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# Fibonacci type sequences and integer multiples of periodic continued fractions

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**Abstract:** We construct a class of quadratic irrationals having continued fractions of period $n \geq 2$ with 'small' partial quotients for which specific integer multiples have periodic continued fractions with the length of the period being 1, 2 or 4, and with 'large' partial quotients. We then show that numbers in the period of the new continued fraction are functions of the numbers in the periods of the original continued fraction. We also show how polynomials arising from generalizations of these continued fractions are related to Chebyshev and Fibonacci polynomials and, in some cases, have hyperbolic root distribution.

**Keywords:** Chebyshev polynomials; Continued fractions; Fibonacci numbers; Fibonacci-like numbers; Lucas numbers; Lucas-like numbers; Fibonacci polynomials.

**MSC:** 11C20; 26C05; 30C15.

## 1. Introduction

Let $[a_0, a_1, a_2, \dots]$ be a simple continued fraction expansion of a real number $\alpha$ where $a_j$'s are positive integers. If $\alpha$ is a quadratic irrational, then by a theorem of Lagrange [1, p. 44] its continued fraction will be periodic, i.e.,

$$\alpha = [a_0, a_1, \dots, a_k, \overline{b_1, b_2, \dots, b_n}],$$

where $b_1, b_2, \dots, b_n$ is the period of the expansion of $\alpha$, and $a_0, a_1, \dots, a_k$ is the non-periodic part. We say that $\alpha$ is of period $k$ if its periodic part has length $k$.

If $\alpha$ is a quadratic irrational, then for any positive integer $N$, $N\alpha$ is a quadratic irrational and has a periodic continuous fraction. Cusick [2] presents an algorithm for obtaining the continued fraction of $N\alpha$ and uses it to estimate the length of the period of the expansion of $N\alpha$ to that of $\alpha$. In this paper, we construct a quadratic irrational $\alpha$ such that when $N$ is a Fibonacci or Lucas number, the continued fraction of $N\alpha$ has a period of length 1, 2 or 4. As we shall see, the length of the period of the new continued fraction depends on the parity of $n$, the length of the period of the original continued fraction.

Fibonacci numbers and Lucas numbers are integers that solve the recurrence relation

$$f_{n+1} = f_n + f_{n-1}\,,$$

under the initial conditions $F_0 = 0$, $F_1 = F_2 = 1$, $L_0 = 2$ and $L_1 = 1$ respectively. Fibonacci polynomials, which are generated by the rational function

$$\frac{t}{1 - xt - t^2} = \sum_{k=1}^{\infty} F_k(x) t^k\,, \tag{1}$$

also define the Fibonacci numbers when $x = 1$. In §2, we construct Fibonacci-like numbers $\tilde{F}_n(m)$ and use them to construct $\alpha_n$, the largest zero of the quadratic

$$x^2 - 2NF_n x - F_n \tilde{F}_n(2N)\,,$$

where $N$ is a positive integer. We then show the relationship between the continued fraction of $\alpha_n$ and

$$F_n \cdot [\overline{2N, 1^{(n-1)}}]\,,$$

where $1^{(n)}$ is the sequence $1,1,1,\dots,1$ with 1 repeated $n$ times.

In §3, we construct Lucas-like numbers $\tilde{L}_n(m)$ and use them to construct $\beta_n$, the largest zero of the quadratic

$$y^2 - 2NL_n y - L_n\tilde{L}_n(2N)\,,$$

where $N = 5k+3$ is a positive integer. We then show the relationship between the continued fraction of $\beta_n$ and the purely periodic continued fraction

$$L_n \cdot [\overline{2N, 1^{(n-2)}, 2, 1, 2k, 1, 2, 1^{(n-2)}}].$$

In §4, we generalize these continued fractions, as well as results of §2. Polynomials arising from convergents of the generalized continued fractions are studied in §5. In particular, we show how these polynomials relate to Fibonacci and Chebyshev polynomials and show that some of them have their roots in hyperbolas. Contents of this paper are a summary of results in the dissertation [3].

## 2. Fibonacci-like numbers and quadratic irrationals

Let $m$ be a positive integer and define the sequence $\tilde{F}_n(m)$ by the recurrence relation

$$\tilde{F}_n(m) = \tilde{F}_{n-1}(m) + \tilde{F}_{n-2}(m)\,, \tag{2}$$

under the initial conditions $\tilde{F}_0(m) = 0$, $\tilde{F}_1(m) = 1$ and $\tilde{F}_2(m) = m$. Using (1) together with the recurrence (2), we obtain

$$\frac{t+(m-1)t^2}{1-t-t^2} = \sum_{n=0}^{\infty} \tilde{F}_n(m)t^n. \tag{3}$$

Clearly, when $m = 1$ we get the usual Fibonacci sequence. From the generating function (3), and the generating function of Fibonacci numbers, we get the relation

$$\tilde{F}_{n+1}(m) = mF_n + F_{n-1}. \tag{4}$$

Now consider the quadratic

$$x^2 - 2NF_n x - F_n\tilde{F}_n(2N)$$

with the zeros

$$\begin{aligned} \alpha_n(N) &= NF_n + \sqrt{N^2F_n^2 + F_n\tilde{F}_n(2N)}, \\ \overline{\alpha}_n(N) &= NF_n - \sqrt{N^2F_n^2 + F_n\tilde{F}_n(2N)}. \end{aligned} \tag{5}$$

Continued fractions of the zeros (5) have some interesting properties which we explore in this paper.

