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Results on the growth of solutions of complex linear differential equations with meromorphic coefficients

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Abstract: The purpose of this paper is the study of the growth of solutions of higher order linear differential equations $f^{(k)} + \left(A_{k-1,1}(z)e^{P_{k-1}(z)} + A_{k-1,2}(z)e^{Q_{k-1}(z)}\right) f^{(k-1)} + \cdots + \left(A_{0,1}(z)e^{P_0(z)} + A_{0,2}(z)e^{Q_0(z)}\right) f = 0$ and $f^{(k)} + \left(A_{k-1,1}(z)e^{P_{k-1}(z)} + A_{k-1,2}(z)e^{Q_{k-1}(z)}\right) f^{(k-1)} + \cdots + \left(A_{0,1}(z)e^{P_0(z)} + A_{0,2}(z)e^{Q_0(z)}\right) f = F(z)$, where $A_{j,i}(z)\,(\not\equiv 0)\,(j = 0, ..., k-1; i = 1, 2)$, $F(z)$ are meromorphic functions of finite order and $P_j(z), Q_j(z)\,(j = 0, 1, ..., k-1; i = 1, 2)$ are polynomials with degree $n \geq 1$. Under some others conditions, we extend the previous results due to Hamani and Belaïdi [1].

Keywords: Order of growth; Hyper-order; Exponent of convergence of zero sequence; Differential equation; Meromorphic function.

MSC: 34M10; 30D35.

1. Introduction and main results

Throughout this work, we assume that the reader knows the standard notations and the fundamental results of the Nevanlinna value distribution theory of meromorphic functions as the order and the hyper-order of growth, the convergence exponents of the zero-sequence and of distinct zeros, the hyper convergence exponents of the zero-sequence and the distinct zeros of a meromorphic function f, see [2–5].

We recall also the following definitions. The linear measure of a set $E \subset [0, +\infty)$ is defined as $m(E) = \int_0^{+\infty} \chi_E(t)\,dt$ and the logarithmic measure of a set $F \subset [1, +\infty)$ is defined by $lm(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t}dt$, where $\chi_H(t)$ is the characteristic function of a set H.

For results on the growth of solutions of the complex linear differential equation

$$f^{(k)} + A_{k-1}(z)e^{P_{k-1}(z)} f^{(k-1)} + \cdots + A_1(z)e^{P_1(z)} f' + A_0(z)e^{P_0(z)} f = 0,$$

where $P_j(z) = a_{j,n}z^n + \cdots + a_{j,0}$ are polynomials with degree $n \geq 1$, $a_{j,q}$ $(j = 0, 1, ..., k-1; q = 0, 1, ..., n)$ are complex numbers and $A_j(z)\,(\not\equiv 0)$, $(j = 0, 1, ..., k-1)$ are entire or meromorphic functions of finite order less than n, the reader is referred to [1,6–8].

Recently, Hamani and Belaïdi [1] studied the order of transcendental meromorphic solutions of the homogeneous and the non-homogeneous linear differential equations

$$f^{(k)} + h_{k-1}(z)e^{P_{k-1}(z)} f^{(k-1)} + \cdots + h_1(z)e^{P_1(z)} f' + h_0(z)e^{P_0(z)} f = 0, \tag{1}$$

$$f^{(k)} + h_{k-1}(z)e^{P_{k-1}(z)} f^{(k-1)} + \cdots + h_1(z)e^{P_1(z)} f' + h_0(z)e^{P_0(z)} f = F, \tag{2}$$

and have proved the following results;

Theorem 1. [1] *Let $k \geq 2$ be an integer and $P_j(z) = \sum_{i=0}^{n} a_{j,i}z^i$ $(j = 0, 1, ..., k-1)$ be nonconstant polynomials with degree $n \geq 1$, where $a_{j,0}$, $a_{j,1}$, ..., $a_{j,n}$ $(j = 0, 1, ..., k-1)$ are complex numbers. Let $h_j(z)$ $(j = 0, 1, ..., k-1)$ be meromorphic functions $\rho(h_j) < n$. Suppose that there exists $s, d \in \{0, 1, ..., k-1\}$ such that $h_s\, h_d \not\equiv 0, a_{s,n} = |a_{s,n}|e^{i\theta_s}$, $a_{d,n} = |a_{d,n}|e^{i\theta_d}, \theta_s, \theta_d \in [0, 2\pi), \theta_s \neq \theta_d$ then for $j \in \{0, 1, ..., k-1\} \setminus \{s, d\}$, $a_{j,n}$ satisfies either $a_{j,n} =$*

$c_j a_{s,n}$ *or* $a_{j,n} = c_j a_{d,n}$ $(0 < c_j < 1)$. *Then every transcendental meromorphic solution* f *whose poles are of uniformly bounded multiplicities of Equation* (1) *is of infinite order and satisfies* $\rho_2(f) = n$.

