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Qualitative study on Hilfer-Katugampola fractional implicit differential equations

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Abstract: This paper solves implicit differential equations involving Hilfer-Katugampola fractional derivatives with nonlocal, boundary, and impulsive conditions. In addition, some sufficient conditions are formulated for the existence and uniqueness of solutions to the given problem, and Hyers-Ulam stability results are also presented.

Keywords: Fractional derivative; Implicit differential equation; Fixed point theorem; Ulam stability.

MSC: 05C07; 05C15; 05C50.

1. Introduction

The year 1695, a communication of Leibniz and L'Hospital, was treated as the origin of fractional calculus. However, the first accurate definition of fractional derivative and ancient was commenced at the end of the nineteenth century by Liouville and Riemann. This calculus of arbitrary order first came into sight as a hypothetical development in mathematical analysis. However, in the past few decades, it has proved to be an exceptional tool in describing many processes occurring naturally. The subject of fractional calculus (integration and differentiation of fractional-order) is enjoying interest among mathematicians, physicists, and engineers. We can find several applications of fractional order differential equations in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. There has been considerable development in ordinary and partial fractional differential equations in recent years; see the monograph of Hilfer [1], Kilbas *et al.*, [2] and Podlubny [3]. Some recent existence-uniqueness results of solutions for fractional differential equations with initial as well as boundary conditions can be found in [4–7] and the references therein.

Jung found many results on the Ulam type stability of linear and nonlinear differential equations and obtained the Hyers-Ulam stability of first-order linear both ordinary and partial differential equations in the series of papers [8–10]. The standard theory of Ulam-Hyers (UH) stability has appropriate significance. If we are dealing with a UH stable system, we do not seek the exact solution. All that is involved is to find a function that satisfies the proper approximation in the equation. This approach is helpful in many applications such as numerical testing and optimization were looking for the exact solution is impossible. Many authors discussed the stability of fractional differential equations, and its significant results could be seen in the papers [7,11,12].

In past decay, differential equations with impulsive effects have been considered by many authors due to their significant applications in various fields of science and technology. Due to its large number of applications, this area has been received great importance and remarkable attention from the researchers see the monographs of Lakshmikantham *et al.*, [13] and Samoilenko *et al.*, [14] and the papers [15–17].

Recently, a new fractional derivative was introduced by Katugampola [18]. Later on, the new fractional derivative is generalized with Hilfer fractional derivative and so-called Hilfer-Katugampola fractional derivative (HKFD), involving basic properties, definitions, and results regarding existence and uniqueness results for the Cauchy type problem is discussed in [19]. This work aims to study the existence, uniqueness, and stability results for implicit differential equations (IDEs) with impulsive, nonlocal, and boundary conditions involving HKFD.

2. Preliminary

For the ease of the readers, we present some basic definitions and lemmas.

Definition 1. [18] The generalized left-sided fractional integral ${}^{\rho}\mathfrak{I}^{\alpha}\mathfrak{g}$ of order $\alpha \in C(\Re(\alpha))$ is defined by

$$\left({}^{\rho}\mathfrak{I}^{\alpha}\mathfrak{g}\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (t^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1} \mathfrak{g}(s) ds, \; t > a, \tag{1}$$

if the integral exists.

The generalized fractional derivative corresponding to the generalized fractional integral (1) is defined for $0 \le a < t$ by

$$\left({}^{\rho}\mathfrak{D}^{\alpha}\mathfrak{g}\right)(t) = \frac{\rho^{\alpha-n-1}}{\Gamma(n-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t (t^{\rho} - s^{\rho})^{n-\alpha+1} s^{\rho-1} \mathfrak{g}(s) ds, \tag{2}$$

if the integral exists.

Definition 2. [19] The HKFD, with respect to t, with $\rho > 0$, is defined by

$$\begin{aligned}\left({}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{g}\right)(t) &= \left({}^{\rho}\mathfrak{J}^{\beta(1-\alpha)} \left(t^{\rho-1}\frac{d}{dt}\right) {}^{\rho}\mathfrak{J}^{(1-\beta)(1-\alpha)}\mathfrak{g}\right)(t) \\ &= \left({}^{\rho}\mathfrak{J}^{\beta(1-\alpha)} \delta_{\rho} {}^{\rho}\mathfrak{J}^{(1-\beta)(1-\alpha)}\mathfrak{g}\right)(t).\end{aligned}$$

Theorem 1. [20]*(Krasnoselskii's fixed point theorem) Let R be a Banach space, B be a bounded closed convex subset of R and $\mathfrak{P}_1, \mathfrak{P}_2$ be mapping from B into R such that $\mathfrak{P}_1\mathfrak{h} + \mathfrak{P}_2\mathfrak{y} \in B$ for every pair $\mathfrak{h}, \mathfrak{y} \in B$. If $\mathfrak{P}_1$ is contraction and $\mathfrak{P}_2$ is completely continuous, then the equation $\mathfrak{P}_1\mathfrak{h} + \mathfrak{P}_2\mathfrak{h} = \mathfrak{h}$ has a solution on B.*

Theorem 2. [20]*(Schaefer's fixed point theorem) Let R be a Banach space and $\mathfrak{P} : R \to R$ be completely continuous operator. If the set $\{\mathfrak{h} \in R : \mathfrak{h} = \delta\mathfrak{P}\mathfrak{h} \text{ for some } \delta \in (0,1)\}$ is bounded, then $\mathfrak{P}$ has a fixed point.*

Theorem 3. [20]*(Banach fixed point theorem) Suppose Q be a non-empty closed subset of a Banach space E. Then any contraction mapping $\mathfrak{P}$ from Q into itself has a unique fixed point.*

3. Impulsive IDEs involving HKFD

Consider the IDEs with HKFD involving impulse effect is of the form

$$\begin{cases} {}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(t) = \mathfrak{g}(t, \mathfrak{h}(t), {}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(t)), \quad t \in J' := J \setminus \{t_1, t_2, ..., t_m\}, \quad J = (a, b], \\ \Delta {}^{\rho}\mathfrak{I}^{1-\gamma}\mathfrak{h}(t)|_{t=t_k} = \psi_k(\mathfrak{h}(t_k)), \quad t = t_k, \quad k = 1, 2, ..., m, \\ {}^{\rho}\mathfrak{I}^{1-\gamma}\mathfrak{h}(t)|_{t=a} = \mathfrak{h}_a, \quad \gamma = \alpha + \beta - \alpha\beta, \end{cases} \tag{3}$$

where ${}^{\rho}\mathfrak{D}^{\alpha,\beta}$ is the Hilfer-Katugampola fractional derivative of order α and of type β, and $\mathfrak{I}^{1-\gamma}$ is generalized fractional integral of order $1-\gamma$. Let $\mathfrak{g} : J \times R \times R \to R$ be a continuous function. Let us denote the space $PC(J)$ be a piecewise continuous space from J into R with the norm

$$PC(J) = \left\{\mathfrak{h} : J \to R : \mathfrak{h}(t) \in C(t_k, t_{k+1}], k = 0, ..., m; \text{ there exists } \mathfrak{h}(t_k^+) \text{ and } \mathfrak{h}(t_k^-)\right\}.$$

