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Ideal theory of interval neutrosophic sets in subtraction algebras

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Abstract: In this paper, we introduce the notion of interval neutrosophic ideals in subtraction algebras. Also, introduce the intersection and union of interval neutrosophic sets in subtraction algebras. We prove intersection of two-interval neutrosophic ideals is also an interval neutrosophic ideal. Some exciting properties and results based on such an ideal are discussed. Moreover, we define the homomorphism and homomorphism of interval neutrosophic sets. We prove the image of an interval neutrosophic subalgebra is also an interval neutrosophic sub-algebra.

Keywords: Neutrosophic sets; Interval neutrosophic sets; Interval neutrosophic ideal; Sub-algebra.

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1. Introduction

Schein [1] considered systems of the form $(\Phi : \circ, \backslash)$, where Φ is a set of functions closed under the composition $'' \circ ''$ of functions and the set-theoretic subtraction $''\backslash''$. Zelinka [2] discussed a problem proposed by B.M. Schein concerning the structure of multiplication in a subtraction semigroup. Jun *et al.*, [3] introduced the notion of ideals in subtraction algebras.

Zadeh [4] defined the concept of fuzzy sets in 1965. Atanassov [5] introduces the intuitionistic fuzzy set as a generalization of fuzzy sets. Fuzzy sets give a degree of membership of an element in a given set, while intuitionistic fuzzy sets give both degrees of membership and non-membership. Both belong to $[0, 1]$, and their sum should not exceed 1. Smarandache [6] introduced and defined the neutrosophic set on three components. The concept of interval neutrosophic set is presented by Wang *et al.*, [7,8].

In this paper, we introduce the notion of interval neutrosophic ideals in subtraction algebras and the intersection and union of interval neutrosophic sets in subtraction algebras. We prove intersection of two-interval neutrosophic ideals is also an interval neutrosophic ideal as well as some exciting properties and results based on such an ideal are discussed. We also define the homomorphism and homomorphism of interval neutrosophic sets and prove that the image of an interval neutrosophic subalgebra is also an interval neutrosophic sub-algebra.

2. Preliminaries

By subtraction algebra we mean an algebra $(\mathfrak{B}, -, 0)$ with a single binary operation $'' - ''$ that satisfies the following conditions:

For any $i, j, l \in \mathfrak{B}$,

(P1) $i - (j - i) = i$,
(P2) $i - (i - j) = j - (j - i)$,
(P3) $(i - j) - l = (i - l) - j$.

The subtraction determines an order relation on $\mathfrak{B} : i \leq j$ if and only if $i - j = 0$, where $0 = i - i$ is an element that does not depend on the choice of $i \in \mathfrak{B}$. The ordered set $(\mathfrak{B}; \leq)$ is a semi-Boolean algebras in the sense of [9], that is, it is a meet semi-lattice with zero 0 in which every interval $[0, i]$ is a boolean algebra with respect to the induced order. Hence $i \wedge j = i - (i - j)$; the complement of an element $j \in [0, i]$ is $i - j$; and if $j, l \in [0, i]$, then

$$j \vee l = (j^{'} \wedge l^{'})^{'} = i - ((i-j) \wedge (i-l)) = i - ((i-j) - (i-j) - (i-l)).$$

In a subtraction algebra, the following are true;

(A1) $(i - j) - j = i - j$,
(A2) $i - 0 = i$ and $0 - i = 0$,
(A3) $(i - j) - i = 0$,
(A4) $i - (i - j) \leq j$.

A nonempty subset $\mathfrak{N}$ of a subtraction algebra $\mathfrak{B}$ is called a subalgebra of $\mathfrak{B}$ if $i - j \in \mathfrak{N}$ for any $i, j \in \mathfrak{N}$. A nonempty subset $\mathfrak{N}$ of a subtraction algebra $\mathfrak{B}$ is called an ideal of $\mathfrak{B}$ if

(I1) $0 \in \mathfrak{B}$,
(I2) $\forall i, j \in \mathfrak{B}$ $i - j, j \in \mathfrak{B}$, imply $i \in \mathfrak{B}$.

A mapping $\theta : X \longrightarrow Y$ of subtraction algebra is called homomorphism if $\theta(i - j) = \theta(i) - \theta(j)$ for all $i, j \in X$.

Definition 1. [8] Let λ and μ be fuzzy sets in $\mathfrak{B}$, we define the join and meet of λ and μ as follows;

$$\lambda \wedge \mu(i) = max\{\lambda(i), \mu(i)\},$$

and

$$\lambda \vee \mu(i) = min\{\lambda(i), \mu(i)\} \text{ for all } i \in \mathfrak{B}.$$

By an interval number we mean a close subinterval $\tilde{j} = [j^-, j^+]$ of $[0,1]$, where $0 \leq j^- \leq j^+ \leq 1$. The interval number $\tilde{j} = [j^-, j^+]$ with $j^- = j^+$ is denoted by j. The set of all interval numbers is denoted by $[0,1]$.

Definition 2. [7] Let $\tilde{j}_1$ and $\tilde{j}_2$ are interval numbers, then we defined the minimum and maximum of $\tilde{j}_1$ and $\tilde{j}_2$ as follows;

$$\min\{\tilde{j}_1, \tilde{j}_2\} = [\min\{j_1^-, j_2^-\}, \min\{j_1^+, j_2^+\}],$$

and

$$\max\{\tilde{j}_1, \tilde{j}_2\} = [\max\{j_1^-, j_2^-\}, \max\{j_1^+, j_2^+\}].$$

Definition 3. [7] Let $\tilde{j}_1$ and $\tilde{j}_2$ be interval numbers. We define the symbols $\leq$, $\geq$ and $=$ in case of $\tilde{j}_1$ and $\tilde{j}_2$ as follows:

$$\tilde{j}_1 \geq \tilde{j}_2 \iff j_1^- \geq j_2^-, \text{ and } j_1^+ \geq j_2^+.$$

Similarly, we define $\tilde{j}_1 \leq \tilde{j}_2$ and $\tilde{j}_1 = \tilde{j}_2$.

