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A two-step block method with two hybrid points for the numerical solution of first order ordinary differential equations

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Abstract: A continuous two-step block method with two hybrid points for the numerical solution of first order ordinary differential equations is proposed. The approximate solution in form of power series and its first ordered derivative are respectively interpolated at the point $x = 0$ and collocated at equally spaced points in the interval of consideration. The application of the method involves using the main scheme derived together with the additional schemes simultaneously to obtain the solution to the problem at the grid points. The analysis of the method and the results obtained from the examples considered show that the method is consistent, zero-stable, convergent and of high accuracy.

Keywords: Hybrid points; Block method; Zero-stability; Consistency; Convergence.

MSC: 65L80.

1. Introduction

Ordinary Differential Equations (ODEs) are important models derived from real-life problems and other natural phenomena. In particular, many problems in engineering, biology, physics and other social sciences have been modelled resulting in ordinary differential equations, [1–3].

The main focus of this paper is to consider an accurate approximate method for the solution of general first order initial value problem of the form

$$y' = f(x, y), \;\; y(x_0) = y_0 \;\; : x \in [a,\, b], \tag{1}$$

where $f(x, y)$ is continuous and satisfies the existence and uniqueness of solution theorem, [4]. Predictor-corrector methods for solving ordinary differential equations of the form (1) proposed by [5] has some demerits which necessitated the introduction of hybrid block methods. Hybrid methods were initially introduced to overcome zero-stability barrier that occurred in block methods, Dahlquist (1956). Apart from the ability to change step size, the benefit of the hybrid block methods is utilizing data off-step points which contributes to the accuracy of the methods, [6]. The standard collocation method at some selected points introduced by [7] was discrete in nature. However, [8] was able to show that the traditional multistep methods including the hybrid ones can be made continuous. This was done through the idea of multistep collocation scheme against the discrete schemes. Through this, better approximation at all interior points and absolute error estimates were obtained.

According to [9], the continuous linear multi-step method has greater advantage over the discrete method in that it gives better error estimation, provides a simplified form of coefficient for further analytical work at different points and guarantees easy appropriation of solutions at all interior points within the interval of integration. [10–12] proposed methods which were implemented in predictor-corrector mode and adopted Taylor series expansion to supply the starting value. Generally, the major setback of the predictor-corrector method is the high cost of implementation, as subroutines are very complicated to write because of the special techniques required to supply starting values, [13]. Therefore, the need to address this setback by proposing a method that melts the properties of both the block method and the predictor-corrector method cannot be over-emphasised. To solve Eq. (1), numerical methods are developed by types of Ordinary Differential Equations (ODEs) such as non-linear, linear, either stiff or non-stiff ODEs. It should be noted that by using inappropriate method for a model may lead to slow convergence or wrong solution, [14,15] observed that Methods such as Adomian decomposition method, variational iteration method, Chebyshev's wavelet method, classical fourth-order Runge-Kutta method, homotopy perturbation method have some setbacks ranging from small convergence/implementation regions to inefficiency in terms of accuracy.

We therefore, present a self-starting continuous two-step hybrid block method with faster rate of convergence and better accuracy for the numerical integration of initial value problems of first order ordinary differential equations. In doing this, the collocation points were evenly selected in the interval of consideration.

2. Derivation of the method

We seek a k-step multistep method of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h[\sum_{j=0}^{k} \beta_j f_{n+j} + \beta_{vi} f_{n+vi}], \tag{2}$$

where α_j and β_j are continuous coefficients and vi are hybrid points.

We assume an approximate solution to Eq. (1) to be a continuous solution of the form

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j. \tag{3}$$

We construct a k-step continuous hybrid multistep method with $x^j \;:\; j = 0(1)5$, $r = 1$, $s = 5$ and $k = 2$. Therefore, from (3) we have

$$\sum_{j=0}^{5} a_j x_{n+i}^j = y_{n+i}\,, \tag{4}$$

and

$$\sum_{j=0}^{5} j a_j x_{n+i}^{j-1} = f_{n+i}. \tag{5}$$

Interpolating (4) at $x = x_n$ such that $i = 0$ and collocating (5) at $x = x_{n+i}$ for $i = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$, we have the following six equations

$$a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5 = y_n, \tag{6}$$

$$a_1 + 2a_2 x_n + 3a_3 x_n^2 + 4a_4 x_n^3 + 5a_5 x_n^4 = f_n, \tag{7}$$

$$a_1 + 2a_2 x_{n+\frac{1}{2}} + 3a_3 x_{n+\frac{1}{2}}^2 + 4a_4 x_{n+\frac{1}{2}}^3 + 5a_5 x_{n+\frac{1}{2}}^4 = f_{n+\frac{1}{2}}, \tag{8}$$