Theorem 2. [1] *Let* $k \geq 2$ *be an integer,* $h_j(z), P_j(z)$ *and* $a_{n,j}$ *satisfy the hypotheses of Theorem . Let* $F(\not\equiv 0)$ *be a meromorphic function of order* $\rho(F) < n$. *Then every transcendental meromorphic solution* f *whose poles are of uniformly bounded multiplicities of Equation* (2) *is of infinite order and satisfies* $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = n$, *with at most one exceptional solution* f_0 *of finite order.*

In this paper, we continue to study the oscillation problem of solutions, we improve and extend Theorem 2 and Theorem 2 for equations of the form

$$f^{(k)} + \left(A_{k-1,1}(z)\, e^{P_{k-1}(z)} + A_{k-1,2}(z)\, e^{Q_{k-1}(z)}\right) f^{(k-1)} + \cdots + \left(A_{0,1}(z)\, e^{P_0(z)} + A_{0,2}(z)\, e^{Q_0(z)}\right) f = 0, \tag{3}$$

and

$$f^{(k)} + \left(A_{k-1,1}(z)\, e^{P_{k-1}(z)} + A_{k-1,2}(z)\, e^{Q_{k-1}(z)}\right) f^{(k-1)} + \cdots + \left(A_{0,1}(z)\, e^{P_0(z)} + A_{0,2}(z)\, e^{Q_0(z)}\right) f = F. \tag{4}$$

We obtain the following results;

Theorem 3. *Let* $k \geq 2$ *be an integer and* $P_j(z) = a_{j,n}z^n + \cdots + a_{j,0}$, $Q_j(z) = b_{j,n}z^n + \cdots + b_{j,0}$ *be polynomials with degree* $n \geq 1$, *where* $a_{j,q}, b_{j,q}$ $(j = 0, 1, ..., k-1; q = 0, 1, ..., n)$ *are complex numbers such that* $a_{j,n}b_{j,n} \neq 0$. *Let* $A_{j,i}(z)$ $(\not\equiv 0)$ $(j = 0, 1, ..., k-1; i = 1, 2)$ *be meromorphic functions such that* $\max\{\rho(A_{j,i}) : j = 0, 1, ..., k-1; i = 1, 2\} < n$. *Suppose that there exist* $s, d \in \{0, 1, ..., k-1\}$ *such that* $A_{s,1}\, A_{d,1} \not\equiv 0, A_{s,2}\, A_{d,2} \not\equiv 0, a_{s,n} = |a_{s,n}|\, e^{i\theta_s}$, $a_{d,n} = |a_{d,n}|\, e^{i\theta_d}$, $b_{s,n} = |b_{s,n}|\, e^{i\varphi}$, $\theta_s, \theta_d, \varphi \in [0, 2\pi)$, $\theta_s \neq \theta_d$, *then for* $j \in \{0, 1, ..., k-1\} \setminus \{s, d\}$, $a_{n,j}$ *and* $b_{j,n}$ *satisfies either* $a_{j,n} = c_j a_{s,n}$ *or* $a_{j,n} = c_j a_{d,n}, b_{j,n} = c_j' b_{s,n}$ $(0 < c_j < 1, 0 < c_j' < 1)$. *Then every transcendental meromorphic solution* f *whose poles are of uniformly bounded multiplicities of Equation* (3) *is of infinite order and satisfies* $\rho_2(f) = n$.