A quadratic irrational $\alpha$ is said to be reduced if $\alpha > 1$ and $-1/\overline{\alpha} > 1$. If $\alpha$ is reduced, then its continued fraction is purely periodic [1, Theorem 2.48]. First we show that when $n$ odd, $1/(\alpha_n(N) - \tilde{F}_{n+1}(2N))$ is reduced.

Let $B_n(N) = N^2F_n^2 + F_n\tilde{F}_n(2N)$, then

$$1/(\alpha_n(N) - \tilde{F}_{n+1}(2N)) = NF_n + F_{n-1} + \sqrt{B_n(N)} > 1. \tag{6}$$

On the other hand,

$$\tilde{F}_{n+1}(2N) - \overline{\alpha}_n(N) = NF_n + F_{n-1} + \sqrt{B_n(N)} > 1.$$

We have used (4) and the well known identity for Fibonacci numbers

$$F_{n-1}^2 - F_nF_{n-2} = (-1)^n. \tag{7}$$

Let $x_n(N) = NF_n + F_{n-1} + \sqrt{B_n(N)}$. Using (4), and the Euclidean algorithm we get

$$x_n(N) = 2\tilde{F}_{n+1}(N) + \frac{1}{x_n(N)},$$

and together with (6) gives

$$\begin{aligned}\alpha_n(N) &= x_n - F_{n-1}\\ &= \tilde{F}_{n+1}(2N) + \frac{1}{x_n(N)}\\ &= [\tilde{F}_{n+1}(2N),\ \overline{2\tilde{F}_{n+1}(N)}].\end{aligned}$$

When $n$ even, and using the identity (7).

$$1/(\alpha_n(N) - \tilde{F}_{n+1}(2N) + 1) = \frac{NF_n + F_{n-1} + \sqrt{B_n(N)}}{NF_n + F_{n-1} + \sqrt{B_n(N)} - 1} > 1.$$

We also have that

$$\tilde{F}_{n+1}(2N) - 1 - \overline{\alpha_n} = NF_n + F_{n-1} + \sqrt{B_n(N)} - 1 > 1.$$

Now let $x_n(N) = 1/(1 - (NF_n + F_{n-1} - \sqrt{B_n(N)}))$, then

$$x_n(N) = 1 + \frac{1}{2\tilde{F}_{n+1}(N) - 2 + 1/x_n(N)}.$$

This implies that when $n$ even,

$$\begin{aligned}\alpha_n(N) &= \tilde{F}_{n+1}(2N) - 1 + \frac{1}{x_n(N)}\\ &= [\tilde{F}_{n+1}(2N) - 1,\ \overline{1, 2\tilde{F}_{n+1}(N) - 2}].\end{aligned}$$

In each case, the length of the period in the continued fraction expansion of $\alpha_n(N)$ is independent of $n$. We have proved

**Theorem 1.** *Let $\alpha_n(N)$ be the largest zero of the quadratic $x^2 - 2NF_nx - F_n\tilde{F}_n(2N)$ and $N > 1$ be a positive integer, then the continued fraction of $\alpha_n(N)$ is given by*

$$\alpha_n(N) = [\tilde{F}_{n+1}(2N), \overline{2\tilde{F}_{n+1}(N)}],$$

*when n is odd, and when n even the continued fraction is given by*

$$\alpha_n(N) = [\tilde{F}_{n+1}(2N) - 1, \overline{1, 2\tilde{F}_{n+1}(N) - 2}].$$

We now show the relationship between the continued fraction of $\alpha_n(N)$ and $F_n \cdot [\overline{2N, 1^{(n-1)}}]$. Let $\alpha$ have the simple continued fraction

$$\alpha = [c_0, c_1, c_2, \dots, c_{k-1}, c_k] = \frac{p_k}{q_k}, \tag{8}$$

where $p_0 = a_0 = \lfloor \alpha \rfloor$, $p_{-1} = 1$, $q_0 = 1$ and $q_{-1} = 0$. We will use the correspondence [1, Lemma 2.8]

$$\begin{pmatrix} c_0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} c_1 & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} c_{k-1} & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} c_k & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix} p_k & p_{k-1}\\ q_k & q_{k-1}\end{pmatrix}. \tag{9}$$

Taking the determinant on both sides of (9) gives the identity

$$p_kq_{k-1} - p_{k-1}q_k = (-1)^{k+1}.$$

Also, taking the transpose of the matrices on both sides of (9) we get

$$\begin{pmatrix} c_k & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} c_{k-1} & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} c_1 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} c_0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix} p_k & q_k\\ p_{k-1} & q_{k-1}\end{pmatrix},$$

from which we deduce

$$\begin{aligned} [c_k, c_{k-1}, \ldots, c_1, c_0] &= \frac{p_k}{p_{k-1}}, \\ [c_k, c_{k-1}, \ldots, c_1] &= \frac{q_k}{q_{k-1}}. \end{aligned}$$