$$a_1 + 2a_2 x_{n+1} + 3a_3 x_{n+1}^2 + 4a_4 x_{n+1}^3 + 5a_5 x_{n+1}^4 = f_{n+1}, \tag{9}$$

$$a_1 + 2a_2 x_{n+\frac{3}{2}} + 3a_3 x_{n+\frac{3}{2}}^2 + 4a_4 x_{n+\frac{3}{2}}^3 + 5a_5 x_{n+\frac{3}{2}}^4 = f_{n+\frac{3}{2}}, \tag{10}$$

and

$$a_1 + 2a_2 x_{n+2} + 3a_3 x_{n+2}^2 + 4a_4 x_{n+2}^3 + 5a_5 x_{n+2}^4 = f_{n+2}. \tag{11}$$

We write Eqs. (6) to (11) in matrix form to have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & \frac{3}{4} & \frac{1}{2} & \frac{5}{16} \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 3 & \frac{27}{4} & \frac{27}{2} & \frac{405}{16} \\ 0 & 1 & 4 & 12 & 32 & 80 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} y_n \\ f_n \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix}. \tag{12}$$

Solving (12) by Gaussian elimination method, we obtain the continuous variables a_i's and the parameters α_j and β_j as the following functions of t.

$$\alpha_0(t) = 1,$$
$$\beta_0(t) = t - \frac{25}{12}t^2 + \frac{35}{18}t^3 - \frac{5}{6}t^4 + \frac{2}{15}t^5,$$
$$\beta_{\frac{1}{2}}(t) = 4t^2 - \frac{52}{9}t^3 + 3t^4 - \frac{8}{15}t^5,$$
$$\beta_1(t) = -3t^2 + \frac{57}{9}t^3 - 4t^4 + \frac{4}{5}t^5,$$
$$\beta_{\frac{3}{2}}(t) = \frac{4}{3}t^2 - \frac{28}{9}t^3 + \frac{1}{2}t^4 - \frac{8}{15}t^5,$$

and

$$\beta_2(t) = -\frac{1}{4}t^2 + \frac{11}{18}t^3 - \frac{7}{4}t^4 + \frac{2}{15}t^5.$$

Eq. (2) is re-written as

$$y(x) = \alpha_0 y_n + h\left[\sum_{j=0}^{2} \beta_j(x) f_{n+j} + \beta_{vi}(x) f_{n+vi}\right]. \tag{13}$$

We evaluate $\beta_j(t)$ at $t = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ 2$ and substitute the values obtained into (13) to obtain the hybrid block methods as

$$y_{n+\frac{1}{2}} - y_n = h[\frac{251}{1440}f_n + \frac{323}{720}f_{n+\frac{1}{2}} - \frac{11}{60}f_{n+1} + \frac{53}{720}f_{n+\frac{3}{2}} - \frac{19}{1440}f_{n+2}], \tag{14}$$
$$y_{n+1} - y_n = h[\frac{29}{180}f_n + \frac{31}{45}f_{n+\frac{1}{2}} + \frac{2}{15}f_{n+1} + \frac{1}{45}f_{n+\frac{3}{2}} - \frac{1}{180}f_{n+2}], \tag{15}$$
$$y_{n+\frac{3}{2}} - y_n = h[\frac{27}{160}f_n + \frac{51}{80}f_{n+\frac{1}{2}} + \frac{9}{20}f_{n+1} + \frac{21}{80}f_{n+\frac{3}{2}} - \frac{3}{160}f_{n+2}], \tag{16}$$

and

$$y_{n+2} - y_n = h[\frac{7}{45}f_n + \frac{32}{45}f_{n+\frac{1}{2}} + \frac{4}{15}f_{n+1} + \frac{32}{45}f_{n+\frac{3}{2}} + \frac{7}{45}f_{n+2}]. \tag{17}$$

3. Analysis of the scheme

The analysis of the derived scheme which includes the order and error constant, consistency, zero stability and convergence is presented here.