Corollary 1. *Let* $k \geq 2$ *be an integer and* $P_j(z) = a_{j,n}z^n + \cdots + a_{j,0}$, $Q_j(z) = b_{j,n}z^n + \cdots + b_{j,0}$ *be polynomials with degree* $n \geq 1$, *where* $a_{j,q}, b_{j,q}$ $(j = 0, 1, ..., k-1; q = 0, 1, ..., n)$ *are complex numbers such that* $a_{j,n}b_{j,n} \neq 0$. *Let* $A_{j,i}(z)$ $(\not\equiv 0)$ $(j = 0, 1, ..., k-1; i = 1, 2)$ *be entire functions such that* $\max\{\rho(A_{j,i}) : j = 0, 1, ..., k-1; i = 1, 2\} < n$. *Suppose that there exist* $s, d \in \{0, 1, ..., k-1\}$ *such that* $A_{s,1}\, A_{d,1} \not\equiv 0, A_{s,2}\, A_{d,2} \not\equiv 0, a_{s,n} = |a_{s,n}|\, e^{i\theta_s}$, $a_{d,n} = |a_{d,n}|\, e^{i\theta_d}$, $b_{s,n} = |b_{s,n}|\, e^{i\varphi}$, $\theta_s, \theta_d, \varphi \in [0, 2\pi)$, $\theta_s \neq \theta_d$, *then for* $j \in \{0, 1, ..., k-1\} \setminus \{s, d\}$, $a_{j,n}$ *and* $b_{j,n}$ *satisfies either* $a_{j,n} = c_j a_{s,n}$ *or* $a_{j,n} = c_j a_{d,n}, b_{j,n} = c_j' b_{s,n}$ $(0 < c_j < 1, 0 < c_j' < 1)$. *Then every transcendental solution* f *of Equation* (3) *is of infinite order and satisfies* $\rho_2(f) = n$.

Example 1. Consider the following differential equation

$$\begin{aligned} f^{(4)} &+ \left(-2ize^{iz^2} + \frac{z^2}{2}e^{-2iz^2}\right) f^{(3)} + \left(2z^2e^{-2iz^2} - iz^3e^{-iz^2}\right) f'' + \left(\left(-24iz^4 + 12iz^3\right) e^{2iz^2}\right. \\ &+ \left.\left(4iz^5 + (6-4i)\, z^3\right) e^{-iz^2}\right) f' + \left(-10e^{iz^2} + \left(4iz^5 + 8z^4 + 6z^3 - 4iz^2\right) e^{-iz^2}\right) f = 0. \end{aligned} \tag{5}$$

Set

$$\begin{cases} A_{0,1}(z) &= -10,\ A_{0,2}(z) = 4iz^5 + 8z^4 + 6z^3 - 4iz^2, \\ A_{1,1}(z) &= -24iz^4 + 12iz^3,\ A_{1,2}(z) = 4iz^5 + (6-4i)\, z^3, \\ A_{2,1}(z) &= 2z^2,\ A_{2,2}(z) = -iz^3, \\ A_{3,1}(z) &= -2iz,\ A_{2,1}(z) = \frac{z^2}{2} \end{cases}$$

and

$$\begin{cases} P_0(z) = P_3(z) = iz^2, \\ P_1(z) = 2iz^2, \\ P_2(z) = -2iz^2, \\ Q_0(z) = Q_1(z) = Q_2(z) = -iz^2, \\ Q_3(z) = -2iz^2. \end{cases}$$

We have $a_{0,2} = i$, $a_{1,2} = 2i = a_{s,2}$, $a_{2,2} = -2i = a_{d,2}$, $a_{3,2} = i$, we can see that

$$\begin{cases} a_{0,2} = i = \frac{1}{2}a_{s,2},\ c_0 = \frac{1}{2},\ 0 < c_0 < 1, \\ a_{3,2} = i = \frac{1}{2}a_{s,2},\ c_3 = \frac{1}{2},\ 0 < c_3 < 1, \\ \arg a_{s,2} \neq \arg a_{d,2}, \end{cases}$$

and $b_{0,2} = b_{1,2} = b_{2,2} = -i, b_{3,2} = -2i$, we can see that

$$\begin{cases} b_{0,2} = -i = \frac{1}{2}b_{3,2}, \\ b_{1,2} = -i = \frac{1}{2}b_{3,2}, \\ b_{2,2} = -i = \frac{1}{2}b_{3,2}, \\ c_j = \frac{1}{2},\ 0 < c_j < 1,\ j = 0,1,2 \end{cases}$$

and $\max\left\{\rho\left(A_{j,i}\right) : j = 0, ..., 3; i = 1, 2\right\} < 2$. Then, according to Corollary 1, every transcendental solution f of Equation (5) satisfies $\rho(f) = +\infty$ and $\rho_2(f) = 2$. We can see that $f(z) = e^{e^{iz^2}}$ represents a solution of Equation (5) that satisfies $\rho(f) = +\infty$ and $\rho_2(f) = 2$.