**Lemma 1.** *For all $n \geq 2$, and $m \geq 1$ we have the continued fraction,*

$$\frac{\tilde{F}_{n+1}(m)}{\tilde{F}_n(m)} = [1^{(n-1)}, m],$$

*and for $n \geq 3$ we have the continued fraction*

$$\frac{\tilde{F}_{n+2}(m)}{\tilde{F}_n(m)} = [2, 1^{(n-2)}, m],$$

*where $1^{(n)} = 1, 1, \ldots, 1$ $n$ times.*

**Proof.** It can easily show by induction on $k$ that for all $k \geq 1$,

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{k-1} \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \tilde{F}_{k+1}(m) & k \\ \tilde{F}_k(m) & F_{k-1} \end{pmatrix},$$

where we have used (4). It follows by (8), and the correspondence (9) that $[1^{(k-1)}, m] = \frac{\tilde{F}_{k+1}(m)}{\tilde{F}_k(m)}$.

Similarly,

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{k-2} \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \tilde{F}_{k+2}(m) & F_{k+1} \\ \tilde{F}_k(m) & F_{k-1} \end{pmatrix},$$

from which we deduce that $[2, 1^{(k-2)}, m] = \frac{\tilde{F}_{k+2}(m)}{\tilde{F}_k(m)}$.

We can observe that all the partial quotients of $[1^{(n)}, m]$ are bounded, and for $j \leq n$, its sequence of convergents $\{p_j/q_j\}$ is the same as the sequence of convergents for the continued fraction of $F_{n+1}/F_n$, which converges to the golden ratio. By comparison, for any positive integer $m$, $\tilde{F}_{n+1}(m)/\tilde{F}_n(m)$ also converges to the golden ration. □

We now show another interesting property of the continued fraction of $\alpha_n(N)$.

**Theorem 2.** *For $n \geq 1$, let*

$$\lambda_n(N) := F_n \cdot [\overline{2N,\ 1^{(n-1)}}],$$

*where the $1^{(n)}$ means that 1 has been repeated $n$ times, and $F_n$ is the nth Fibonacci number. Then $\lambda_n(N)$ is an algebraic integer, and when $n$ is odd*

$$\lambda_n(N) = [\tilde{F}_{n+1}(2N), \overline{2\tilde{F}_{n+1}(N)}],$$

*while when $n$ is even*

$$\lambda_n(N) = [\tilde{F}_{n+1}(2N) - 1,\ \overline{1,\ 2\tilde{F}_{n+1}(N) - 2}].$$

One way of approaching this question is by the use of Chatelet algorithm for integer multiples of a continued fraction described by Cusick [2] in order to get the continued fraction of $\lambda_n(N)$. This approach is tedious and very involving. We will employ a rather straightforward approach by using the correspondence (9).

Let $x = [\overline{2N, 1^{(n-1)}}]$ and recall that

$$x = [2N, 1^{(n-1)}, x] = \frac{xp_n + p_{n-1}}{xq_n + q_{n-1}},$$

where $[2N, 1^{(n-1)}] = \frac{p_n}{q_n}$. By the correspondence (9),

$$\begin{pmatrix} 2N & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} = \begin{pmatrix} F_{n+1}(2N) & F_n(2N) \\ F_n & F_{n-1} \end{pmatrix},$$

from which we get

$$[2N, 1^{(n-1)}] = \frac{\tilde{F}_{n+1}(2N)}{F_n},$$

and hence

$$x = \frac{x\tilde{F}_{n+1}(2N) + \tilde{F}_n(2N)}{x\tilde{n}_n + F_{n-1}}.$$

Clearly $x$ is the largest zero of the quadratic

$$F_n x^2 - 2NF_n x - \tilde{F}_n(2N),$$

and is given by

$$x = N + \sqrt{N^2 + \tilde{F}_n(2N)/F_n}.$$

Both $x$ and $\lambda_n(N) = F_n x$ are algebraic integers. We also have that $F_n x = \alpha_n(N)$, and its continued fraction has been described in theorem 1.

## 3. Lucas-like numbers and quadratic irrationals

Let $m$ be a positive integer and define the Lucas-like sequence $\tilde{L}_n(m)$ by

$$\tilde{L}_n(m) = mL_{n-1} + L_{n-2}, \tag{10}$$

where $\tilde{L}_n(1)$ is the Lucas sequence. The numbers $\tilde{L}_n(m)$ solves the recurrence relation

$$\tilde{L}_n(m) = \tilde{L}_{n-1}(m) + \tilde{L}_{n-2}(m),$$

with initial conditions $\tilde{L}_0(m) = 3 - m$, $\tilde{L}_1(m) = -1 + 2m$ and $\tilde{L}_2(m) = 2 + m$.