The weighted space $PC_{\gamma,\rho}(J)$ of functions $\mathfrak{g}$ on J is defined by

$$PC_{\gamma,\rho}(J) = \left\{\mathfrak{g} : (a, b] \to R : \left(\frac{t^{\rho} - a^{\rho}}{\rho}\right)^{\gamma} \mathfrak{g}(t) \in PC(J)\right\}, 0 \le \gamma < 1,$$

with the norm

$$\|\mathfrak{g}\|_{PC_{\gamma,\rho}} = \left\|\left(\frac{t^{\rho} - a^{\rho}}{\rho}\right)^{\gamma} \mathfrak{g}(t)\right\|_{PC[a,b]} = \max_{t \in J} \left|\left(\frac{t^{\rho} - a^{\rho}}{\rho}\right)^{\gamma} \mathfrak{g}(t)\right|.$$

Now, we shall give the definitions for UH and UHR stability for IDEs with impulsive effect under HK fractional derivative. Let ϵ be a positive number and $\varphi : J \to R^+$ be a continuous function, for every $t \in J'$ and $k = 1, 2, ..., m$, we have the following inequalities

$$\begin{cases} \left|{}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(t) - \mathfrak{g}(t, \mathfrak{v}(t), {}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(t))\right| & \leq \epsilon, \\ \left|\Delta^\rho\mathfrak{J}^{1-\gamma}\mathfrak{v}(t)|_{t=t_k} - \psi_k(\mathfrak{v}(t_k))\right| & \leq \epsilon, \end{cases} \tag{4}$$

$$\begin{cases} \left|{}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(t) - \mathfrak{g}(t, \mathfrak{v}(t), {}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(t))\right| & \leq \epsilon\varphi(t), \\ \left|\Delta^\rho\mathfrak{J}^{1-\gamma}\mathfrak{v}(t)|_{t=t_k} - \psi_k(\mathfrak{v}(t_k))\right| & \leq \epsilon\varphi(t), \end{cases} \tag{5}$$

$$\begin{cases} \left|{}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(t) - \mathfrak{g}(t, \mathfrak{v}(t), {}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(t))\right| & \leq \varphi(t), \\ \left|\Delta^\rho\mathfrak{J}^{1-\gamma}\mathfrak{v}(t)|_{t=t_k} - \psi_k(\mathfrak{v}(t_k))\right| & \leq \varphi(t), \end{cases} \tag{6}$$

Definition 3. The Eq. (3) is UH stable if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $\mathfrak{v} \in PC_{1-\gamma,\rho}(J)$ of the inequality (4) there exists a solution $\mathfrak{h} \in PC_{1-\gamma,\rho}(J)$ of Eq. (3) with

$$|\mathfrak{v}(t) - \mathfrak{h}(t)| \leq C_f\, \epsilon, \quad t \in J.$$

Definition 4. The Eq. (3) is generalized UH stable if there exist $\varphi \in PC_{1-\gamma,\rho}(J)$, $\varphi_f(0) = 0$ such that for each solution $\mathfrak{v} \in PC_{1-\gamma,\rho}(J)$ of the inequality (4) there exists a solution $\mathfrak{h} \in PC_{1-\gamma,\rho}(J)$ of Eq. (3) with

$$|\mathfrak{v}(t) - \mathfrak{h}(t)| \leq \varphi_f\, \epsilon, \quad t \in J.$$

Definition 5. The Eq. (3) is UHR stable with respect to $\varphi \in PC_{1-\gamma,\rho}(J)$ if there exists a real number $C_{f,\varphi} > 0$ such that for each $\epsilon > 0$ and for each solution $\mathfrak{v} \in PC_{1-\gamma,\rho}(J)$ of the inequality (5) there exists a solution $\mathfrak{h} \in PC_{1-\gamma,\rho}(J)$ of Eq. (3) with

$$|\mathfrak{v}(t) - \mathfrak{h}(t)| \leq C_{f,\varphi}\, \epsilon\varphi(t), \quad t \in J.$$

Definition 6. The Eq. (3) is generalized UHR stable with respect to $\varphi \in PC_{1-\gamma,\rho}(J)$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $\mathfrak{v} \in PC_{1-\gamma,\rho}(J)$ of the inequality (6) there exists a solution $\mathfrak{h} \in PC_{1-\gamma,\rho}(J)$ of Eq. (3) with

$$|\mathfrak{v}(t) - \mathfrak{h}(t)| \leq C_{f,\varphi}\varphi(t), \quad t \in J.$$

Lemma 1. *Let $\mathfrak{g} \in C_{1-\gamma;\rho}(J)$. Then the linear problem*

$$\begin{cases} {}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(t) = \mathfrak{g}(t), \\ {}^\rho\mathfrak{J}^{1-\gamma}\mathfrak{h}(t)|_{t=a} = \mathfrak{h}_a, \end{cases} \tag{7}$$

has a unique solution which is given by

$$\mathfrak{h}(t) = \frac{\mathfrak{h}_a}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\mathfrak{g}(s)ds. \tag{8}$$

Lemma 2. *Let $\mathfrak{g} : J \to R$ be continuous. A function $\mathfrak{h} \in PC_{1-\gamma}(J)$ is a solution of the fractional differential equation*

$$\begin{aligned} &{}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(t) = \mathfrak{g}(t), \quad t \in J' \\ &{}^\rho\mathfrak{J}^{1-\gamma}\mathfrak{h}(t_i) = \mathfrak{h}_{t_i}, \end{aligned}$$

if and only if $\mathfrak{h}$ is a solution of the integral equation

$$\mathfrak{h}(t) = \frac{\mathfrak{h}_{t_i}}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \frac{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}}{\Gamma(\gamma)\Gamma(1-\beta(1-\alpha))}\int_a^{t_i}\left(\frac{t_i^\rho - s^\rho}{\rho}\right)^{(1-\beta(1-\alpha))-1} s^{\rho-1}\mathfrak{g}(s)ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathfrak{g}(s)ds. \tag{9}$$

Lemma 3. *Let $\mathfrak{g}: J\times R\times R\to R$ be continuous. A function $\mathfrak{h}$ is a solution of the fractional integral equation*

$$\mathfrak{h}(t)=\frac{\mathfrak{h}_a}{\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}+\frac{\sum_{0<t_k<t}\psi_k(\mathfrak{h}(t_k))}{\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}$$
$$+\frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathfrak{g}(s,\mathfrak{h}(s),{}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(s))ds \tag{10}$$

if and only if $\mathfrak{h}$ is a solution of the Problem (3).

Lemma 4. [21] *Let $a(t)$ be a nonnegative function locally integrable on $a\le t<b$ for some $b\le\infty$, and let $g(t)$ be a nonnegative, nondecreasing continuous function defined on $a\le t<b$, such that $g(t)\le K$ for some constant K. Further let $\mathfrak{h}(t)$ be a nonnegative locally integrable on $a\le t<b$ function satisfying*

$$|\mathfrak{h}(t)|\le a(t)+g(t)\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathfrak{h}(s)ds,\ \ t\in[a,b)$$

with some $\alpha>0$. Then

$$|\mathfrak{h}(t)|\le a(t)+\int_a^t\left[\sum_{n=1}^{\infty}\frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}\left(\frac{t^\rho-s^\rho}{\rho}\right)^{n\alpha-1}\right]s^{\rho-1}a(s)ds,\ \ a\le t<b.$$

Remark 1. Under the hypothesis of Lemma 4, let $a(t)$ be a nondecreasing function on $[0,T)$. Then $\mathfrak{h}(t)\le a(t)E_\alpha(g(t)\Gamma(\alpha)t^\alpha)$, where E_α is the Mittag-Leffler function defined by

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)},\ z\in C,\ Re(\alpha)>0.$$

Lemma 5. *Let $\mathfrak{h}\in PC_{1-\gamma}(J)$ satisfies the following inequality*

$$|\mathfrak{h}(t)|\le c_1+c_2\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}|\mathfrak{h}(t)|\,ds+\sum_{0<t_k<t}\psi_k\,|\mathfrak{h}(t_k)|\,,$$

where c_1 is a nonnegative, continuous and nondecreasing function and c_2,ψ_i are constants. Then

$$|\mathfrak{h}(t)|\le c_1\left(1+\psi E_\alpha(c_2\Gamma(\alpha)t^\alpha)^kE_\alpha(c_2\Gamma(\alpha)t^\alpha\right)\ \ for\ t\in(t_k.t_{k+1}],$$

where $\psi=\sup\{\psi_k:k=1,2,3,...,m\}$.