3.1. Orders and error constants

Consider the linear operator, L associated with the hybrid block method (2) which is defined as

$$L[y(x_n) : h] = \sum_{j=0}^{k} \left[\alpha_j y(x_n + jh) - h(\beta_j y'(x_n + jh) + \beta_{vi} y'(x_n + vih))\right], \tag{18}$$

where $y(x)$ is an arbitrary test function that is continuously differentiable in the interval $[a, b]$. Expanding $y(x_n + jh)$, $y'(x_n + jh)$ and $y'(x_n + vih)$ in Taylor series about x_n and collecting the coefficients of $h^{(q)}\ : q = 0, 1, 2, 3, ...$ to get

$$L[y(x_n) : h] = c_0 y(x_n) + c_1 h y'(x_n) + c_2 h^2 y''(x_n) + ... + c_q h^q y^{(q)}(x_n) + ... \tag{19}$$

where c_q are vectors. From Eq. (19), if we can obtain that

$$c_0 = c_1 = c_2 = ... = c_q = 0 : c_{q+1} \neq 0$$

then the hybrid block method (2) is said to be of order q and its error constant is c_{q+1}, [16,17].

Now, expanding Eq. (14) in Taylor's series around $y(x = x_{n+\frac{1}{2}})$ and collecting like terms in $h^q y^{(q)}(x_n)$: $q = 0, 1, 2, 3, ..$ we obtain

$$\begin{aligned} L[y(x_{n+\frac{1}{2}}) : h] &= y(x_n) - y(x_n) + \left(\frac{1}{2} - \frac{251}{1440} - \frac{323}{720} + \frac{11}{60} - \frac{53}{720} + \frac{19}{1440}\right) hy'(x_n) \\ &+ \left(\frac{1}{8} - \frac{323}{1440} + \frac{11}{60} - \frac{159}{1440} + \frac{19}{720}\right) h^2 y''(x_n) + \left(\frac{1}{48} - \frac{323}{5760} + \frac{11}{120} - \frac{477}{5760} + \frac{19}{720}\right) h^3 y'''(x_n) \\ &+ \left(\frac{1}{384} - \frac{323}{34560} + \frac{11}{360} - \frac{1431}{34560} + \frac{19}{1080}\right) h^4 y^{iv}(x_n) + \left(\frac{1}{3840} - \frac{323}{276480} + \frac{11}{1440} - \frac{4293}{276480} + \frac{19}{2160}\right) h^5 y^{v}(x_n) \\ &+ \left(\frac{1}{46080} - \frac{323}{2764800} + \frac{11}{7200} - \frac{12879}{2764800} + \frac{19}{5400}\right) h^6 y^{vi}(x_n) + ... \end{aligned} \tag{20}$$

Eq. (20) is simplified further to obtain

$$L[y(x_{n+\frac{1}{2}}) : h] = c_0 y(x_n) + c_1 y'(x_n) + c_2 y''(x_n) + ... + c_q y^{(q)}(x_n) + c_{q+1} y^{(q+1)}(x_n) + ... \tag{21}$$

In Eq. (21),

$$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = 0 \, : \, c_6 = \frac{3h^6}{10240}.$$

In a similar manner, we obtained from Eqs. (15), (16) and (17) by evaluating respectively at $y(x = x_{n+1})$, $y(x = x_{n+\frac{3}{2}})$ and $y(x = x_{n+2})$

$$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = 0 \, : \, c_6 = \frac{h^6}{5760},$$

$$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = 0 \, : \, c_6 = \frac{3h^6}{10240},$$

$$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0 \, : \, c_7 = -\frac{h^7}{15120}.$$

Therefore, the orders of our scheme are obtained as $(5, 5, 5, 6)^T$ and error constants as

$$\left(\frac{3}{10240}, \frac{1}{5760}, \frac{3}{10240}, -\frac{1}{15120}\right)^T$$

3.2. Consistency

A linear multistep method is said to be consistent if it has an order of convergence, $q \geq 1$ [18]. The derived hybrid block method is consistent since the orders are all greater than

3.3. Zero stability

A linear multistep method of the form (2) is said to be zero stable if no roots of the first characteristic polynomial $\rho(R)$ has modulus greater than one, and if every root of modulus one is simple, [18,19], Eqs (114) to (17) are put in block form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+2} \\ y_{n+\frac{3}{2}} \\ y_{n+1} \\ y_n \end{bmatrix} + h \begin{bmatrix} \frac{323}{720} & -\frac{11}{60} & \frac{53}{720} & -\frac{19}{1440} \\ \frac{31}{45} & \frac{2}{15} & \frac{1}{45} & -\frac{1}{180} \\ \frac{51}{80} & \frac{9}{20} & \frac{21}{80} & -\frac{3}{160} \\ \frac{32}{45} & \frac{4}{5} & \frac{32}{45} & \frac{7}{45} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix}$$
$$+ h \begin{bmatrix} 0 & 0 & 0 & \frac{7}{45} \\ 0 & 0 & 0 & \frac{27}{160} \\ 0 & 0 & 0 & \frac{29}{180} \\ 0 & 0 & 0 & \frac{251}{1440} \end{bmatrix} \begin{bmatrix} f_{n+2} \\ f_{n+\frac{3}{2}} \\ f_{n+1} \\ f_n \end{bmatrix} \tag{22}$$