Definition 4. [6] Let X be a nonempty universe. A neutrosophic set $\mathfrak{N}$ of X is defined by a truth-membership function $\mathfrak{T}_{\mathfrak{N}}$, an indeterminacy $\mathfrak{I}_{\mathfrak{N}}$ and a falsity-membership function $\mathfrak{F}_{\mathfrak{N}}$. Then a neutrosophic set is defined by

$$\mathfrak{N} = \{\langle i, \mathfrak{T}_{\mathfrak{N}}(i), \mathfrak{I}_{\mathfrak{N}}(i), \mathfrak{F}_{\mathfrak{N}}(i)\rangle \,|i \in X\}$$

with the condition $0 \leq \mathfrak{T}_{\mathfrak{N}}(i) + \mathfrak{I}_{\mathfrak{N}}(i) + \mathfrak{F}_{\mathfrak{N}}(i) \leq 3$ where $\mathfrak{T}_{\mathfrak{N}}(i), \mathfrak{I}_{\mathfrak{N}}(i), \mathfrak{F}_{\mathfrak{N}}(i) \in [0,1]$.

Definition 5. [6] Let $\mathfrak{N}$ be a neutrosophic set in a subtraction algebra $\mathfrak{B}$ then the level set of $\mathfrak{N}$ is defined by

$$\mathfrak{N}_{\alpha,\beta,\gamma} = \{i, \mathfrak{T}(i) \geq \alpha, \mathfrak{I}(i) \geq \beta, \mathfrak{F}(i) \leq \gamma | i \in X\}.$$

Definition 6. [7] Let X be a nonempty universe. An interval neutrosophic set $\tilde{\mathfrak{N}}$ of X is defined by a truth-membership function $\mathfrak{T}_{\tilde{\mathfrak{N}}}$, an indeterminacy $\mathfrak{I}_{\tilde{\mathfrak{N}}}$ and a falsity-membership function $\mathfrak{F}_{\tilde{\mathfrak{N}}}$. Then a neutrosophic set is defined by

$$\tilde{\mathfrak{N}} = \{\langle i, \mathfrak{T}_{\tilde{\mathfrak{N}}}(i), \mathfrak{I}_{\tilde{\mathfrak{N}}}(i), \mathfrak{F}_{\tilde{\mathfrak{N}}}(i)\rangle \,|i \in X\}$$

with the condition $0 \leq \mathfrak{T}_{\tilde{\mathfrak{N}}}(i) + \mathfrak{I}_{\tilde{\mathfrak{N}}}(i) + \mathfrak{F}_{\tilde{\mathfrak{N}}}(i) \leq 3$ where $\mathfrak{T}_{\tilde{\mathfrak{N}}}(i), \mathfrak{I}_{\tilde{\mathfrak{N}}}(i), \mathfrak{F}_{\tilde{\mathfrak{N}}}(i) \in [0,1]$.

Definition 7. [7] Let $\tilde{\mathfrak{N}}$ be an interval neutrosophic set in a subtraction algebra $\mathfrak{B}$ then the level set of $\tilde{\mathfrak{N}}$ is defined by

$$\tilde{\mathfrak{N}}_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}} = \left\{i, \mathfrak{T}_{\tilde{\mathfrak{N}}}(i) \geq \tilde{\alpha}, \mathfrak{I}_{\tilde{\mathfrak{N}}}(i) \geq \tilde{\beta}, \mathfrak{F}_{\tilde{\mathfrak{N}}}(i) \leq \tilde{\gamma} | i \in X\right\}.$$

3. Interval Neutrosophic ideals in subtraction algebra

Definition 8. An interval neutrosophic set $\tilde{\mathfrak{N}}$ in $\mathfrak{B}$ is said to be an interval neutrosophic sub-algebra of $\mathfrak{B}$ if for all $i, j \in \mathfrak{B}$,

$$\begin{aligned} &\mathfrak{T}_{\tilde{\mathfrak{N}}}(i-j) \geq min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}}(i), \mathfrak{T}_{\tilde{\mathfrak{N}}}(j)\right\}, \\ &\mathfrak{I}_{\tilde{\mathfrak{N}}}(i-j) \geq min\left\{\mathfrak{I}_{\tilde{\mathfrak{N}}}(i), \mathfrak{I}_{\tilde{\mathfrak{N}}}(j)\right\}, \quad \text{and}, \\ &\mathfrak{F}_{\tilde{\mathfrak{N}}}(i-j) \leq max\left\{\mathfrak{F}_{\tilde{\mathfrak{N}}}(i), \mathfrak{F}_{\tilde{\mathfrak{N}}}(j)\right\}. \end{aligned}$$

Example 1. Let $\mathfrak{B} = \{0, 1, 2, 3\}$ be a subtraction algebra with the following multiplication table;

-	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	1
3	3	2	1	0

Define an interval neutrosophic set $\tilde{\mathfrak{N}}$ as follows;
For $i = 0, 1, 2, 3$

$$\mathfrak{T}_{\tilde{\mathfrak{N}}}(i) = \begin{cases} [.4, .6], \\ [.3, .6], \\ [.2, .4], \\ [.1, .2]. \end{cases}$$

$$\mathfrak{I}_{\tilde{\mathfrak{N}}}(i) = \begin{cases} [.5, .6], \\ [.4, .6], \\ [.4, .6], \\ [.5, .7]. \end{cases}$$

$$\mathfrak{F}_{\tilde{\mathfrak{N}}}(i) = \begin{cases} [.4, .5], \\ [.4, .5], \\ [.4, .7], \\ [.6, .8]. \end{cases}$$

Then $\tilde{\mathfrak{N}}$ is an interval neutrosophic sub-algebra of $\mathfrak{B}$.