Let us introduce the following assumptions which are useful in proving the results

(H1) Let $\mathfrak{g}:J\times R\times R\to R$ be a continuous function and there exists positive constantS $\ell,\ k>0$, such that

$$|\mathfrak{g}(t,\mathfrak{h}_1,\mathfrak{h}_2)-\mathfrak{g}(t,\mathfrak{v}_1,\mathfrak{v}_2)|\le\ell\,|\mathfrak{h}_1-\mathfrak{v}_1|+k\,|\mathfrak{h}_2-\mathfrak{v}_2|\,,\ \text{for all}\ \mathfrak{h}_1,\mathfrak{h}_2,\mathfrak{v}_1,\mathfrak{v}_2\in R.$$

(H2) There exist $l,\ m,\ n\ :J\to R^+$ with $l^*=\sup_{t\in J}l(t)<1$ such that

$$|\mathfrak{g}(t,\mathfrak{h},\mathfrak{v})|\le l(t)+m(t)\,|\mathfrak{h}|+n(t)\,|\mathfrak{v}|\,.$$

(H3) Let the functions $\psi_k:R\to R$ are continuous and there exists a constant $\ell_k^*>0$, such that

$$|\psi_k(\mathfrak{h})-\psi_k(\mathfrak{v})|\le\ell^*\,|\mathfrak{h}-\mathfrak{y}|\,,\ \text{for all}\ \mathfrak{h},\mathfrak{v}\in R,\ k=1,2,...,m.$$

(H4) Let the functions $\psi_k:R\to R$ are continuous and there exists a constant $\mathfrak{h}^*>0$, such that

$$|\psi_k(\mathfrak{h})|\le\mathfrak{h}^*(t),\ \text{for all}\ \mathfrak{h}\in R,\ k=1,2,...,m,$$

for $t \in J$ and $\mathfrak{h}, \mathfrak{v} \in R$.

(H5) There exists an increasing finctions $\varphi \in PC_{1-\gamma}(J)$ and there exists $\lambda_\varphi > 0$ such that for any $t \in J$,

$$\mathfrak{I}^{\alpha}\varphi(t) \leq \lambda_\varphi \varphi(t).$$

Theorem 4 (Existence)**.** *Assume that [H1] - [H4] are satisfied. Then, Eq.(3) has at least one solution.*

Proof. Consider the operator $\mathfrak{P} : PC_{1-\gamma}(J) \to PC_{1-\gamma}(J)$. The operator form of integral equation (7) is written as follows

$$\mathfrak{h}(t) = \mathfrak{P}\mathfrak{h}(t),$$

where

$$\mathfrak{P}\mathfrak{h}(t) = \frac{\mathfrak{h}_a}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{\sum_{0<t_k<t}\psi_k(\mathfrak{h}(t_k))}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{h}}(s)ds. \quad (11)$$

For sake of brevity, we take $g_{\mathfrak{h}}(t) := {}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(t) = \mathfrak{g}(t, \mathfrak{h}(t), g_{\mathfrak{h}}(t))$.

$$\begin{aligned} |g_{\mathfrak{h}}(t)| &= l(t) + m(t)\,|\mathfrak{h}(t)| + n(t)\,|g_{\mathfrak{h}}(t)| \\ &\leq l^* + m^*\,|\mathfrak{h}(t)| + n^*\,|g_{\mathfrak{h}}(t)| \\ &\leq \left[\frac{l^* + m^*\,|\mathfrak{h}(t)|}{1 - n^*}\right]. \end{aligned}$$

First, we prove that the operator $\mathfrak{P}$ defined by (11) verifies the conditions of Theorem 2.

Claim 1: The operator $\mathfrak{P}$ is continuous.

Let $\mathfrak{h}_n$ be a sequence such that $\mathfrak{h}_n \to \mathfrak{h}$ in $PC_{1-\gamma}[J, R]$. Then for each $t \in J$,

$$\begin{aligned} \left|(\mathfrak{P}\mathfrak{h}_n(t) - \mathfrak{P}\mathfrak{h}(t))\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\right| \leq& \frac{1}{\Gamma(\gamma)}\sum_{0<t_k<t}|\psi_k(\mathfrak{h}_n(t_k)) - \psi_k(\mathfrak{h}(t_k))| \\ &+ \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\,|g_{\mathfrak{h}_n}(s) - g_{\mathfrak{h}}(s)|\,ds. \end{aligned}$$

Since $\mathfrak{g}$ is continuous, then we have

$$\|\mathfrak{P}\mathfrak{h}_n - \mathfrak{P}\mathfrak{h}\|_{PC_{1-\gamma}} \to 0 \text{ as } n \to \infty.$$

This proves the continuity of $\mathfrak{P}$.

Claim 2: The operator $\mathfrak{P}$ maps bounded sets into bounded sets in $PC_{1-\gamma}(J)$.

Indeed, it is enough to show that for $r > 0$, there exists a positive constant $\tilde{l}$ such that $B_r = \left\{\mathfrak{h} \in PC_{1-\gamma}(J) : \|\mathfrak{h}\|_{PC_{1-\gamma}} \leq r\right\}$, we have $\|\mathfrak{P}\|_{PC_{1-\gamma}} \leq \tilde{l}$.

$$\begin{aligned} \left|(\mathfrak{P}\mathfrak{h})(t)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\right| \leq& \frac{|\mathfrak{h}_a|}{\Gamma(\gamma)} + \frac{\sum_{0<t_k<t}|\psi_k(x(t_k))|}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\,|g_{\mathfrak{h}}(s)|\,ds \\ \leq& \frac{|\mathfrak{h}_a|}{\Gamma(\gamma)} + \frac{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}\sum_{0<t_k<t}\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} h^*(t)\right|}{\Gamma(\gamma)} \\ &+ \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left[\frac{l^* + m^*\,|\mathfrak{h}(s)|}{1 - n^*}\right]ds \\ \leq& \frac{\mathfrak{h}_a}{\Gamma(\gamma)} + \frac{m\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}}{\Gamma(\gamma)}\|\mathfrak{h}^*\|_{PC_{1-\gamma}} + \frac{l^*\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}}{(1 - n^*)\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{m^*\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}}{(1-n^*)\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\left|\mathfrak{h}(s)\right|ds\\
\leq&\frac{|\mathfrak{h}_a|}{\Gamma(\gamma)}+\frac{m\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}}{\Gamma(\gamma)}\|\mathfrak{h}^*\|_{PC_{1-\gamma}}+\frac{l^*\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}}{(1-n^*)\Gamma(\alpha+1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha}\\
&+\frac{m^*\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}}{(1-n^*)\Gamma(\alpha)}B(\gamma,\alpha)\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha+\gamma-1}\|\mathfrak{h}\|_{PC_{1-\gamma}}\\
\leq&\frac{\mathfrak{h}_a}{\Gamma(\gamma)}+\frac{m\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\gamma-1}}{\Gamma(\gamma)}\|\mathfrak{h}^*\|_{PC_{1-\gamma}}+\frac{l^*}{(1-n^*)\Gamma(\alpha+1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha+\gamma-1}\\
&+\frac{m^*}{(1-n^*)\Gamma(\alpha)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}B(\gamma,\alpha)\|\mathfrak{h}\|_{PC_{1-\gamma}}\\
=&\tilde{l}.
\end{aligned}
$$

That is $\mathfrak{P}$ is bounded.