For a fixed positive integer $N$, consider the quadratic

$$x^2 - 2NL_n x - L_n\tilde{L}_n(2N). \tag{11}$$

The zeros of the quadratic are given by

$$\begin{aligned} \beta_n(N) &= NL_n + \sqrt{N^2L_n^2 + L_n\tilde{L}_n(2N)}, \\ \overline{\beta}_n(N) &= NL_n - \sqrt{N^2L_n^2 + L_n\tilde{L}_n(2N)}. \end{aligned} \tag{12}$$

Just like in the previous section, we are going to examine the continued fractions of these quadratic irrationals. We will use the well known identity

$$L_{n-1}^2 - L_nL_{n-2} = (-1)^{n-1}5, \tag{13}$$

as well as identities that relate the Fibonacci sequence to the Lucas sequence

$$L_n = F_{n+1} + F_{n-1}, \tag{14}$$

and

$$L_n + 2L_{n-1} = 5F_n. \tag{15}$$

First we show that when $n$ is even, $1/(\beta_n(N) - \tilde{L}_{n+1}(2N))$ is reduced.

Let $C_n(N) = N^2L_n^2 + L_n\tilde{L}_n(2N)$,

$$1/(\beta_n(N) - \tilde{L}_{n+1}(2N)) = \frac{1}{5}\left(NL_n + L_{n-1} + \sqrt{C_n(N)}\right) > 1.$$

Where we have used (13). On the other hand,

$$\tilde{L}_{n+1}(2N)-\overline{\beta_n(N)}=NL_n+L_{n-1}+\sqrt{C_n(N)}>1.$$

Let $N=5k+3$ for an integer $k\geq 0$, and $y_n(N)=1/(\beta_n(N)-\tilde{L}_{n+1}(2N))$. Using (10) and the Euclidean algorithm we obtain

$$y_n(N)=2kL_n+2F_{n+1}+\frac{1}{NL_n+L_{n-1}+\sqrt{C_n(N)}}.$$

It can also be shown that

$$\begin{aligned}NL_n+L_{n-1}+\sqrt{C_n(N)} &= 2\tilde{L}_{n+1}(2N)+\left(\sqrt{C_n(N)}-NL_n-L_{n-1})\right)\\ &= 2\tilde{L}_{n+1}(2N)+\frac{-(L_{n-1}^2-L_nL_{n-2})}{NL_n+L_{n-1}+\sqrt{C_n(N)}}\\ &= 2\tilde{L}_{n+1}(N)+\frac{1}{y_n(N)}.\end{aligned}$$

We now have that when $n$ is even,

$$\begin{aligned}\beta_n(N) &= 5y_n(N)-L_{n-1}\\ &= \tilde{L}_{n+1}(2N)+\cfrac{1}{2kL_n+2F_{n+1}+\cfrac{1}{2\tilde{L}_{n+1}(N)+\cfrac{1}{y_n(N)}}}\\ &= \left[\tilde{L}_{n+1}(2N),\,\overline{2kL_n+2F_{n+1},\,2\tilde{L}_{n+1}(N)}\,\right].\end{aligned}$$

When $n$ is odd, and using identity (13), we get

$$1/(\beta_n(N)-\tilde{L}_{n+1}(2N)+1)=\frac{NL_n+L_{n-1}+\sqrt{C_n(N)}}{NL_n+L_{n-1}+\sqrt{C_n(N)}-5}>1,$$

for $n>1$. We also have that

$$\tilde{L}_{n+1}(2N)-1-\overline{\beta_n}=NL_n+L_{n-1}+\sqrt{C_n(N)}-1>1.$$

It can easily be verified that for all $n\geq 1$, $1<y_n(N)<2$. Let $N=5k+3$ where $k\geq 0$ is an integer, and

$$y_n(N)=1/\left(1-\left(NL_n+L_{n-1}-\sqrt{C_n(N)}\right)\right).$$

Then,

$$\begin{aligned}y_n(N) &= 1+\frac{1}{\frac{1}{5}\left(NL_n+L_{n-1}+\sqrt{C_n(N)}-5\right)}\\ &= 1+\frac{1}{2kL_n+2F_{n+1}-2+\frac{1}{5}\left(5-NL_n-L_{n-1}+\sqrt{C_n(N)}\right)}.\end{aligned}$$

It is easy to show (by induction on $n$) that

$$0<\frac{1}{5}\left(5-NL_n-L_{n-1}+\sqrt{C_n(N)}\right)<1.$$

Now,

$$\frac{1}{\frac{1}{5}\left(5-NL_n-L_{n-1}+\sqrt{C_n(N)}\right)}=1+\frac{1}{NL_n+L_{n-1}+\sqrt{C_n(N)}-1}=1+\cfrac{1}{2\tilde{L}_{n+1}(N)-2+\cfrac{1}{y_n(N)}}.$$

This implies that when $n$ is odd,

$$\beta_n(N)=[\tilde{L}_{n+1}(2N)-1,\ \overline{1,\ 2kL_n+2F_{n+1}-2,\ 1\ 2\tilde{L}_{n+1}(N)-2}].$$

We can also make the simplification

$$2kL_n+2F_{n+1}=\frac{2}{5}L_{n+1}(N),$$

in the two statements above. We have proved the following result:

**Theorem 3.** *Let $N=5k+3$ for $k\geq 0$, and $\beta_n(k)$ be the largest zero of the quadratic $x^2-2NL_nx-L_n\tilde{L}_n(2N)$, then the continued fraction of $\beta_n(k)$ is given by*

$$\beta_n(N)=[\tilde{L}_{n+1}(2N)-1,\overline{1,\frac{2}{5}L_{n+1}(N)-2,1,2\tilde{L}_{n+1}(N)-2}],$$

*when n is odd, and is given by*

$$\beta_n(N)=[\tilde{L}_{n+1}(2N),\overline{\frac{2}{5}L_{n+1}(N),2\tilde{L}_{n+1}(N)}],$$

*when n is even.*

There are some purely periodic continued fractions with periods of arbitrary length which when multiplied by $L_n$, give periodic continued fractions with period of length 2 or 4 depending on the parity of $n$. We give some of these continued fractions below.