Proposition 1. *Every interval neutrosophic sub-algebra of $\mathfrak{B}$ satisfies $\mathfrak{T}_{\tilde{\mathfrak{N}}}(0) \geq \mathfrak{T}_{\tilde{\mathfrak{N}}}(i)$, $\mathfrak{I}_{\tilde{\mathfrak{N}}}(0) \geq \mathfrak{I}_{\tilde{\mathfrak{N}}}(i)$ and $\mathfrak{F}_{\tilde{\mathfrak{N}}}(0) \leq \mathfrak{F}_{\tilde{\mathfrak{N}}}(i)$ for all $i \in \mathfrak{B}$.*

Theorem 1. *Let $\tilde{\mathfrak{N}}$ be an interval neutrosophic set in $\mathfrak{B}$ and let $\tilde{a}, \tilde{b}, \tilde{c} \in D[0,1]$ with $0 \leq \tilde{a} + \tilde{b} + \tilde{c} \leq 3$. Then $\tilde{\mathfrak{N}}$ is an interval neutrosophic sub-algebra of $\mathfrak{B}$ iff the level set $\tilde{\mathfrak{N}}_{\tilde{a},\tilde{b},\tilde{c}}$ are sub-algebras of $\mathfrak{B}$ when $\tilde{\mathfrak{N}}_{\tilde{a},\tilde{b},\tilde{c}} \neq \emptyset$.*

Proof. Since $\tilde{\mathfrak{N}}$ is an interval neutrosophic sub-algebra of $\mathfrak{B}$. Let $i, j \in \tilde{\mathfrak{N}}_{\tilde{a},\tilde{b},\tilde{c}}$. Then

$$\begin{aligned} &\mathfrak{T}_{\tilde{\mathfrak{N}}}(i) \geq \tilde{a}, \quad \mathfrak{T}_{\tilde{\mathfrak{N}}}(j) \geq \tilde{a}, \\ &\mathfrak{I}_{\tilde{\mathfrak{N}}}(j) \geq \tilde{b}, \quad \mathfrak{I}_{\tilde{\mathfrak{N}}}(i) \geq \tilde{b}, \\ &\mathfrak{F}_{\tilde{\mathfrak{N}}}(j) \leq \tilde{c}, \quad \mathfrak{F}_{\tilde{\mathfrak{N}}}(i) \leq \tilde{c}. \end{aligned}$$

By definition of subtraction algebra we have,

$$\mathfrak{T}_{\tilde{\mathfrak{N}}}(i-j) \geq min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}}(i), \mathfrak{T}_{\tilde{\mathfrak{N}}}(j)\right\} \geq \tilde{a},$$

$$\mathfrak{I}_{\tilde{\mathfrak{N}}}(i-j) \geq min\{\mathfrak{I}_{\tilde{\mathfrak{N}}}(i), \mathfrak{I}_{\tilde{\mathfrak{N}}}(j)\} \geq \tilde{b},$$

$$\mathfrak{F}_{\tilde{\mathfrak{N}}}(i-j) \leq max\{\mathfrak{F}_{\tilde{\mathfrak{N}}}(i), \mathfrak{F}_{\tilde{\mathfrak{N}}}(j)\} \leq \tilde{c}.$$

Thus $i-j \in \tilde{\mathfrak{N}}_{\tilde{a},\tilde{b},\tilde{c}}$. Hence $\tilde{\mathfrak{N}}_{\tilde{a},\tilde{b},\tilde{c}}$ is a sub-algebra of $\mathfrak{B}$.

Conversely let us take, $\tilde{\mathfrak{N}}_{\tilde{a},\tilde{b},\tilde{c}}$ is a sub-algebra of $\mathfrak{B}$. Assume that there exist $\tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2, \tilde{a}_3, \tilde{b}_3 \in \mathfrak{B}$ such that

$$\begin{aligned} \mathfrak{T}_{\tilde{\mathfrak{N}}}(\tilde{a}_1 - \tilde{b}_1) &< min\{\mathfrak{T}_{\tilde{\mathfrak{N}}}(\tilde{a}_1), \mathfrak{T}_{\tilde{\mathfrak{N}}}(\tilde{b}_1)\}, \\ \mathfrak{I}_{\tilde{\mathfrak{N}}}(\tilde{a}_2 - \tilde{b}_2) &< min\{\mathfrak{I}_{\tilde{\mathfrak{N}}}(\tilde{a}_2), \mathfrak{I}_{\tilde{\mathfrak{N}}}(\tilde{b}_2)\}, \\ \mathfrak{F}_{\tilde{\mathfrak{N}}}(\tilde{a}_3 - \tilde{b}_3) &> max\{\mathfrak{F}_{\tilde{\mathfrak{N}}}(\tilde{a}_3), \mathfrak{I}_{\tilde{\mathfrak{N}}}(\tilde{b}_3)\}. \end{aligned}$$