Claim 3: The operator $\mathfrak{P}$ maps bounded sets into equicontinuous set of $PC_{1-\gamma}(J)$.

Let $t_1, t_2 \in J, t_1 > t_2, B_r$ be a bounded set of $PC_{1-\gamma}(J)$ as in Claim 2, and $\mathfrak{h} \in B_r$. Then,

$$
\begin{aligned}
&\left|\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{1-\gamma}(\mathfrak{P}\mathfrak{h})(t_1)-\left(\frac{t_2^\rho-a^\rho}{\rho}\right)(\mathfrak{P}\mathfrak{h})(t_2))\right|\\
\leq&\left|\frac{\sum_{0<t_k<t_1}\psi_k(x(t_k))}{\Gamma(\gamma)}+\frac{1}{\Gamma(\alpha)}\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{1-\gamma}\int_a^{t_1}\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}g_{\mathfrak{h}}(s)ds\right|\\
&-\left|\frac{\sum_{0<t_k<t_2}\psi_k(x(t_k))}{\Gamma(\gamma)}-\frac{1}{\Gamma(\alpha)}\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{1-\gamma}\int_a^{t_2}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}g_{\mathfrak{h}}(s)ds\right|.
\end{aligned}
$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. As a consequence of Claim 1 - Claim 3 together with Arzelä-Ascoli theorem, we can conclude that $\mathfrak{P} : PC_{1-\gamma}(J) \to PC_{1-\gamma}(J)$ is continuous and completely continuous.

It is continuous and bounded from Claim 1 - Claim 3. Now, it remains to show that the set

$$\omega = \left\{\mathfrak{h} \in PC_{1-\gamma}(J) : \mathfrak{h} = \tau\mathfrak{P}(\mathfrak{h}), 0 < \tau < 1\right\}$$

is bounded set.

Let $\mathfrak{h} \in \omega$, $\mathfrak{h} = \tau\mathfrak{P}(\mathfrak{h})$ for some $0 < \tau < 1$. Thus for each $t \in J$ we have

$$\mathfrak{h}(t) = \tau\left[\frac{\mathfrak{h}_a}{\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}+\frac{\sum_{0<t_k<t}\psi_k(\mathfrak{h}(t_k))}{\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}g_{\mathfrak{h}}(s)ds\right].$$

This shows that the set ω is bounded. As a consequence of Theorem 2, we deduce that $\mathfrak{P}$ has a fixed point which is a solution of Problem (3). □

Theorem 5. *(Uniqueness) Assume that [H1] and [H3] are satisfied. If*

$$\rho = \left(\frac{m\ell^*}{\Gamma(\gamma)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma}+\frac{\ell}{(1-k)\Gamma(\alpha)}B(\gamma,\alpha)\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}\right) < 1, \tag{12}$$

then, the Eq. (3) *has a unique solution.*

Theorem 6. *The assumptions [H1], [H3], [H5] and* (12) *are satisfied. Then, Eq.*(3) *is generalized UHR stable.*

Proof. Let $\mathfrak{v}$ be solution of inequality (6) and by Theorem 5, $\mathfrak{h}$ is a unique solution of the problem

$$\begin{aligned} {}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(t) &= \mathfrak{g}(t,\mathfrak{h}(t),{}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(t)),\\ \Delta^{\rho}\mathfrak{J}^{1-\gamma}\mathfrak{h}(t)|_{t=t_k} &= \psi_k(\mathfrak{h}(t_k)),\\ {}^{\rho}\mathfrak{J}^{1-\gamma}\mathfrak{h}(t)|_{t=a} &= {}^{\rho}\mathfrak{J}^{1-\gamma}\mathfrak{v}(t)|_{t=a} = \mathfrak{h}_a. \end{aligned}$$

Then, we have

$$\mathfrak{h}(t) = \frac{\mathfrak{h}_a}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{\sum\limits_{0<t_k<t}\psi_k(\mathfrak{h}(t_k))}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{h}}(s)ds.$$

By differentiating inequality (6), for each $t \in (t_k, t_{k+1}]$, we have

$$\begin{aligned} &\left|\mathfrak{v}(t) - \frac{\mathfrak{h}_a}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \frac{\sum\limits_{0<t_k<t}\psi_k(\mathfrak{v}(t_k))}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{v}}(s)ds\right| \\ &\qquad \le \left|\frac{\sum\limits_{0<t_k<t} g_k}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\varphi(t)ds\right| \\ &\qquad \le m\varphi(t)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \lambda_\varphi\varphi(t) \\ &\qquad \le \left(m\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \lambda_\varphi\right)\varphi(t). \end{aligned}$$

Hence for each $t \in (t_k, t_{k+1}]$, it follows

$$\begin{aligned} &|\mathfrak{v}(t) - \mathfrak{h}(t)| \\ &\le \left|\mathfrak{v}(t) - \frac{\mathfrak{h}_a}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \frac{\sum\limits_{0<t_k<t}\psi_k(\mathfrak{h}(t_k))}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{h}}(s)ds\right| \\ &\le \left|\mathfrak{v}(t) - \frac{\mathfrak{h}_a}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \frac{\sum\limits_{0<t_k<t}\psi_k(\mathfrak{v}(t_k))}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{v}}(s)ds\right| \\ &\quad + \frac{\sum\limits_{0<t_k<t}|\psi_k(\mathfrak{v}(t_k)) - \psi_k(\mathfrak{h}(t_k))|}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left|g_{\mathfrak{v}}(s) - g_{\mathfrak{h}}(s)\right| ds \\ &\le \left(m\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \lambda_\varphi\right)\varphi(t) + \frac{m\ell^*}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}|\mathfrak{v}(t) - \mathfrak{h}(t)| \\ &\quad + \frac{\ell}{(1-k)\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left|\mathfrak{v}(s) - \mathfrak{h}(s)\right| ds \end{aligned}$$

By Lemma 4, there exists a constant $K > 0$ independent of $\lambda_\varphi\varphi(t)$ such that

$$|\mathfrak{v}(t) - \mathfrak{h}(t)| \le K\left(m\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \lambda_\varphi\right)\varphi(t) := C_{f,\varphi}\varphi(t).$$

Thus, Eq.(3) is generalized UHR stable. □

4. Nonlocal IDEs involving HK fractional derivative

Nonlocal IDEs with HK fractional derivative is given by

$$\begin{cases} {}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(t) = \mathfrak{g}(t,\mathfrak{h}(t),{}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(t)),\ t\in J:=(a,b],\\ {}^{\rho}\mathfrak{J}^{1-\gamma}\mathfrak{h}(t)|_{t=a} = \sum_{k=1}^{m} c_i\mathfrak{h}(\tau_i), \end{cases} \tag{13}$$

$\tau_i, i = 0,1,...,m$ are prefixed points satisfying $a < \tau_1 \le ... \le \tau_m < b$ and c_i is real numbers. We remark that nonlocal condition $\mathfrak{J}^{1-\gamma}\mathfrak{h}(0) = \sum_{i=1}^{m} c_i\mathfrak{h}(\tau_i)$ can be applied in physical problems yields better effect than the initial conditions $\mathfrak{J}^{1-\gamma}\mathfrak{h}(0) = \mathfrak{h}_0$, in [22].