**Theorem 4.** *Let $n\geq 2$, $k\geq 0$ and*

$$\mu_n(k):=L_n\cdot[\overline{2(5k+3),1^{(n-2)},2,1,2k,1,2,1^{(n-2)}}],$$

*where the $1^{(n)}$ means that 1 has been repeated n times, and $L_n$ is the nth Lucas number. Then $\mu_n(k)$ is an algebraic integer, and when n is odd*

$$\mu_n(k)=\left[\tilde{L}_{n+1}(10k+6)-1,\overline{1,\frac{2}{5}L_{n+1}(5k+3)-2,1,2\tilde{L}_{n+1}(5k+3)-2}\right],$$

*while when n is even it is given by*

$$\mu_n(k)=[\tilde{L}_{n+1}(10k+6),\overline{\frac{2}{5}L_{n+1}(5k+3),2\tilde{L}_{n+1}(5k+3)}].$$

**Proof.** Fix $N=5k+3$ and let

$$y=[\overline{2N,1^{(n-2)},2,1,2k,1,2,1^{(n-2)}}].$$

Then

$$y=[2N,1^{(n-2)},2,1,2k,1,2,1^{(n-2)},y]=\frac{yp_n+p_{n-1}}{yq_n+q_{n-1}},$$

where $[2N,1^{(n-2)},2,1,2k,1,2,1^{(n-2)},y]=\frac{p_n}{q_n}$.

By the correspondence (9),

$$\begin{pmatrix}2N&1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\1&0\end{pmatrix}^{n-2}\begin{pmatrix}2&1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}2k&1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}2&1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\1&0\end{pmatrix}^{(n-2)}$$

$$
\begin{aligned}
&= \begin{pmatrix} 2N & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} & F_{n-2} \\ F_{n-2} & F_{n-3} \end{pmatrix} \begin{pmatrix} 6(3k+2) & 6k+5 \\ 6k+5 & 2k+2 \end{pmatrix} \begin{pmatrix} F_{n-1} & F_{n-2} \\ F_{n-2} & F_{n-3} \end{pmatrix} \\
&= \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix},
\end{aligned}
$$

with $p_n,\ p_{n-1},\ q_n$ and $q_{n-1}$ given by

$$
\begin{aligned}
p_n &= \frac{4}{5}N^2L_n^2 + \frac{6}{5}NL_nL_{n-1} + \frac{1}{5}\left(L_n^2 - L_nL_{n-1} + L_{n-1}^2\right), \\
p_{n-1} &= \frac{2}{5}\tilde{L}_{n+1}(N)\tilde{L}_n(2N), \\
q_n &= \frac{2}{5}L_n\tilde{L}_{n+1}(N) \\
q_{n-1} &= (2k+1)L_nL_{n-1} + \frac{1}{5}\left(L_n^2 + L_{n-1}^2\right).
\end{aligned}
$$

In evaluating the matrix multiplication, we used the identities (10), (14) and (15). Clearly, $y$ is the largest zero of the quadratic

$$y^2q_n + (q_{n-1} - p_n)y - p_{n-1}.$$

Here,

$$q_{n-1} - p_n = -\frac{4}{5}NL_n\tilde{L}_{n+1}(N),$$

and so the quadratic can also be written as

$$\frac{2}{5}\tilde{L}_{n+1}(N)\left(y^2L_n - 2NL_ny - \tilde{L}_n(2N)\right).$$

The largest zero of the quadratic is given by

$$y = \frac{1}{L_n}\left(NL_n + \sqrt{N^2L_n^2 + L_n\tilde{L}_n(2N)}\right).$$

Clearly, $y$ is an algebraic integer, and so is $\mu_n(k) = L_ny$. The continued fraction of $\mu_n(k) = L_ny$ follows from theorem 3. $\square$

## 4. Generalizations

We now examine the continued fraction of $G_k(N,x)$ defined by

$$G_k(N,x) := F_k(x) \cdot [\overline{2N, x^{(k-1)}}], \tag{16}$$

where $x \geq 1$, $x^{(k)} = x, x, \ldots, x$ repeated $k$ times and $n$ is a non-zero positive integer.