Then

$$\begin{aligned} \mathfrak{T}_{\tilde{\mathfrak{N}}}(\tilde{a}_1 - \tilde{b}_1) &< \tilde{\alpha} \leq min\{\mathfrak{T}_{\tilde{\mathfrak{N}}}(\tilde{a}_1), \mathfrak{T}_{\tilde{\mathfrak{N}}}(\tilde{b}_1)\}, \\ \mathfrak{I}_{\tilde{\mathfrak{N}}}(\tilde{a}_2 - \tilde{b}_2) &< \tilde{\beta} \leq min\{\mathfrak{I}_{\tilde{\mathfrak{N}}}(\tilde{a}_2), \mathfrak{I}_{\tilde{\mathfrak{N}}}(\tilde{b}_2)\}, \\ \mathfrak{F}_{\tilde{\mathfrak{N}}}(\tilde{a}_3 - \tilde{b}_3) &> \tilde{\gamma} \geq max\{\mathfrak{F}_{\tilde{\mathfrak{N}}}(\tilde{a}_3), \mathfrak{I}_{\tilde{\mathfrak{N}}}(\tilde{b}_3)\}. \end{aligned}$$

Hence $\tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2 \in \tilde{\mathfrak{N}}_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}$ and $\tilde{a}_3, \tilde{b}_3 \in \tilde{\mathfrak{N}}_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}$. But $\tilde{a}_1 - \tilde{b}_1, \tilde{a}_2 - \tilde{b}_2 \notin \tilde{\mathfrak{N}}_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}$ and $\tilde{a}_3 - \tilde{b}_3 \notin \tilde{\mathfrak{N}}_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}$, which is contradiction. Hence $\tilde{\mathfrak{N}}$ is an interval neutrosophic sub-algebra of $\mathfrak{B}$. □

Definition 9. An interval neutrosophic set in $\mathfrak{B}$ is called an interval neutrosophic ideal of $\mathfrak{B}$ if for all $i, j \in \mathfrak{B}$,

$$\mathfrak{T}_{\tilde{\mathfrak{N}}}(i) \geq min\{\mathfrak{T}_{\tilde{\mathfrak{N}}}(i-j), \mathfrak{T}_{\tilde{\mathfrak{N}}}(j)\},$$

$$\mathfrak{I}_{\tilde{\mathfrak{N}}}(i) \geq min\{\mathfrak{I}_{\tilde{\mathfrak{N}}}(i-j), \mathfrak{I}_{\tilde{\mathfrak{N}}}(j)\}$$

and,

$$\mathfrak{F}_{\tilde{\mathfrak{N}}}(i) \leq max\{\mathfrak{F}_{\tilde{\mathfrak{N}}}(i-j), \mathfrak{F}_{\tilde{\mathfrak{N}}}(j)\}.$$

Example 2. Let $\mathfrak{B} = \{0, a, b, c\}$ be a subtraction algebra with the following multiplication table;

-	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

Define an interval neutrosophic set $\tilde{\mathfrak{N}}$ as follows:
For $i = 0, a, b, c$

$$\mathfrak{T}_{\tilde{\mathfrak{N}}}(i) = \begin{cases} [.5, .6] \\ [.4, .7] \\ [.3, .4] \\ [.3, .4] \end{cases}$$

$$\mathfrak{I}_{\tilde{\mathfrak{N}}}(i) = \begin{cases} [.7, .8] \\ [.6, .8] \\ [.6, .8] \\ [.5, .7] \end{cases}$$

$$\mathfrak{F}_{\tilde{\mathfrak{N}}}(i) = \begin{cases} [.6, .9] \\ [.5, .7] \\ [.4, .7] \\ [.6, .8] \end{cases}$$

Then $\tilde{\mathfrak{N}}$ is an interval neutrosophic ideal of $\mathfrak{B}$.

Proposition 2. *Every interval neutrosphic ideal of* $\mathfrak{B}$ *is an interval neutrosophic sub-algebra of* $\mathfrak{B}$.

Proof. Assume that $\mathfrak{N}$ be an interval neutrosophic ideal of $\mathfrak{B}$. Then by definition

$$\begin{cases} \mathfrak{T}_{\tilde{\mathfrak{N}}}(i) \geq min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}}(i-j), \mathfrak{T}_{\tilde{\mathfrak{N}}}(j)\right\}, \\ \mathfrak{I}_{\tilde{\mathfrak{N}}}(i) \geq min\left\{\mathfrak{I}_{\tilde{\mathfrak{N}}}(i-j), \mathfrak{I}_{\tilde{\mathfrak{N}}}(j)\right\}, \\ \mathfrak{F}_{\tilde{\mathfrak{N}}}(i) \leq max\left\{\mathfrak{F}_{\tilde{\mathfrak{N}}}(i-j), \mathfrak{F}_{\tilde{\mathfrak{N}}}(j)\right\}. \end{cases} \tag{1}$$

By putting $i = i - j$ and $j = i$ in Eq. (1), we have

$$\mathfrak{T}_{\tilde{\mathfrak{N}}}(i-j) \geq min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}}((i-j)-i), \mathfrak{T}_{\tilde{\mathfrak{N}}}(i)\right\} \geq min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}}(0), \mathfrak{T}_{\tilde{\mathfrak{N}}}(i)\right\} \geq min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}}(i), \mathfrak{T}_{\tilde{\mathfrak{N}}}(j)\right\}$$

for any $i, j \in \mathfrak{B}$.