Now set the space $C(J)$ be a continuous space from J into R with the norm

$$\|\mathfrak{h}\| = \sup\{|\mathfrak{h}(t)| : t\in J\}.$$

The weighted space $C_{\gamma,\rho}(J)$ of functions $\mathfrak{g}$ on J is defined by

$$C_{\gamma,\rho}(J) = \left\{\mathfrak{g} : (a,b] \to R : \left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma}\mathfrak{g}(t)\in C(J)\right\}, 0\le\gamma<1,$$

with the norm

$$\|\mathfrak{g}\|_{C_{\gamma,\rho}} = \left\|\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma}\mathfrak{g}(t)\right\|_{C[a,b]} = \max_{t\in J}\left|\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma}\mathfrak{g}(t)\right|.$$

Next, we shall give the definitions and the criteria of UH stability and UHR stability for IDEs involving HK fractional derivative. Let ϵ be a positive number and $\varphi : J \to R^{+}$ be a continuous function, for every $t\in J$ and $k = 1,2,...,m$, we have the following inequalities

$$\left|{}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(t) - \mathfrak{g}(t,\mathfrak{v}(t),{}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(t))\right| \le \epsilon. \tag{14}$$

$$\left|{}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(t) - \mathfrak{g}(t,\mathfrak{v}(t),{}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(t))\right| \le \epsilon\varphi(t). \tag{15}$$

$$\left|{}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(t) - \mathfrak{g}(t,\mathfrak{v}(t),{}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(t))\right| \le \varphi(t). \tag{16}$$

Definition 7. The Eq. (13) is UH stable if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $\mathfrak{v}\in C_{1-\gamma,\rho}(J)$ of the inequality (14) there exists a solution $\mathfrak{h}\in C_{1-\gamma,\rho}(J)$ of Eq. (13) with

$$|\mathfrak{v}(t) - \mathfrak{h}(t)| \le C_f\,\epsilon, \quad t\in J.$$

Definition 8. The Eq. (13) is generalized UH stable if there exist $\varphi\in C_{1-\gamma,\rho}(J)$, $\varphi_f(0) = 0$ such that for each solution $\mathfrak{v}\in C_{1-\gamma,\rho}(J)$ of the inequality (14) there exists a solution $\mathfrak{h}\in C_{1-\gamma,\rho}(J)$ of Eq. (13) with

$$|\mathfrak{v}(t) - \mathfrak{h}(t)| \le \varphi_f\,\epsilon, \quad t\in J.$$

Definition 9. The Eq. (13) is UHR stable with respect to $\varphi\in C_{1-\gamma,\rho}(J)$ if there exists a real number $C_{f,\varphi} > 0$ such that for each $\epsilon > 0$ and for each solution $\mathfrak{v}\in C_{1-\gamma,\rho}(J)$ of the inequality (15) there exists a solution $\mathfrak{h}\in C_{1-\gamma,\rho}(J)$ of Eq. (13) with

$$|\mathfrak{v}(t) - \mathfrak{h}(t)| \le C_{f,\varphi}\,\epsilon\varphi(t), \quad t\in J.$$

Definition 10. The Eq. (13) is generalized UHR stable with respect to $\varphi\in C_{1-\gamma,\rho}(J)$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $\mathfrak{v}\in C_{1-\gamma,\rho}(J)$ of the inequality (16) there exists a solution $\mathfrak{h}\in C_{1-\gamma,\rho}(J)$ of Eq. (13) with

$$|\mathfrak{v}(t) - \mathfrak{h}(t)| \le C_{f,\varphi}\varphi(t), \quad t\in J.$$

Lemma 6. *Let $\mathfrak{g} : J \times R \times R \to R$ be continuous. A function $\mathfrak{h}$ is a solution of the fractional integral equation*

$$\mathfrak{h}(t) = \frac{T}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} \sum_{i=1}^m c_i \int_a^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\mathfrak{g}(s,\mathfrak{h}(s), {}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(s))ds + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\mathfrak{g}(s,\mathfrak{h}(s), {}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(s))ds \tag{17}$$

if and only if $\mathfrak{h}$ is a solution of the Problem (13).

Theorem 7. *(Existence) Assume that [H1] and [H2] are satisfied. Then, Eq.(13) has at least one solution.*

Consider the operator $\tilde{\mathfrak{P}} : C_{1-\gamma,\rho}(J) \to C_{1-\gamma,\rho}(J)$, it is well defined and given by

$$\tilde{\mathfrak{P}}\mathfrak{h}(t) = \frac{T}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} \sum_{i=1}^m c_i \int_a^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{h}}(s)ds + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{h}}(s)ds. \tag{18}$$

Consider the ball $B_r = \left\{\mathfrak{h} \in C_{1-\gamma,\rho}[a,b] : \|\mathfrak{h}\|_{C_{1-\gamma,\rho}} \le r\right\}$. Now we subdivide the operator $\tilde{\mathfrak{P}}$ into two operator $\tilde{\mathfrak{P}}_1$ and $\tilde{\mathfrak{P}}_2$ on B_r as follows

$$\tilde{\mathfrak{P}}_1\mathfrak{h}(t) = \frac{T}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} \sum_{i=1}^m c_i \int_a^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{h}}(s)ds$$

and

$$\tilde{\mathfrak{P}}_2\mathfrak{h}(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{h}}(s)ds.$$

The proof is divided into several steps.
Claim 1: $\tilde{\mathfrak{P}}_1\mathfrak{h} + \tilde{\mathfrak{P}}_2\mathfrak{v}(t) \in B_r$ for every $\mathfrak{h}, \mathfrak{v} \in B_r$.

$$\begin{aligned}
\left|\tilde{\mathfrak{P}}_1\mathfrak{h}(t)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\right| &\le \frac{|T|}{\Gamma(\alpha)} \sum_{i=1}^m c_i \int_a^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left|g_{\mathfrak{h}}(s)\right| ds \\
&\le \frac{|T|}{\Gamma(\alpha)} \sum_{i=1}^m c_i \int_a^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left[\frac{l^* + m^*\left|\mathfrak{h}(s)\right|}{1-n^*}\right] ds \\
&\le \frac{|T|}{\Gamma(\alpha)} \sum_{i=1}^m c_i \int_a^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left[\frac{l^*}{1-n^*}\right] ds + \frac{|T|}{\Gamma(\alpha)} \sum_{i=1}^m c_i \int_a^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left[\frac{m^*\left|\mathfrak{h}(s)\right|}{1-n^*}\right] ds \\
&\le \frac{|T|\, l^*}{(1-n^*)\Gamma(\alpha+1)} \sum_{i=1}^m c_i \left(\frac{\tau_i^\rho - a^\rho}{\rho}\right)^{\alpha} \frac{m^*\,|T|}{(1-n^*)\Gamma(\alpha)} B(\gamma,\alpha) \sum_{i=1}^m c_i \left(\frac{\tau_i^\rho - a^\rho}{\rho}\right)^{\alpha+\gamma-1} \|\mathfrak{h}\|_{C_{1-\gamma,\rho}}
\end{aligned}$$