Normalizing Eq. (22) by multiplying with the inverse of A^0 and the first characteristic polynomial of the hybrid block method is obtained as

$$\rho(R) = det[RA^0 - A^1] \tag{23}$$

where $A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $A^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Therefore,

$$\rho(R) = det\left(R \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) = det \begin{bmatrix} R & 0 & 0 & -1 \\ 0 & R & 0 & -1 \\ 0 & 0 & R & -1 \\ 0 & 0 & 0 & R-1 \end{bmatrix}$$

Implies, $\rho(R) = R^4 - R^3$. Since $\rho(R) = 0$, so $R^4 - R^3 = 0$. Therefore, the roots of the first characteristics polynomial are $R_1 = R_2 = R_3 = 0$ and $R_4 = 1$ and $\rho(R)$ satisfies the condition $|R| \leq 1$ and $|R| = 1$ is simple, then our block hybrid scheme given by Eqs. (14) - (17) is zero stable.

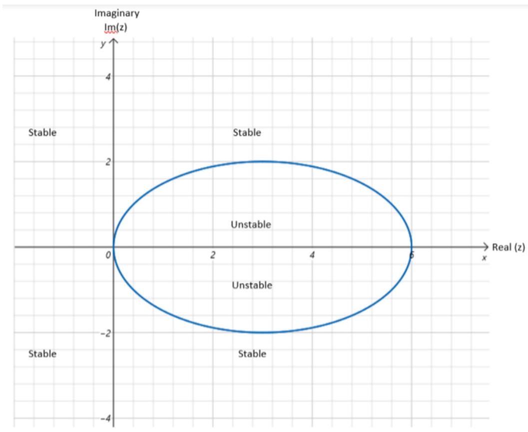

Figure 1. Stability region of the two-step with three hybrid points method

3.4. Convergence

The convergence of the hybrid two-step method is considered in the light of its basic properties, that is the consistency and zero-stability; in conjunction with the fundamental theorem of Dahlquist for linear multi-step method which states that "the necessary and sufficient condition for a multi-step method to be convergent is for it to be consistent and zero-stable", [4]. Then the hybrid block method discussed is convergent since it is consistent and zero-stable.

3.5. Numerical Implementation of the Scheme

In this section, we test the effectiveness and validity of our derived scheme by applying it to some first order differential equations. Unlike the predictor-corrector method which requires that the starting values y_{n+j}, $j = 0, 1, 2, ...$ be generated first, the proposed method is self starting thereby reducing the amount of work in the computation. For error calculation, the error formula is given by

$$E(x_n) = |y(x) - y(x_n)|. \tag{24}$$

In (18), $y(x)$ is the exact solution for the problem considered and $y(x_n)$ is the approximate solution obtained using derived methods.

All computations and programs are carried out with the aid of Maple 13 software.

Example 1 ([19]). Consider the following nonlinear Ordinary Differential Equation

$$y' = -2xy^2, \ \ y(0) = 1 \ : \ 0 \leq x \leq 1. \tag{25}$$

The exact solution is $y(x) = (x^2 + 1)^{-1}$.

Table 1 shows the comparison between our method and the Euler method with.

Table 1. Numerical results for Example 1: Comparison between the absolute errors in our method and Euler method

x	Exact solution	Euler method error		Our method error	
0.0	1.00000	1.00000	0.000000	1.00000	0.000000
0.1	0.99010	1.00000	9.900e-3	0.99010	0.000000
0.2	0.96154	0.98000	1.846e-2	0.96153	1.000e-5
0.3	0.91743	0.94158	2.415e-2	0.91744	1.000e-5
0.4	0.86207	0.88839	2.632e-2	0.86207	0.000000
0.5	0.80000	0.82525	2.525e-2	0.80000	0.000000
0.6	0.73529	0.75715	2.186e-2	0.73529	0.000000
0.7	0.67114	0.68835	1.721e-2	0.67113	1.000e-5
0.8	0.60976	0.62202	1.226e-2	0.60976	0.000000
0.9	0.55249	0.56011	7.620e-3	0.55250	1.000e-5
1.0	0.50000	0.50364	3.640e-3	0.50000	0.000000

Example 2 ([19]). Consider the following linear Ordinary Differential Equation

$$y' = x + y, \;\; y(0) = 1 \;:\; 0 \leq x \leq 1. \tag{26}$$

The exact solution is $y(x) = 2e^x - x - 1$.