Similarly we can prove for $\mathfrak{I}_{\tilde{\mathfrak{N}}}$ and $\mathfrak{F}_{\tilde{\mathfrak{N}}}$, hence $\tilde{\mathfrak{N}}$ be an interval neutrosophic sub-algebra of $\mathfrak{B}$. □

The converse of the above proposition is not valid in general. Example 1 shows that $\mathfrak{B}$ is an interval neutrosophic subtraction algebra, but it is not an interval neutrosophic ideal of $\mathfrak{B}$.

Theorem 2. *Let* $\tilde{\mathfrak{N}}$ *be an interval neutrosophic set in* $\mathfrak{B}$ *and let* $\tilde{a}, \tilde{b}, \tilde{c} \in D[0,1]$ *with* $0 \leq \tilde{a} + \tilde{b} + \tilde{c} \leq 3$. *Then* $\tilde{\mathfrak{N}}$ *is an interval neutrosophic ideal of* $\mathfrak{B}$ *if and only if the level set* $\tilde{\mathfrak{N}}_{\tilde{a},\tilde{b},\tilde{c}}$ *are ideals of* $\mathfrak{B}$ *when* $\tilde{\mathfrak{N}}_{\tilde{a},\tilde{b},\tilde{c}} \neq \emptyset$.

Proof. Since $\tilde{\mathfrak{N}}$ is an interval neutrosophic ideal of $\mathfrak{B}$. Let $i, j \in \tilde{\mathfrak{N}}_{\tilde{a},\tilde{b},\tilde{c}}$. Then

$$\mathfrak{T}_{\tilde{\mathfrak{N}}}(i-j) \geq \tilde{a}, \quad \mathfrak{T}_{\tilde{\mathfrak{N}}}(j) \geq \tilde{a}, \quad \mathfrak{I}_{\tilde{\mathfrak{N}}}(i-j) \geq \tilde{b}, \quad \mathfrak{I}_{\tilde{\mathfrak{N}}}(j) \geq \tilde{b}, \quad \mathfrak{F}_{\tilde{\mathfrak{N}}}(i-j) \leq \tilde{c}, \quad \mathfrak{F}_{\tilde{\mathfrak{N}}}(j) \leq \tilde{c}.$$

By definition of ideal we have,

$$\begin{aligned} \mathfrak{T}_{\tilde{\mathfrak{N}}}(0) \geq \mathfrak{T}_{\tilde{\mathfrak{N}}}(i) \geq min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}}(i-j), \mathfrak{T}_{\tilde{\mathfrak{N}}}(j)\right\} \geq \tilde{a}, \\ \mathfrak{I}_{\tilde{\mathfrak{N}}}(0) \geq \mathfrak{I}_{\tilde{\mathfrak{N}}}(i) \geq min\left\{\mathfrak{I}_{\tilde{\mathfrak{N}}}(i-j), \mathfrak{I}_{\tilde{\mathfrak{N}}}(j)\right\} \geq \tilde{b}, \\ \mathfrak{F}_{\tilde{\mathfrak{N}}}(0) \leq \mathfrak{F}_{\tilde{\mathfrak{N}}}(i) \leq max\left\{\mathfrak{F}_{\tilde{\mathfrak{N}}}(i-j), \mathfrak{F}_{\tilde{\mathfrak{N}}}(j)\right\} \leq \tilde{c}. \end{aligned}$$

Thus $i - j \in \tilde{\mathfrak{N}}_{\tilde{a},\tilde{b},\tilde{c}}$. Hence $\tilde{\mathfrak{N}}_{\tilde{a},\tilde{b},\tilde{c}}$ is an ideal of $\mathfrak{B}$.

Conversely let us take, $\tilde{\mathfrak{N}}_{\tilde{a},\tilde{b},\tilde{c}}$ is an ideal of $\mathfrak{B}$. Assume that there exist $\tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2, \tilde{a}_3, \tilde{b}_3 \in \mathfrak{B}$ such that

$$\begin{aligned} \mathfrak{T}_{\tilde{\mathfrak{N}}}(\tilde{a}_1 - \tilde{b}_1) < min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}}(\tilde{a}_1), \mathfrak{T}_{\tilde{\mathfrak{N}}}(\tilde{b}_1)\right\}, \\ \mathfrak{I}_{\tilde{\mathfrak{N}}}(\tilde{a}_2 - \tilde{b}_2) < min\left\{\mathfrak{I}_{\tilde{\mathfrak{N}}}(\tilde{a}_2), \mathfrak{I}_{\tilde{\mathfrak{N}}}(\tilde{b}_2)\right\}, \\ \mathfrak{F}_{\tilde{\mathfrak{N}}}(\tilde{a}_3 - \tilde{b}_3) > max\left\{\mathfrak{F}_{\tilde{\mathfrak{N}}}(\tilde{a}_3), \mathfrak{I}_{\tilde{\mathfrak{N}}}(\tilde{b}_3)\right\}. \end{aligned}$$

Then

$$\begin{aligned} \mathfrak{T}_{\tilde{\mathfrak{N}}}(\tilde{a}_1 - \tilde{b}_1) < \tilde{\alpha} \leq min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}}(\tilde{a}_1), \mathfrak{T}_{\tilde{\mathfrak{N}}}(\tilde{b}_1)\right\}, \\ \mathfrak{I}_{\tilde{\mathfrak{N}}}(\tilde{a}_2 - \tilde{b}_2) < \tilde{\beta} \leq min\left\{\mathfrak{I}_{\tilde{\mathfrak{N}}}(\tilde{a}_2), \mathfrak{I}_{\tilde{\mathfrak{N}}}(\tilde{b}_2)\right\}, \\ \mathfrak{F}_{\tilde{\mathfrak{N}}}(\tilde{a}_3 - \tilde{b}_3) > \tilde{\gamma} \geq max\left\{\mathfrak{F}_{\tilde{\mathfrak{N}}}(\tilde{a}_3), \mathfrak{I}_{\tilde{\mathfrak{N}}}(\tilde{b}_3)\right\}. \end{aligned}$$