This gives

$$\left\|\tilde{\mathfrak{P}}_1\mathfrak{h}\right\|_{C_{1-\gamma,\rho}} \le \frac{|T|\, l^*}{(1-n^*)\Gamma(\alpha+1)} \sum_{i=1}^m c_i \left(\frac{\tau_i^\rho - a^\rho}{\rho}\right)^{\alpha} \frac{m^*\,|T|}{(1-n^*)\Gamma(\alpha)} B(\gamma,\alpha) \sum_{i=1}^m c_i \left(\frac{\tau_i^\rho - a^\rho}{\rho}\right)^{\alpha+\gamma-1} \|\mathfrak{h}\|_{C_{1-\gamma,\rho}}. \tag{19}$$

For operator $\tilde{\mathfrak{P}}_2$

$$\begin{aligned}
\left|\mathfrak{P}_2\mathfrak{h}(t)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\right| &\le \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left|g_{\mathfrak{h}}(s)\right| ds \\
&\le \frac{l^*}{(1-n^*)\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} ds \\
&\quad + \frac{m^*}{1-n^*\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left|\mathfrak{h}(s)\right| ds
\end{aligned}$$

$$\leq \frac{l^*}{(1-n^*)\Gamma(\alpha+1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha} + \frac{m^*}{(1-n^*)\Gamma(\alpha)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma} B(\gamma,\alpha)\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha+\gamma-1} \|\mathfrak{h}\|_{C_{1-\gamma,\rho}}.$$

Thus we obtain

$$\|\tilde{\mathfrak{P}}_2\mathfrak{h}\|_{C_{1-\gamma,\rho}} \leq \frac{l^*}{(1-n^*)\Gamma(\alpha+1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha-\gamma+1} + \frac{m^*}{(1-n^*)\Gamma(\alpha)}B(\gamma,\alpha)\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}\|\mathfrak{h}\|_{C_{1-\gamma,\rho}}. \tag{20}$$

Linking (19) and (20), for every $x, y \in B_r$,

$$\|\tilde{\mathfrak{P}}_1\mathfrak{h}+\tilde{\mathfrak{P}}_2\mathfrak{v}\|_{C_{1-\gamma,\rho}} \leq \|\tilde{\mathfrak{P}}_1\mathfrak{h}\|_{C_{1-\gamma,\rho}} + \|\tilde{\mathfrak{P}}_2\mathfrak{v}\|_{C_{1-\gamma,\rho}} \leq r.$$

Claim 2: $\tilde{\mathfrak{P}}_1$ is a contraction mapping.
For any $\mathfrak{h},\mathfrak{v} \in B_r$

$$\begin{aligned}\left|\left(\tilde{\mathfrak{P}}_1\mathfrak{h}(t)-\tilde{\mathfrak{P}}_1\mathfrak{v}(t)\right)\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}\right| &\leq \frac{|T|}{\Gamma(\alpha)}\sum_{i=1}^m c_i\int_a^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left|g_{\mathfrak{h}}(s)-g_{\mathfrak{v}}(s)\right|ds\\ &\leq \frac{|T|}{\Gamma(\alpha)}\sum_{i=1}^m c_i\int_a^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left(\frac{\ell}{1-k}\right)\left|\mathfrak{h}(s)-\mathfrak{v}(s)\right|ds\\ &\leq \frac{|T|}{\Gamma(\alpha)}B(\gamma,\alpha)\sum_{i=1}^m c_i\left(\frac{\tau_i^\rho-a^\rho}{\rho}\right)^{\alpha+\gamma-1}\left(\frac{\ell}{1-k}\right)\|\mathfrak{h}-\mathfrak{v}\|_{C_{1-\gamma,\rho}}.\end{aligned}$$

This gives

$$\|\tilde{\mathfrak{P}}_1\mathfrak{h}-\tilde{\mathfrak{P}}_1\mathfrak{v}\|_{C_{1-\gamma,\rho}} \leq \frac{|T|}{\Gamma(\alpha)}B(\gamma,\alpha)\left(\frac{\ell}{1-k}\right)\sum_{i=1}^m c_i\left(\frac{\tau_i^\rho-a^\rho}{\rho}\right)^{\alpha+\gamma-1}\|\mathfrak{h}-\mathfrak{v}\|_{C_{1-\gamma,\rho}}.$$

The operator $\tilde{\mathfrak{P}}_1$ is contraction mapping due to hypothesis [H2].
Claim 3: The operator $\tilde{\mathfrak{P}}_2$ is compact and continuous.
According to Step 1, we know that

$$\|\tilde{\mathfrak{P}}_2\|_{C_{1-\gamma,\rho}} \leq \frac{l^*}{(1-n^*)\Gamma(\alpha+1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha-\gamma+1} + \frac{m^*}{(1-n^*)\Gamma(\alpha)}B(\gamma,\alpha)\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}\|\mathfrak{h}\|_{C_{1-\gamma,\rho}}.$$

So operator $\tilde{\mathfrak{P}}_2$ is uniformly bounded. Now we prove the compactness of operator $\tilde{\mathfrak{P}}_2$. For $a < t_1 < t_2 < b$, we have

$$\begin{aligned}\left|\tilde{\mathfrak{P}}_2\mathfrak{h}(t_1)-\tilde{\mathfrak{P}}_2\mathfrak{h}(t_2)\right| &\leq \left|\frac{1}{\Gamma(\alpha)}\int_a^{t_1}\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}f(s,x(s))ds - \frac{1}{\Gamma(\alpha)}\int_a^{t_2}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}g_{\mathfrak{h}}(s)ds\right|\\ &\leq \frac{\|g_{\mathfrak{h}}\|_{C_{1-\gamma,\rho}}}{\Gamma(\alpha)}B(\gamma,\alpha)\left|\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\alpha+\gamma-1}-\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\alpha+\gamma-1}\right|\end{aligned}$$

tending to zero as $t_1 \to t_2$. Thus $\tilde{\mathfrak{P}}_2$ is equicontinuous. Hence, the operator $\tilde{\mathfrak{P}}_2$ is compact on B_r by the Arzelǎ-Ascoli theorem. It follows Theorem 1 that the problem (13) has at least one solution.

Theorem 8. *(Uniqueness) Assume that [H1] and [H3] are satisfied. If*

$$\rho_1 = \frac{\ell B(\gamma,\alpha)}{(1+k)\Gamma(\alpha)}\left(\sum_{i=1}^m c_i\left(\frac{\tau_i^\rho-a^\rho}{\rho}\right)^{\alpha+\gamma-1}+\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}\right) < 1, \tag{21}$$

then, the Eq. (13) *has a unique solution.*

Theorem 9. *The assumptions [H1], [H3], [H5] and* (21) *hold. Then, Eq.*(13) *is generalized UHR stable.*

Proof. Let $\mathfrak{v}$ be solution of inequality (16) and by Theorem 5, $\mathfrak{h}$ is a unique solution of the problem

$$\begin{aligned} {}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(t) &= \mathfrak{g}(t,\mathfrak{h}(t),{}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(t)),\\ {}^{\rho}\mathfrak{J}^{1-\gamma}\mathfrak{h}(t)|_{t=a} &= \sum_{k=1}^{m} c_i\mathfrak{h}(\tau_i). \end{aligned}$$

Then, we have

$$\mathfrak{h}(t) = \mathfrak{f}_h + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{h}}(s)ds,$$

where

$$\mathfrak{f}_h = \frac{T}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}\sum_{i=1}^{m} c_i \int_a^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{h}}(s)ds.$$