Let $\tilde{G}_k(N,x) = [\overline{2N, x^{(k-1)}}]$ so that

$$\tilde{G}_k(N,x) = [2N, x^{(k-1)}, \tilde{G}_k(N,x)] = \frac{\tilde{G}_k(N,x)p_k(x) + p_{k-1}(x)}{\tilde{G}_k(N,x)q_k(x) + q_{k-1}(x)},$$

where $[2N, x^{(k-1)}] = \frac{p_k(x)}{q_k(x)}$. It can easily be shown by induction on $k$ that

$$\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}^k = \begin{pmatrix} F_{k+1}(x) & F_k(x) \\ F_k(x) & F_{k-1}(x) \end{pmatrix},$$

from which we get

$$\begin{pmatrix} 2N & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}^{k-1} = \begin{pmatrix} 2N & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_k(x) & F_{k-1}(x) \\ F_{k-1}(x) & F_{k-2}(x) \end{pmatrix}$$
$$= \begin{pmatrix} 2NF_k(x) + F_{k-1}(x) & 2NF_{k-1}(x) + F_{k-2}(x) \\ F_k(x) & F_{k-1}(x) \end{pmatrix}.$$

This implies that

$$[2N, x^{(k-1)}] = \frac{2NF_k(x) + F_{k-1}(x)}{F_k(x)},$$

and

$$\tilde{G}_k(N,x) = \frac{\tilde{G}_k(N,x)\,(2NF_k(x) + F_{k-1}(x)) + 2NF_{k-1}(x) + F_{k-2}(x)}{\tilde{G}_k(N,x)F_k(x) + F_{k-1}(x)}.$$

$\tilde{G}_k(N,x)$ is the largest zero of the quadratic

$$F_k(x)z^2 - 2NF_k(x)z - (2NF_{k-1}(x) + F_{k-2}(x)),$$

and is given by

$$\tilde{G}_k(N,x) = N + \sqrt{N^2 + (2NF_{k-1}(x) + F_{k-2}(x))/F_k(x)}.$$

We can now write

$$G_k(N,x) = NF_k(x) + \sqrt{N^2F_k^2(x) + F_k(x)(2NF_{k-1}(x) + F_{k-2}(x))}. \tag{17}$$

As we will see below, the continued fraction of $G_k(N,x)$ depends on the parity of $k$ because of the identity

$$F_{k-1}^2(x) - F_k(x)F_{k-2}(x) = (-1)^k. \tag{18}$$

To simplify the notation, let

$$\beta_k(N,x) = N^2F_k^2(x) + F_k(x)(2NF_{k-1}(x) + F_{k-2}(x))\,,$$

so that $G_k(N,x) = NF_k(x) + \sqrt{\beta_k(N,x)}$. We first examine the continued fraction of $G_k(N,x)$ for $k$ odd.

For all $k \geq 1$ and $x \geq 1$, we have by (18) that $\beta_k(N,x) = (NF_k(x) + F_{k-1}(x))^2 + 1$ so that

$$G_k(N,x) - (2NF_k(x) + F_{k-1}(x)) = \sqrt{\beta_k(N,x)} - (NF_k(x) + F_{k-1}(x)) > 0.$$

$$1/(G_k(N,x) - (2NF_k(x) + F_{k-1}(x))) = NF_k(x) + F_{k-1}(x) + \sqrt{\beta_k(N,x)} > 0.$$

Denote this by $G_k^{(1)}(N,x)$ so that

$$G_k^{(1)}(N,x) - (2NF_k(x) + 2F_{k-1}(x)) = \sqrt{\beta_k(N,x)} - (NF_k(x) + F_{k-1}(x)).$$

We now have that

$$1/(G_k^{(1)}(N,x) - (2NF_k(x) + 2F_{k-1}(x))) = G_k^{(1)}(N,x).$$

Hence the continued fraction of $G_k^{(1)}(N,x)$ is purely periodic with period of length 1 given by $\{2NF_k(x) + 2F_{k-1}(x)\}$. For all $k > 1$ odd, the continued fraction of $G_k(N,x)$ is given by,

$$G_k(N,x) = [2NF_k(x) + F_{k-1}(x),\ \overline{2NF_k(x) + 2F_{k-1}(x)}].$$

Now for the case when $k > 1$ is even, by identity (18), $\beta_k(N,x) = (NF_k(x) + F_{k-1}(x))^2 - 1$ so that

$$G_k(N,x) - (2NF_k(x) + F_{k-1}(x) - 1) = \sqrt{\beta_k(N,x)} - (NF_k(x) + F_{k-1}(x)) + 1 > 0.$$

Denote this by $G_k^{(2)}(N,x)$ so that

$$\begin{aligned}
1/G_k^{(2)}(N,x) &= 1+\frac{NF_k(x)+F_{k-1}(x)-\sqrt{\beta_k(N,x)}}{G_k^{(2)}(N,x)}\\
&= 1+\frac{F_{k-1}^2(x)-F_k(x)F_{k-2}(x)}{NF_k(x)+F_{k-1}(x)+\sqrt{\beta_k(N,x)}-(F_{k-1}^2(x)-F_k(x)F_{k-2}(x))}\\
&= 1+\frac{1}{NF_k(x)+F_{k-1}(x)+\sqrt{\beta_k N,(x)}-1}\\
&= 1+\frac{1}{2NF_k(x)+2F_{k-1}(x)-2+G_k^{(2)}(N,x)}.
\end{aligned}$$