Hence $\tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2 \in \tilde{\mathfrak{N}}_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}$ and $\tilde{a}_3, \tilde{b}_3 \in \tilde{\mathfrak{N}}_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}$. But $\tilde{a}_1 - \tilde{b}_1, \tilde{a}_2 - \tilde{b}_2 \notin \tilde{\mathfrak{N}}_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}$ and $\tilde{a}_3 - \tilde{b}_3 \notin \tilde{\mathfrak{N}}_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}$, which is contradiction. Hence $\tilde{\mathfrak{N}}$ is an interval neutrosophic ideal of $\mathfrak{B}$. □

Proposition 3. *Every interval neutrosophic ideal of* $\mathfrak{B}$ *satisfies*

1. *For all* $i, j \in \mathfrak{B}$, $i \leq j$ *implies* $\mathfrak{T}_{\tilde{\mathfrak{N}}}(i) \geq \mathfrak{T}_{\tilde{\mathfrak{N}}}(j)$, $\mathfrak{I}_{\tilde{\mathfrak{N}}}(i) \geq \mathfrak{I}_{\tilde{\mathfrak{N}}}(j)$, $\mathfrak{F}_{\tilde{\mathfrak{N}}}(i) \leq \mathfrak{T}_{\tilde{\mathfrak{N}}}(j)$.
2. *For all* $i, j, l \in \mathfrak{B}$, $i - j \leq l$ *implies* $\mathfrak{T}_{\tilde{\mathfrak{N}}}(i) \geq min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}}(j), \mathfrak{T}_{\tilde{\mathfrak{N}}}(l)\right\}$, $\mathfrak{I}_{\tilde{\mathfrak{N}}}(i) \geq min\left\{\mathfrak{I}_{\tilde{\mathfrak{N}}}(j), \mathfrak{I}_{\tilde{\mathfrak{N}}}(l)\right\}$ *and* $\mathfrak{F}_{\tilde{\mathfrak{N}}}(i) \leq max\left\{\mathfrak{F}_{\tilde{\mathfrak{N}}}(j), \mathfrak{F}_{\tilde{\mathfrak{N}}}(l)\right\}$.

Proof. 1. Let $i, j \in \mathfrak{B}$ be such that $i \leq j$, then $i - j = 0$. By using Definition 9 and Proposition 1, we have

$$\mathfrak{T}_{\tilde{\mathfrak{N}}}(i) \geq min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}}(i-j), \mathfrak{T}_{\tilde{\mathfrak{N}}}(j)\right\} = min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}}(0), \mathfrak{T}_{\tilde{\mathfrak{N}}}(j)\right\} = \mathfrak{T}_{\tilde{\mathfrak{N}}}(j).$$

Similarly we can prove for $\mathfrak{I}_{\tilde{\mathfrak{N}}}(i) \geq \mathfrak{I}_{\tilde{\mathfrak{N}}}(j)$ and $\mathfrak{F}_{\tilde{\mathfrak{N}}}(i) \leq \mathfrak{F}_{\tilde{\mathfrak{N}}}(j)$.

2. Let $i, j, l \in \mathfrak{B}$ be such that $i - j \leq l$. By using Definition 9 and Proposition 1, we have

$$\mathfrak{T}_{\tilde{\mathfrak{N}}}(i-j) \geq min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}}((i-j)-l), \mathfrak{T}_{\tilde{\mathfrak{N}}}(l)\right\} = min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}}(0), \mathfrak{T}_{\tilde{\mathfrak{N}}}(l)\right\} = \mathfrak{T}_{\tilde{\mathfrak{N}}}(l).$$

Thus,

$$\mathfrak{T}_{\tilde{\mathfrak{N}}}(i) \geq min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}}(i-j), \mathfrak{T}_{\tilde{\mathfrak{N}}}(j)\right\} \geq min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}}(j), \mathfrak{T}_{\tilde{\mathfrak{N}}}(l)\right\}.$$

Similarly we can prove $\mathfrak{I}_{\tilde{\mathfrak{N}}}(i) \geq min\left\{\mathfrak{I}_{\tilde{\mathfrak{N}}}(j), \mathfrak{I}_{\tilde{\mathfrak{N}}}(l)\right\}$ and $\mathfrak{F}_{\tilde{\mathfrak{N}}}(i) \leq max\left\{\mathfrak{F}_{\tilde{\mathfrak{N}}}(j), \mathfrak{F}_{\tilde{\mathfrak{N}}}(l)\right\}$.
□

Corollary 3. *Every interval neutrosophic ideal of $\mathfrak{B}$ satisfies; for all $i, j_1, j_2, ...j_n \in \mathfrak{B}$ $(....(i-j_1)....) - j_n = 0$ implies $\mathfrak{T}_{\tilde{\mathfrak{N}}}(i) \geq \bigwedge_{k=1}^{n} \mathfrak{T}_{\tilde{\mathfrak{N}}}(j_k)$, $\mathfrak{I}_{\tilde{\mathfrak{N}}}(i) \geq \bigwedge_{k=1}^{n} \mathfrak{I}_{\tilde{\mathfrak{N}}}(j_k)$ and $\mathfrak{F}_{\tilde{\mathfrak{N}}}(i) \leq \bigvee_{k=1}^{n} \mathfrak{F}_{\tilde{\mathfrak{N}}}(j_k)$.*