On the other hand, $\mathfrak{h}(\tau_i) = \mathfrak{y}(\tau_i)$, then we get $\mathfrak{f}_h = \mathfrak{f}_y$. Thus

$$\mathfrak{h}(t) = \mathfrak{f}_v + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{h}}(s)ds.$$

By differentiating inequality (16) for each $t \in J$, we have

$$\left|\mathfrak{v}(t) - \mathfrak{f}_v - \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{v}}(s)ds\right| \le \left(T\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} m\sum_{i=1}^{m} c_i + 1\right)\lambda_\varphi\varphi(t).$$

Hence it follows

$$\begin{aligned} |\mathfrak{v}(t) - \mathfrak{h}(t)| &\le \left|\mathfrak{v}(t) - \mathfrak{f}_v - \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{h}}(s)ds\right|\\ &\le \left|\mathfrak{v}(t) - \mathfrak{f}_v - \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{y}}(s)ds\right|\\ &\quad + \left|\frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{y}}(s)ds - \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{h}}(s)ds\right|\\ &\le \left(T\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} m\sum_{i=1}^{m} c_i + 1\right)\lambda_\varphi\varphi(t) + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left|g_{\mathfrak{v}}(s) - g_{\mathfrak{h}}(s)\right| ds\\ &\le \left(T\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} m\sum_{i=1}^{m} c_i + 1\right)\lambda_\varphi\varphi(t)\\ &\quad + \left(\frac{\ell}{1-k}\right)\frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left|\mathfrak{v}(s) - \mathfrak{h}(s)\right| ds\\ &:= C_{f,\varphi}\varphi(t). \end{aligned}$$

Thus, Eq.(13) is generalized UHR stable. □

5. Boundary value problem involving HK fractional derivative

Boundary valve problem for IDEs with HK fractional derivative is given by

$$\begin{cases} {}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(t) = \mathfrak{g}(t,\mathfrak{h}(t),{}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(t)),\ t \in := (a,b]\\ {}^{\rho}\mathfrak{J}^{1-\gamma}\mathfrak{h}(t)|_{t=a} = \mathfrak{h}_a, \qquad {}^{\rho}\mathfrak{J}^{1-\gamma}\mathfrak{h}(t)|_{t=b} = \mathfrak{h}_b. \end{cases} \tag{22}$$

Lemma 7. *Let $\mathfrak{g} : J \times R \times R \to R$ be continuous. A function $\mathfrak{h}$ is a solution of the fractional integral equation*

$$\mathfrak{h}(t) = {}^{\rho}\mathfrak{J}^{\alpha}\mathfrak{g}(t, \mathfrak{h}(t), {}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(t)) + \frac{\mathfrak{h}_a}{\Gamma\gamma}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma-1} + \left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{1-2\beta}\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\left(\mathfrak{h}_b - \mathfrak{h}_a - {}^{\rho}\mathfrak{J}^{1-\beta+\alpha\beta}\mathfrak{g}(b, \mathfrak{h}(b), {}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(b))\right)\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma+2\beta-2}, \quad (23)$$

if and only if $\mathfrak{h}$ is a solution of the Problem (22)*.*

Theorem 10. *Assume that [H1] and [H2] are satisfied. Then,* (22) *has at least one solution.*

Proof. Consider the operator $\breve{\mathfrak{P}} : C_{1-\gamma,\rho}(J) \to C_{1-\gamma,\rho}(J)$. The equivalent integral equation (23) which can be written in the operator form

$$\mathfrak{h}(t) = \breve{\mathfrak{P}}\mathfrak{h}(t)$$

where

$$\breve{\mathfrak{P}}\mathfrak{h}(t) = {}^{\rho}\mathfrak{J}^{\alpha}g_{\mathfrak{h}}(t) + \frac{\mathfrak{h}_a}{\Gamma\gamma}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma-1} + \left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{1-2\beta}\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\left(\mathfrak{h}_b - \mathfrak{h}_a - {}^{\rho}\mathfrak{J}^{1-\beta+\alpha\beta}g_{\mathfrak{h}}(b)\right)\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma+2\beta-2}. \quad (24)$$

We shall show that the operator $\breve{\mathfrak{P}}$ is continuous and completely continuous.
Claim 1: $\breve{\mathfrak{P}}$ is continuous.

Let $\mathfrak{h}_n$ be a sequence such that $\mathfrak{h}_n \to \mathfrak{h}$ in $C_{1-\gamma,\rho}[a,b]$. Then for each $t \in J$,

$$\begin{aligned}
&\left|(\breve{\mathfrak{P}}\mathfrak{h}_n(t) - \breve{\mathfrak{P}}\mathfrak{h}(t))\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{1-\gamma}\right| \\
&\leq {}^{\rho}\mathfrak{J}^{\alpha}\left|g_{\mathfrak{h}_n}(t) - g_{\mathfrak{h}}(t)\right| + \left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{1-2\beta}\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\left({}^{\rho}\mathfrak{J}^{1-\beta+\alpha\beta}\left|g_{\mathfrak{h}_n}(b) - g_{\mathfrak{h}}(b)\right|\right)\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{2\beta-1} \\
&\leq \left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{\alpha}\left(\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{B(\gamma, 1-\beta(1-\alpha))}{\Gamma(1-\beta(1-\alpha))} + \frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\right)\|g_{\mathfrak{h}_n}(\cdot) - g_{\mathfrak{h}}(\cdot)\|_{C_{1-\gamma,\rho}}.
\end{aligned}$$

Since $\mathfrak{g}$ is continuous, then we have

$$\|(\breve{\mathfrak{P}}\mathfrak{h}_n - \breve{\mathfrak{P}}\mathfrak{h})\|_{C_{1-\gamma,\rho}} \to 0 \text{ as } n \to \infty.$$

Claim 2 : $\breve{\mathfrak{P}}$ maps bounded sets into bounded sets in $C_{1-\gamma,\rho}(J)$.

Indeed, it is enough to show that for $r > 0$, there exists a positive constant l such that $B_r = \left\{\mathfrak{h} \in C_{1-\gamma,\rho}(J) : \|\mathfrak{h}\|_{C_{1-\gamma,\rho}} \leq r\right\}$,

$$\begin{aligned}
&\left|\breve{\mathfrak{P}}\mathfrak{h}(t)\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{1-\gamma}\right| \\
&\leq \left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{1-\gamma}{}^{\rho}\mathfrak{J}^{\alpha}\left|g_{\mathfrak{h}}(t)\right| + \frac{\mathfrak{h}_a}{\Gamma\gamma} \\
&\quad + \left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{1-2\beta}\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\left(\mathfrak{h}_b + \mathfrak{h}_a + {}^{\rho}\mathfrak{J}^{1-\beta+\alpha\beta}\left|g_{\mathfrak{h}}(b)\right|\right)\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{2\beta-1} \\
&\leq \left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{1-\gamma}{}^{\rho}\mathfrak{J}^{\alpha}\left[\frac{l^* + m^*\left|\mathfrak{h}(t)\right|}{1-n^*}\right] + \frac{\mathfrak{h}_a}{\Gamma\gamma} \\
&\quad + \left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{1-2\beta}\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\left(\mathfrak{h}_b + \mathfrak{h}_a + {}^{\rho}\mathfrak{J}^{1-\beta+\alpha\beta}\left[\frac{l^* + m^*\left|\mathfrak{h}(b)\right|}{1-n^*}\right]\right)\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{2\beta-1} \\
&\leq \frac{l^*}{(1-n^*)\Gamma(\alpha+1)}\left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-\gamma+1} + \frac{m^*B(\gamma,\alpha)}{(1-n^*)\Gamma(\alpha)}\left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{\alpha}\|\mathfrak{h}\|_{C_{1-\gamma,\rho}} + \frac{\mathfrak{h}_a}{\Gamma(\gamma)}
\end{aligned}$$

$$+\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}(\mathfrak{h}_b+\mathfrak{h}_a)+\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{l^*}{(1-n^*)\Gamma(2-\beta(1-\alpha))}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\beta+\alpha\beta}$$
$$+\frac{m^*\Gamma(2\beta)}{(1-n^*)\Gamma(\gamma+2\beta-1)}\frac{B(\gamma,1-\beta(1-\alpha))}{\Gamma(1-\beta(1-\alpha))}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}\|\mathfrak{h}\|_{C_{1-\gamma,\rho}}$$
$$= l.$$