Table 2 shows the comparison between our method; third order Adams-Moulton (A-M) method and fourth order Milne-Simpson (M-S) method. Applying the Adams-Moulton and Milne-Simpson methods, the starting value y_1 was determined using Taylor series method of appropriate order.

Table 2. Numerical results for Example 2: Comparison between the absolute errors in our method and other methods in literature

x	Exact solution	Error in our method	Error in A-M	Error in M-S
0.0	1.0000000	0.0000000	0.0000000	0.0000000
0.1	1.1103418	0.0000000	8.5.000e-6	2.0000e-7
0.2	1.2428055	0.0000000	3.0000e-7	2.0000e-7
0.3	1.3997176	1.0000e-7	1.1200e-5	1.0000e-7
0.4	1.5836494	0.0000000	2.4300e-5	5.0000e-7
0.5	1.7974425	0.0000000	4.0200e-5	5.0000e-7

Example 3. Consider the nonlinear Ordinary Differential Equation

$$y' = \frac{1}{2}(1 - y), \;\; y(0) = \frac{1}{2}, \quad x \in [0,\, 1]. \tag{27}$$

The exact solution is $y(x) = 1 - \frac{1}{2}e^{-\frac{1}{2}x}$.

Table 3. Numerical results for Example 3: Comparison between the absolute errors in our method and other methods in literature

x	Error in our method	Error in OSBM3H	Error in OSHBM	Error in CM3HAM
0.0	0.000000000	0.000000000	0.000000000	0.000000000
0.1	3.01260E-17	1.99840E-15	1.71400E-14	6.78013E-13
0.2	5.49357E-17	3.88578E-15	3.26000E-14	6.35936E-13
0.3	5.83702E-17	5.44009E-15	4.65300E-14	6.38045E-13
0.4	5.89498E-17	6.99441E-15	5.90200E-14	1.18994E-12
0.5	7.20996E-17	8.21565E-15	7.01800E-14	1.12410E-12
0.6	8.43851E-17	9.54792E-15	8.01100E-14	1.09901E-12
0.7	8.85311E-17	1.05471E-14	8.89100E-14	1.54798E-12
0.8	9.33604E-17	1.13243E-14	9.66500E-14	1.46805E-12
0.9	2.67745E-16	1.22125E-14	1.03420E-13	1.41909E-12
1.0	2.98500E-16	1.28786E-14	1.09310E-13	1.78202E-12

Example 4. Consider the nonlinear Ordinary Differential Equation

$$y' = -10(y-1)^2, \ \ y(0) = 2, \quad x \in [0,\ 0.1]. \tag{28}$$

The exact solution is $y(x) = 1 + (1+10x)^{-1}$.

Table 4 shows the comparison between our method; Block Method with One Hybrid Point (BM1HP) by [22], One-Sixth Hybrid Block Method (OSHBM) derived by using the Chebyshev polynomials by [21] and One-Step Block Method with Three Hybrid (OSBM3H) points by [6].

Table 4. Numerical results for Example 4: Comparison between the absolute errors in our method and other methods in literature

x	Error in our method	Error in OSBM3H	Error in BM1HP	Error in OSHBM
0.0	0.0000000	0.0000000	0.0000000	0.0000000
0.01	1.61332E-14	1.61332E-10	2.82900E-7	1.55825E-6
0.02	2.14045E-14	2.14045E-10	4.04578E-7	2.39975E-6
0.03	2.22905E-14	2.22905E-10	4.47254E-7	2.83045E-6
0.04	2.14208E-14	2.14208E-10	4.50903E-7	3.02094E-6
0.05	1.99082E-14	1.99082E-10	4.35625E-7	3.06956E-6
0.06	1.82326E-14	1.82326E-10	4.11764E-7	3.03457E-6
0.07	1.65980E-14	1.65980E-10	3.84699E-7	2.95115E-6
0.08	1.50853E-14	1.50853E-10	3.57218E-7	2.84088E-6
0.09	1.37194E-13	1.37194E-10	3.30725E-7	2.71713E-6
0.10	1.25003E-13	1.25003E-10	3.05879E-7	2.58816E-6

4. Conclusion

A two-step block method with two hybrid points for the numerical solution of first order ordinary differential equations has been proposed and discussed. The method was derived through interpolation of the assumed power series solution at the point $x = x_n$ and collocation of its first ordered derivative at equally spaced points in the interval of consideration. The consistency, zero-stability and applicability of the proposed method were considered and well discussed. The analysis of the method showed that it is convergent since consistency and zero-stability properties are satisfied. Numerical results as presented in Tables 1-4 show that the method performs better than most of the existing methods in literature. Furthermore, the method produces results that are very close to the exact solutions for all the problems considered.

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