We can now see that $1/G_k^{(2)}(N,x)$ is purely periodic with a period of length 2 given by $\{1,\ 2NF_k(x)+2F_{k-1}(x)-2\}$. For all $k>1$ even, the continued fraction of $G_k(N,x)$ is given by,

$$G_k(N,x)=[2NF_k(x)+F_{k-1}(x)-1,\ \overline{1,\ 2NF_k(x)+2F_{k-1}(x)-2}].$$

We have proved the following result:

**Theorem 5.** *For all $k\geq 1$ and $x\geq 1$, the product of the kth Fibonacci polynomial $F_k(x)$ and the periodic continued fraction $[\overline{2N,\ x^{(k-1)}}]$ gives the periodic continued fraction*

$$[2NF_k(x)+F_{k-1}(x),\overline{2NF_k(x)+2F_{k-1}(x)}]\,,$$

*when k is odd, and*

$$[2NF_k(x)+F_{k-1}(x)-1,\overline{1,2NF_k(x)+2F_{k-1}(x)-2}]\,,$$

*when k is even.*

If we set $x=1$ in the above theorem, we get the result of Theorem 2.

## 5. Polynomials from convergents of $[\overline{N,x^{(k)}}]$

In this section, we show how polynomials arising from the convergents of $[\overline{N,x^{(k)}}]$ are related to Chebyshev and Fibonacci polynomials. We describe the polynomials for $k=1,2$ and 3 and show how the roots of these polynomials are distributed.

For a fixed $k=1$, let $p_n(N,x)/q_n(N,x)$ be the convergents of $[\overline{N,x}]$. When $n\equiv(0 \mod 2)$ and $n\equiv(1 \mod 2)$, $q_{2n}(N,x)$ and $q_{2n+1}(N,x)$ are respectively generated by the rational functions

$$\frac{1-t}{1-(Nx+2)t+t^2}\quad\text{and}\quad\frac{x}{1-(Nx+2)t+t^2}. \tag{19}$$

Comparing these generating functions to the generating function of Chebyshev polynomials of the second kind below

$$\frac{1}{1-2xt+t^2}=\sum_{n=0}^{\infty}U_n(x)t^n, \tag{20}$$

we have

$$q_{2n}(N,x)=U_n(N/2x+1)-U_{n-1}(N/2x+1)\quad\text{and}\quad q_{2n+1}(N,x)=xU_n(N/2x+1).$$

To determine the roots of $q_k(x)$, first note that $U_n(x)$ has all its roots in the interval $(-1,1)$ given by $x_k=\cos\left(\frac{k}{n+1}\pi\right)$ [4, section 2.2]. It follows that $q_{2n+1}(N,x)$ has real roots in the interval $(-4/N,0]$ and are given by $x_k=\frac{2}{N}\left(\cos\left(\frac{k}{n+1}\pi\right)-1\right)$. It can also be shown that $q_{2n}(N,x)$ has $n-1$ real roots in the interval $(-4/N,0]$ and one real root outside this interval.

Now for the case when $k = 2$, let $p_n(N, x)/q_n(N, x)$ be the convergents of $[\overline{N, x, x}]$. When $n \equiv (0 \mod 3)$ and $n \equiv (1 \mod 3)$, $q_{3n}(N, x)$ and $q_{3n+1}(N, x)$ are respectively generated by the rational functions

$$\frac{x+t}{1-(Nx^2+2x+N)t-t^2} \quad \text{and} \quad \frac{x}{1-((Nx^2+2x+N))t-t^2}, \tag{21}$$

and their zeros seem to lie close to the hyperbola $y^2 - x^2 = 1 - 1/N^2$.

Let $p_n(N, x)/q_n(N, x)$ be the convergents of $[\overline{N, x, x}]$. When $n \equiv (2 \mod 3)$, $q_n(N, x)$ are generated by the rational function

$$\frac{1+x^2}{1-(Nx^2+2x+N)t-t^2},$$

and have a factor of $x^2 + 1$. By eliminating this factor and making a change of variable $x \mapsto x - 1/N$ we get (after clearing denominators) polynomials $Q_n(N, x)$ that are generated by

$$\frac{1}{1-(N^2x^2+N^2-1)t-N^2t^2} = \sum_{n=0}^{\infty} Q_n(N, x)t^n. \tag{22}$$

This shifting of the polynomial reduces the number of terms as can be seen in the generating function above. Comparing this generating function to that of the Fibonacci polynomials (1), we have

$$Q_k(N, x) = N^k F_k\left(Nx^2 + N - 1/N\right).$$

**Theorem 6.** *For any non-zero $k \in \mathbb{R}$, let $Q_n(N, x)$ be as defined in equation (22). Then for all $n \geq 1$, all the zeros $Q_n(N, x)$ lie on the hyperbola*

$$H1: \quad y^2 - x^2 = \frac{N^2-1}{N^2}.$$

**Proof.** Bicknell and Hoggatt [5] proved that if $F_n(x) = 0$ then $x = 2i\cos(j\pi/n)$ for $j = 1, 2, \ldots, n-1$. Let

$$z = \frac{\sqrt{N^2-1}}{N}(\sinh\phi + i\cosh\phi),$$

where $0 < N$. Then

$$\begin{aligned} Nz^2 + N - 1/N &= (N-1/N)(2i\sinh\phi\cosh\phi - 1) + N - 1/N \\ &= (N-1/N)i\sinh 2\phi. \end{aligned}$$