Claim 3 : $\breve{\mathfrak{P}}$ maps bounded sets into equicontinuous set of $C_{1-\gamma,\rho}(J)$.

Let $t_1, t_2 \in J, t_1 > t_2, B_r$ be a bounded set of $C_{1-\gamma,\rho}(J)$ as in claim 2, and $\mathfrak{h} \in B_r$. Then,

$$\left|(\breve{\mathfrak{P}}\mathfrak{h}(t_1)-\breve{\mathfrak{P}}\mathfrak{h}(t_2))\right| \le {}^\rho\mathfrak{J}^\alpha\left|g_{\mathfrak{h}}(t_1)-g_{\mathfrak{h}}(t_2)\right|+\frac{\mathfrak{h}_a}{\Gamma\gamma}\left(\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\gamma-1}-\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\gamma-1}\right)$$
$$+\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-2\beta}\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\left(\mathfrak{h}_b+\mathfrak{h}_a+{}^\rho\mathfrak{J}^{1-\beta+\alpha\beta}\left|f(b,x(b))\right|\right)$$
$$\times\left(\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\gamma+2\beta-2}-\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\gamma+2\beta-2}\right)$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. As a consequence of claim 1 to 3, together with Arzelӓ-Ascoli theorem, we can conclude that $\breve{\mathfrak{P}} : C_{1-\gamma,\rho}(J) \to C_{1-\gamma,\rho}(J)$ is continuous and completely continuous.

Claim 4: A priori bounds.

Now it remains to show that the set

$$\omega = \left\{\mathfrak{h} \in C_{1-\gamma,\rho}(J) : \mathfrak{h} = \delta\breve{\mathfrak{P}}(\mathfrak{h}), 0 < \delta < 1\right\}$$

is bounded set. Let $\mathfrak{h} \in \omega$, $\mathfrak{h} = \delta\breve{\mathfrak{P}}(\mathfrak{h})$ for some $0 < \delta < 1$. Thus for each $t \in J$ we have

$$\mathfrak{h}(t) = \delta\Bigg[{}^\rho\mathfrak{J}^\alpha g_{\mathfrak{h}}(t)+\frac{\mathfrak{h}_a}{\Gamma\gamma}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}$$
$$+\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-2\beta}\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\left(\mathfrak{h}_b-\mathfrak{h}_a-{}^\rho\mathfrak{J}^{1-\beta+\alpha\beta}g_{\mathfrak{h}}(b)\right)\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma+2\beta-2}\Bigg].$$

This shows that the set ω is bounded. As a consequence of Theorem 2, we deduce that $\breve{\mathfrak{P}}$ has a fixed point which is a solution of Problem (22). □

Theorem 11. *Assume that hypothesis (H1) is fulfilled. If*

$$\left(\frac{\ell B(\gamma,\alpha)}{(1-k)\Gamma(\alpha)}+\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{\ell B(\gamma,1-\beta+\alpha\beta)}{(1-k)\Gamma(1-\beta(1-\alpha))}\right)\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha}<1$$

then, Eq. (22) *has unique solution.*

Theorem 12. *The assumptions [H1], [H3], [H5] and* (21) *hold. Then, Eq.*(22) *is generalized UHR stable.*

Proof. Let $\mathfrak{v}$ be solution of inequality (16) and by Theorem 11, $\mathfrak{h}$ is a unique solution of the problem

$${}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(t) = \mathfrak{g}(t,\mathfrak{h}(t),{}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{h}(t)),$$
$${}^\rho\mathfrak{J}^{1-\gamma}\mathfrak{h}(t)|_{t=a} = \mathfrak{h}_a, \qquad {}^\rho\mathfrak{J}^{1-\gamma}\mathfrak{h}(t)|_{t=b} = \mathfrak{h}_b.$$

Then, we have

$$\mathfrak{h}(t) = \mathfrak{g}_h+\frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}g_{\mathfrak{h}}(s)ds,$$

where

$$\mathfrak{g}_h = \frac{\mathfrak{h}_a}{\Gamma\gamma}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-2\beta} \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\left(\mathfrak{h}_b - \mathfrak{h}_a - {}^\rho\mathfrak{J}_{a^+}^{1-\beta+\alpha\beta} g_{\mathfrak{h}}(b)\right)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma+2\beta-2}.$$

On the other hand, ${}^\rho\mathfrak{J}^{1-\gamma}\mathfrak{h}(t)|_{t=a} = {}^\rho\mathfrak{J}^{1-\gamma}\mathfrak{v}(t)|_{t=a}, {}^\rho\mathfrak{J}^{1-\gamma}\mathfrak{h}(t)|_{t=b} = {}^\rho\mathfrak{J}^{1-\gamma}\mathfrak{v}(t)|_{t=b}$, then we get $\mathfrak{g}_h = \mathfrak{g}_v$. Thus

$$\mathfrak{h}(t) = \mathfrak{g}_v + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{h}}(s)ds.$$

By differentiating inequality (16) for each $t \in J$, we have

$$\left|\mathfrak{v}(t) - \mathfrak{g}_v - \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{v}}(s)ds\right| \leq \left(\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\gamma-1} + 1\right)\lambda_\varphi\varphi(t).$$

Hence it follows

$$\begin{aligned}
|\mathfrak{v}(t) - \mathfrak{h}(t)| &\leq \left|\mathfrak{v}(t) - \mathfrak{g}_v - \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{h}}(s)ds\right| \\
&\leq \left|\mathfrak{v}(t) - \mathfrak{g}_v - \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{v}}(s)ds\right| \\
&\quad + \left|\frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{v}}(s)ds - \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathfrak{h}}(s)ds\right| \\
&\leq \left(\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\gamma-1} + 1\right)\lambda_\varphi\varphi(t) + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left|g_{\mathfrak{v}}(s) - g_{\mathfrak{h}}(s)\right| ds \\
&\leq \left(\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\gamma-1} + 1\right)\lambda_\varphi\varphi(t) \\
&\quad + \left(\frac{\ell}{1-k}\right)\frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left|\mathfrak{v}(s) - \mathfrak{h}(s)\right| ds \\
&:= C_{f,\varphi}\varphi(t).
\end{aligned}$$

Thus, Eq.(22) is generalized UHR stable. □

6. Conclusion

Fractional implicit differential equations be used to model the many real-world problems. This paper looks at the HKFD for the proposed problem with the impulsive, nonlocal, and boundary conditions. We investigated the essential requirements for the existence, uniqueness, and stability of solutions using classical fixed point theorems.

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