For the case of non-homogeneous equation, we have the following result;

Theorem 4. *Let $k \geq 2$ be an integer, $P_j(z)$, $Q_j(z)$, $A_{j,i}$, $a_{j,n}$, $b_{j,n}$, $(j = 0, 1, ..., k-1)$ satisfy the hypotheses of Theorem 3. Let $F (\not\equiv 0)$ be a meromorphic function of order $\rho(f) < n$. Then every transcendental meromorphic solution f whose poles are of uniformly bounded multiplicities of Equation (4) satisfies $\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$ and $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = n$, with at most one exceptional solution f_0 of finite order.*

Corollary 2. *Let $k \geq 2$ be an integer, $P_j(z)$, $Q_j(z)$, $A_{j,i}$, $a_{j,n}$, $b_{j,n}$, $(j = 0, 1, ..., k-1)$ satisfy the hypotheses of Corollary 1. Let $F (\not\equiv 0)$ be an entire function of order $\rho(f) < n$. Then every transcendental solution f of Equation (4) satisfies $\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$ and $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = n$, with at most one exceptional solution f_0 of finite order.*

Example 2. Consider the following differential equation

$$\begin{aligned} f^{(3)} &+ \left((z-1)e^{-z} + \left(z^2 + z + 2 + \frac{1}{z}\right)e^{-2z}\right)f'' + \left(\left(z^2 + z\right)e^{2z} + \left(-2z^2 - 4z - \frac{2}{z} - 1\right)e^{-z}\right)f' \\ &+ \left(\left(z^2 - z + 1\right)e^{z} - \left(z^2 + z + 2 + \frac{1}{z}\right)e^{-z}\right)f = 2z^3 + 2z^2 + z + 1. \end{aligned} \tag{6}$$

Set

$$\begin{cases} A_{0,1}(z) = z^2 - z + 1,\ A_{0,2}(z) = -z^2 - z - 2 - \frac{1}{z}, \\ A_{1,1}(z) = z^2 + z,\ A_{1,2}(z) = -2z^2 - 4z - \frac{2}{z} - 1, \\ A_{2,1}(z) = z - 1,\ A_{2,2}(z) = z^2 + z + 2 + \frac{1}{z} \end{cases}$$

and

$$\begin{cases} P_0(z) = z,\ P_1(z) = 2z, \\ P_2(z) = -z, \\ Q_0(z) = Q_1(z) = -z,\ Q_2(z) = -2z, \\ F(z) = 2z^3 + 2z^2 + z + 1. \end{cases}$$

We have $a_{0,1} = 1$, $a_{1,1} = 2 = a_{S,1}$, $a_{2,1} = -1 = a_{d,1}$, we can see that

$$\begin{cases} a_{0,1} = 1 = \frac{1}{2}a_{s,1},\ c_0 = \frac{1}{2},\ 0 < c_0 < 1, \\ \arg a_{s,1} \neq \arg a_{d,1}, \end{cases}$$

and $b_{0,1} = b_{1,1} = -1, b_{2,1} = -2$, we can see that

$$\begin{cases} b_{0,1} = -1 = \frac{1}{2}b_{2,1}, \\ b_{1,1} = -1 = \frac{1}{2}b_{2,1}, \\ c_j = \frac{1}{2}, 0 < c_j < 1, j = 0,1. \end{cases}$$

and $\max\left\{\rho\left(A_{j,i}\right) : j = 0,...,2; i = 1,2, \rho\left(F\right)\right\} < 1$. Then, according to Corollary 2, every transcendental solution f of Equation (6) satisfies $\rho(f) = +\infty$ and $\overline{\lambda}_2\left(f\right) = \lambda_2\left(f\right) = \rho_2(f) = 1$ with at most one exceptional solution f_0 of finite order. We can see that $f\left(z\right) = z + e^{e^z}$ represents a solution of Equation (6) that satisfies $\rho(f) = +\infty$ and $\overline{\lambda}_2\left(f\right) = \lambda_2\left(f\right) = \rho_2(f) = 1$.

