i}{n-1} + \frac{j}{n} - \frac{1}{(n-1)\,n}, \qquad i\,,j = 0,\dots,n-2\,,$$

where $i$ is related to $Y_i^{(n)}(x)$ . As for the lower parameters, we have the following behaviour, associated to the respective relevant function $Y_i^{(n)}(x)$ :

$$\begin{array}{llll} \frac{1}{n-1}, & \dots, & \frac{n-2}{n-1} & \qquad Y_0^{(n)} \\ \frac{2}{n-1}, & \dots, & \frac{n-2}{n-1}, \; \frac{n}{n-1} & \qquad Y_1^{(n)} \\ \vdots & & & \qquad \vdots \\ \frac{n}{n-1}, & \dots, & \frac{2n-3}{n-1} & \qquad Y_{n-1}^{(n)} \end{array}$$

Observe that the value 1 is forbidden to appear in the lower parameters. The principal solution of (9) is found to be:

$$Y_1^{(n)}(x) = x \; {}_{n-1}F_{n-2}\left( \left. \begin{matrix} \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n} \\ \frac{2}{n-1}, \dots, \frac{n-2}{n-1}, \frac{n}{n-1} \end{matrix} \right| -\frac{n^n\; x^{n-1}}{(n-1)^{n-1}} \right). \tag{32}$$

Equation (32) is inspired by the sequence of coefficients deduced in the previous cases: it can be verified by comparison with the derivatives obtained from the Implicit Function Theorem [34] applied to (32).

## 5. Generalized algorithm

We now provide an outline of the implementation of the generalization of Besso's algorithm within the *Mathematica* environment. Here, Equation (9) is considered with $n \in \mathbb{N}, n \geq 3$. The pseudo–code presented can more easily be followed by taking into account the quintic ($n = 5$) example worked–out in Section 2.2.

To form the right–hand side in identity (4), the following *replacement rules* (33), (34) and (35) are applied in an iterative manner, alternating them with differentiation steps. The notation adopted is that of *Mathematica*, where a **rule** is indicated by the *arrow* symbol:

$$lhs \longrightarrow rhs$$

which means that, if the **rule** is employed in conjunction with a **replacement** operator, then, whenever the *lhs* pattern is encountered within a current expression, it has to be substituted by the *rhs* content. In (33) and (35), the parameter $p \in \mathbb{N}$ is assumed to be $p \geq n-3$.

The following replacement, which we call **identity rule**, is used to enforce the identity (16) iteratively:

$$\left(y'\,y\right)y^{p-1} \longrightarrow \left(\frac{n}{n-1}\,x\,y' - \frac{1}{n-1}\,y\right)y^{p-1} \qquad\qquad p = n-3\,,\ldots,1. \tag{33}$$

A second replacement is what we named **basic rule**, that allows to eliminate dependency on $x$, as it is done, for example, to obtain (13):

$$x \longrightarrow y^n + y. \tag{34}$$

A last replacement, that we called **general rule**, serves to eliminate $p$–th derivatives of $y$, lowering the degree to reach linearity, as performed, for instance, to arrive to (2.2):

$$y^p \longrightarrow y^{(p \bmod n)}\,(x-y)^{\frac{p}{n}}. \tag{35}$$

Given the three replacement rules above, we are now ready to set–up the overall iteration; the differentiations required are performed invoking the **DSolve** differential equation solver available in *Mathematica* [35].

An *inizialization* phase is performed, consisting of four operations:

- eliminate the dependency on $x$ using the basic rule (34);
- derive the current expression with respect to $y$;
- eliminate $p$–th derivatives of $y$ using the general rule (35);
- re–insert the dependency on $x$ in $y$ and its derivatives.

The *core* of the iterative procedure is formed by the following steps, that get repeated $n-3$ times:

- derive the current expression with respect to $x$;
- eliminate the dependency on $x$ using the basic rule (34);
- eliminate non–linearity, repeatedly using the identity rule (33);
- re–insert the dependency on $x$ in $y$ and its derivatives.

All this yields the right–hand side in identity (4).

To form the left–hand side in the same equation, a *finalising* phase is performed:

- derive repeatedly ($n-3$ times in total) the left–hand side component of (4), namely:

$$D\left(\frac{d\,y(x)}{d\,x}\right)^2,$$

taking care of eliminating $y'$ from the current expression after the very first derivation.

At this point, left–hand and right–hand sides of (4) can be equated, and a last application of the differential equation solver yields the desidered principal solution to (9).

## 6. Conclusion

The solution of algebraic equations of degree greater than four, by means of differential equations, is considered as a well–known issue among the experts. In the specialised literature, however, few articles are actually operational. Our contribution goes in the direction of filling this gap and proposes to express, in explicit terms, via a generalised hypergeometric series, the principal solution to the family of trinomial equations with secondary exponent equal to 1; in the quintic case and possibly under Bring–Jerrard transformations, the solution to the latter family represents resolving the problem in its generality.

The method we propose, and have implemented within the *Mathematica* environment, constitute a generalization of Besso's algorithm, which is based on an identity due to Brioschi and Betti: one of aims of this work is, in fact, to bring to the attention of the scientific community the fundamental contributions, to the theory of equations, of these nineteenth–century Italian mathematicians.

It is important to note that Besso's method, unlike other approaches, does not rely on the Implicit Functions Theorem: in our case, the latter theorem is used only as a tool to verify and validate the results obtained.

Although the methods we recalled cannot be said to be innovative, their reinterpretation through the power of symbolic calculus allows them to be put into practice, in applications that can numerically take advantage of the throughput of the hypergeometric series we obtained.

As a final note, we mention that the subject dealt with in this paper has also a considerable didactic value, for advanced students, as applications of special functions are presented that are of immediate impact in concrete problems.

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