Definition 10. Let $\tilde{\mathfrak{N}}_1$ and $\tilde{\mathfrak{N}}_2$ be any two interval neutrosophic sets in $\mathfrak{B}$. Then the union of these two sets is defined as:

$$\tilde{\mathfrak{N}}_1 \cup \tilde{\mathfrak{N}}_2 = \left\{\left(i, \mathfrak{T}_{\tilde{\mathfrak{N}}_1 \cup \tilde{\mathfrak{N}}_2}(i), \mathfrak{I}_{\tilde{\mathfrak{N}}_1 \cup \tilde{\mathfrak{N}}_2}(i), \mathfrak{F}_{\tilde{\mathfrak{N}}_1 \cup \tilde{\mathfrak{N}}_2}(i)\right) | i \in \mathfrak{B}\right\},$$

here $\mathfrak{T}_{\tilde{\mathfrak{N}}_1 \cup \tilde{\mathfrak{N}}_2}(i) = max\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}_1}(i), \mathfrak{T}_{\tilde{\mathfrak{N}}_2}(i)\right\}$, $\mathfrak{I}_{\tilde{\mathfrak{N}}_1 \cup \tilde{\mathfrak{N}}_2}(i) = max\left\{\mathfrak{I}_{\tilde{\mathfrak{N}}_1}(i), \mathfrak{I}_{\tilde{\mathfrak{N}}_2}(i)\right\}$, and $\mathfrak{F}_{\tilde{\mathfrak{N}}_1 \cup \tilde{\mathfrak{N}}_2}(i) = min\left\{\mathfrak{F}_{\tilde{\mathfrak{N}}_1}(i), \mathfrak{F}_{\tilde{\mathfrak{N}}_2}(i)\right\}$ for $i \in \mathfrak{B}$.

Definition 11. Let $\tilde{\mathfrak{N}}_1$ and $\tilde{\mathfrak{N}}_2$ be any two interval neutrosophic sets in $\mathfrak{B}$. Then the intersection of these two sets is defined as;

$$\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2 = \left\{\left(i, \mathfrak{T}_{\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2}(i), \mathfrak{I}_{\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2}(i), \mathfrak{F}_{\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2}(i)\right) | i \in \mathfrak{B}\right\},$$

here $\mathfrak{T}_{\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2}(i) = min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}_1}(i), \mathfrak{T}_{\tilde{\mathfrak{N}}_2}(i)\right\}$, $\mathfrak{I}_{\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2}(i) = min\left\{\mathfrak{I}_{\tilde{\mathfrak{N}}_1}(i), \mathfrak{I}_{\tilde{\mathfrak{N}}_2}(i)\right\}$, and $\mathfrak{F}_{\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2}(i) = max\left\{\mathfrak{F}_{\tilde{\mathfrak{N}}_1}(i), \mathfrak{F}_{\tilde{\mathfrak{N}}_2}(i)\right\}$ for $i \in \mathfrak{B}$.

Theorem 4. *The intersection of two interval neutrosophic ideals of $\mathfrak{B}$ is also an interval neutrosophic ideal of $\mathfrak{B}$.*

Proof. Let $\tilde{\mathfrak{N}}_1$ and $\tilde{\mathfrak{N}}_2$ be any two interval neutrosophic ideals in $\mathfrak{B}$. By using Proposition 3 we prove that for any $l \in \mathfrak{B}$ we have

$$\mathfrak{T}_{\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2}(0) = min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}_1}(0), \mathfrak{T}_{\tilde{\mathfrak{N}}_2}(0)\right\} \geq min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}_1}(l), \mathfrak{T}_{\tilde{\mathfrak{N}}_2}(l)\right\} = \mathfrak{T}_{\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2}(l).$$

Similarly, we can prove $\mathfrak{I}_{\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2}(0) \geq \mathfrak{I}_{\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2}(l)$ and $\mathfrak{F}_{\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2}(0) \leq \mathfrak{F}_{\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2}(l)$.

Also let $i, k \in \mathfrak{B}$ then we have,

$$\begin{aligned}
\mathfrak{T}_{\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2}(i) &= min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}_1}(i), \mathfrak{T}_{\tilde{\mathfrak{N}}_2}(i)\right\} \\
&\geq min\left\{min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}_1}(i-k), \mathfrak{T}_{\tilde{\mathfrak{N}}_1}(k)\right\}, min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}_2}(i-k), \mathfrak{T}_{\tilde{\mathfrak{N}}_2}(k)\right\}\right\} \\
&= min\left\{min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}_1}(i-k), \mathfrak{T}_{\tilde{\mathfrak{N}}_2}(i-k)\right\}, min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}_1}(k), \mathfrak{T}_{\tilde{\mathfrak{N}}_2}(k)\right\}\right\} \\
&= min\left\{\mathfrak{T}_{\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2}(i-k), \mathfrak{T}_{\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2}(k)\right\}.
\end{aligned}$$

Similarly, we can prove

$$\begin{aligned}
\mathfrak{I}_{\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2}(i) &\geq min\left\{\mathfrak{I}_{\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2}(i-k), \mathfrak{I}_{\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2}(k)\right\}, \\
\mathfrak{F}_{\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2}(i) &\leq max\left\{\mathfrak{F}_{\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2}(i-k), \mathfrak{F}_{\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2}(k)\right\}.
\end{aligned}$$