Now $Q_n(N, z) = 0$ implies that

$$\sinh 2\phi = \frac{2N}{N^2-1}\cos\theta_j,$$

where $\theta_j = j\pi/(n+1)$ for $j = 1, 2, \ldots, n-1$. Using the identity

$$\sinh^{-1} x = \log(x + \sqrt{x^2+1}) \qquad -\infty < x < \infty, \tag{23}$$

in which we consider the principal branch of the log, we get

$$\phi_j = \frac{1}{2}\log\left|2N\cos\theta_j + \sqrt{4N^2\cos^2\theta_j + (N^2-1)^2}\right| - \frac{1}{2}\log|N^2-1|. \tag{24}$$

The $2n$ zeros of $Q_n(N, z)$ are now given by

$$z_j = \pm\sqrt{(1-1/N^2)}(\sinh\phi_j + i\cosh\phi_j), \tag{25}$$

for $j = 1, 2, \ldots, n-1$ where $\phi_j$ is given by (24). It is straightforward to check that all these points lie on $H1$. □

Numerical calculations suggest that polynomials generated by the rational functions (22) have their roots close to $H1$.

Polynomials arising from the convergents of $[\overline{N, x^{(k)}}]$ seem to get more complicated as $k$ gets larger. For an example, let $k = 3$, and $p_n(N,x)/q_n(N,x)$ be the convergents of $[\overline{N,x,x,x}]$ then $q_{4n+3}(N,x)$ are generated by

$$\frac{x(x^2+2)}{1-(Nx^3+2x^2+2Nx+2)t+t^2}.$$

Now for $k = 4$, $q_{5n+4}(N,x)$ are generated by

$$\frac{x^4+3^2+1}{1-(Nx^4+2x^3+3Nx^2+4x+N)t-t^2}.$$

In general, for a fixed $k$, let $q_n(N,x)$ be the denominator of the convergents of $[\overline{N, x^{(k)}}]$ when $n \equiv -1 \mod (k+1)$. To simplify the notation, denote them by $Q_m(x)$. Then $Q_m(x)$ are generated by

$$\frac{F_{k+1}(x)}{1-g_k(N,x)t-(-1)^k t^2} = \sum_{m=0}^{\infty} Q_m(N,x)t^m, \tag{26}$$

where $g_k(N,x)$ are generated by the rational function

$$\frac{N+2t}{1-xt-t^2} = \sum_{k=0}^{\infty} g_k(N,x)t^k. \tag{27}$$

By (1), we get the explicit expression for $g_k(N,x)$ as

$$g_k(N,x) = NF_k(x) + 2F_{k-1}(x).$$

We can apply a theorem of Khang Tran [6, Theorem 1], to determine the curve on which the roots of $Q_m(N,x)$ lie. As $k$ increases however, so does the difficulty in describing this curve. As an example, for $k = 4$ the roots of $Q_m(3,x)/F_5(x)$ lie on the curve given by

$$3x^4 - 18x^2y^2 + 3y^4 + 2x^3 - 6xy^2 + 9x^2 - 9y^2 + 4x + 3 = 0.$$

The relationship to Fibonacci and Chebyshev polynomials of the second kind follows from the generating function (26). For a fixed $k$ that is even,

$$Q_n(N,x) = F_{k+1}(x)F_n(g_k(N,x)),$$

while for a fixed $k$ that is odd,

$$Q_n(N,x) = F_{k+1}(x)U_n(g_k(N,x)).$$

## Conclusion

In conclusion, we pose a question. Are there any other integer sequences $f_n$ such that $\tilde{f}_{n+1}(m) := mf_n + f_{n-1}$, and for which the continued fraction of

$$\frac{1}{f_n}\left(Nf_n + \sqrt{N^2 f_n^2 + f_n\tilde{f}_n(2N)}\right)$$

is purely periodic with length of the period depending on $n$. And the continued fraction of

$$Nf_n + \sqrt{N^2 f_n^2 + f_n\tilde{f}_n(2N)}$$

is periodic with a fixed period length independent of $n$?

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## References

[1] Borwein, J., Van Der Poorten, A., Shallit, J., & Zudilin, W. (2014). *Neverending Fractions: An Introduction to Continued Fractions* (No. 23). Cambridge University Press.

[2] Cusick, T. (1978). Integer multiples of periodic continued fractions. *Pacific Journal of Mathematics*, *78*(1), 47-60.

[3] Oyengo, M. O. (2018). *Chebyshev-Like Polynomials, Conic Distribution of Roots, and Continued Fractions*. Doctoral dissertation, University of Illinois at Urbana-Champaign.

[4] Mason, J. C., & Handscomb, D. C. (2002). *Chebyshev Polynomials*. Chapman and Hall/CRC.

[5] Hoggatt Jr, V. E., & Bicknell, M. (1973). Roots of Fibonacci polynomials. *The Fibonacci Quarterly*, *11*(3), 271-274.

[6] Tran, K. (2015). The root distribution of polynomials with a three-term recurrence. *Journal of Mathematical Analysis and Applications*, *421*(1), 878-892.