Hence $\tilde{\mathfrak{N}}_1 \cap \tilde{\mathfrak{N}}_2$ is an interval neutrosophic ideal of $\mathfrak{B}$. □

Corollary 5. *If $\{\mathfrak{N}_g | g \in \mathbb{N}\}$ is a family of interval neutrosophic ideals in $\mathfrak{B}$, then $\bigcap_{g \in \mathbb{N}} \mathfrak{N}_g$ is also an interval neutrosophic ideal.*

Definition 12. If θ is a homomorphism from $\mathfrak{H}_1$ to $\mathfrak{H}_2$. Let $\tilde{\mathfrak{N}}$ be an interval neutrosophic set in $\mathfrak{H}_2$. Then the inverse image of $\tilde{\mathfrak{N}}$ under θ is an interval neutrosophic set defined as;

$$\theta^{-1}(\tilde{\mathfrak{N}}) = \left\{ \left(l, \theta^{-1}(\mathfrak{T}_{\tilde{\mathfrak{N}}})(l), \theta^{-1}(\mathfrak{I}_{\tilde{\mathfrak{N}}})(l), \theta^{-1}(\mathfrak{F}_{\tilde{\mathfrak{N}}})(l) \right) | l \in \mathfrak{B} \right\}$$

where $\theta^{-1}(\mathfrak{T}_{\tilde{\mathfrak{N}}})(l) = \mathfrak{T}_{\tilde{\mathfrak{N}}}(\theta(l))$, $\theta^{-1}(\mathfrak{I}_{\tilde{\mathfrak{N}}})(l) = \mathfrak{I}_{\tilde{\mathfrak{N}}}(\theta(l))$ and $\theta^{-1}(\mathfrak{F}_{\tilde{\mathfrak{N}}})(l) = \mathfrak{F}_{\tilde{\mathfrak{N}}}(\theta(l))$ for all $l \in \mathfrak{B}$.

Definition 13. If θ is an onto homomorphism from $\mathfrak{H}_1$ to $\mathfrak{H}_2$. Let $\tilde{\mathfrak{N}}$ be an interval neutrosophic set in $\mathfrak{H}_2$. Then the image of $\tilde{\mathfrak{N}}$ under θ is an interval neutrosophic set defined as;

$$\theta(\tilde{\mathfrak{N}}) = \{ (l, \theta(\mathfrak{T}_{\tilde{\mathfrak{N}}})(l), \theta(\mathfrak{I}_{\tilde{\mathfrak{N}}})(l), \theta(\mathfrak{F}_{\tilde{\mathfrak{N}}})(l)) | l \in \mathfrak{B} \}$$

where $\theta(\mathfrak{T}_{\tilde{\mathfrak{N}}})(l) = \bigvee_{l \in \theta^{-1}(m)} \mathfrak{T}_{\tilde{\mathfrak{N}}}(\theta(l))$, $\quad \theta(\mathfrak{I}_{\tilde{\mathfrak{N}}})(l) = \bigvee_{l \in \theta^{-1}(m)} \mathfrak{I}_{\tilde{\mathfrak{N}}}(\theta(l))$ and $\theta(\mathfrak{F}_{\tilde{\mathfrak{N}}})(l) = \bigwedge_{l \in \theta^{-1}(m)} \mathfrak{F}_{\tilde{\mathfrak{N}}}(\theta(l))$ for all $l \in \mathfrak{B}$.

Theorem 6. *Let $\theta : \mathfrak{H}_1 \longrightarrow \mathfrak{H}_2$ be a homomorphism of $\mathfrak{B}$. If $\tilde{\mathfrak{N}}$ is an interval neutrosophic sub-algebra of $\mathfrak{H}_2$. Then the inverse image of $\tilde{\mathfrak{N}}$ under θ is an interval neutrosophic sub-algebra of $\mathfrak{B}$.*

Proof. Let $i, j \in \mathfrak{B}$, then we have

$$\theta^{-1}(\mathfrak{T}_{\tilde{\mathfrak{N}}}(i-j)) = \mathfrak{T}_{\tilde{\mathfrak{N}}}(\theta(i-j)) = \mathfrak{T}_{\tilde{\mathfrak{N}}}(\theta(i)-\theta(j)) \geq min\{\mathfrak{T}_{\tilde{\mathfrak{N}}}(\theta(i)), \mathfrak{T}_{\tilde{\mathfrak{N}}}(\theta(j))\} = min\left\{\theta^{-1}(\mathfrak{T}_{\tilde{\mathfrak{N}}}(i)), \theta^{-1}(\mathfrak{T}_{\tilde{\mathfrak{N}}}(j))\right\}.$$

Similarly, we can prove

$$\theta^{-1}(\mathfrak{I}_{\tilde{\mathfrak{N}}}(i-j)) \geq min\left\{\theta^{-1}(\mathfrak{I}_{\tilde{\mathfrak{N}}}(i)), \theta^{-1}(\mathfrak{I}_{\tilde{\mathfrak{N}}}(j))\right\} \theta^{-1}(\mathfrak{F}_{\tilde{\mathfrak{N}}}(i-j)) \leq max\left\{\theta^{-1}(\mathfrak{F}_{\tilde{\mathfrak{N}}}(i)), \theta^{-1}(\mathfrak{F}_{\tilde{\mathfrak{N}}}(j))\right\}.$$

Hence $\theta(\tilde{\mathfrak{N}})$ is an interval neutrosophic sub-algebra of $\mathfrak{B}$. □

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