2. Auxiliary lemmas for the proofs of the theorems

To prove our theorems, we need the following lemmas;

Lemma 1. [9] *Let $P_j\left(z\right)\left(j = 0,1,...,k\right)$ be polynomials with $\deg P_0 = n\ (n \geq 1)$ and $\deg P_j \leq n\ (j = 1,...,k)$. Let $A_j(z)\ (j = 0,1,...,k)$ be meromorphic functions with finite order and $\max\left\{\rho\left(A_j\right) : j = 0,1,...,k\right\} < n$ such that $A_0\left(z\right) \not\equiv 0$. We denote*

$$F\left(z\right) = A_k e^{P_k(z)} + A_{k-1}e^{P_{k-1}(z)} + ... + A_1 e^{P_1(z)} + A_0 e^{P_0(z)}.$$

If $\deg\left(P_0\left(z\right) - P_j\left(z\right)\right) = n$ for all $j = 1,...,k$, then F is a nontrivial meromorphic function with finite order that satisfies $\rho\left(F\right) = n$.

Lemma 2. [10] *Let f be a transcendental meromorphic function, and let $\alpha > 1$ and $\varepsilon > 0$ be given constants. Then there exist a set $E_1 \subset (1,+\infty)$ having finite logarithmic measure and a constant $B > 0$, that depends only on α and (n,m) (n,m positive integers with $n > m \geq 0$) such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have*

$$\left|\frac{f^{(n)}\left(z\right)}{f^{(m)}\left(z\right)}\right| \leq B\left(\frac{T\left(\alpha r, f\right)}{r}\left(\log^{\alpha} r\right)\log T\left(\alpha r, f\right)\right)^{n-m}.$$

Lemma 3. [11] *Let $f(z) = \frac{g(z)}{d(z)}$ be a meromorphic function with $\rho(f) = \rho \leq +\infty$, where $g(z)$ and $d(z)$ are entire functions satisfying one of the following conditions:*

(i) g being transcendental and d being polynomial,
(ii) g, d all being transcendental and $\lambda\left(d\right) = \rho\left(d\right) = \beta < \rho\left(g\right) = \rho$.

For each sufficiently large $|z| = r$, let $z_r = re^{i\theta_r}$ be a point satisfying $|g(z_r)| = M(r,g)$. Then there exist a constant δ_r (> 0), a sequence $\left\{r_m\right\}_{m\in\mathbb{N}}$, $r_m \to +\infty$ and a set E_2 of finite logarithmic measure such that the estimation

$$\left|\frac{f(z)}{f^{(n)}(z)}\right| \leq r_m^{2n}\ (n \geq 1 \text{ is an integer})$$

holds for all z satisfying $|z| = r_m \notin E_2$, $r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r,\ \theta_r + \delta_r]$.

Lemma 4. [12] *Let $P\left(z\right) = \left(\alpha + i\beta\right)z^n + \cdots$ (α, β are real numbers, $|\alpha| + |\beta| \neq 0$) be a polynomial with degree $n \geq 1$ and $A(z)$ be a meromorphic function with $\rho\left(A\right) < n$. Set $f\left(z\right) = A(z)e^{P(z)}$, $\left(z = re^{i\theta}\right)$, $\delta\left(P,\theta\right) = \alpha\cos n\theta - \beta\sin n\theta$. Then for any given $\varepsilon > 0$, there is a set $E_3 \subset [1,+\infty)$ having finite logarithmic measure such that for any $\theta \in [0,2\pi) \setminus H$ ($H = \left\{\theta \in [0,2\pi) : \delta\left(P,\theta\right) = 0\right\}$) for $|z| = r \notin [0,1] \cup E_3, r \to +\infty$, we have*

(i) if $\delta\left(P,\theta\right) > 0$, then

$$\exp\left\{\left(1-\varepsilon\right)\delta\left(P,\theta\right)r^n\right\} \leq \left|f\left(re^{i\theta}\right)\right| \leq \exp\left\{\left(1+\varepsilon\right)\delta\left(P,\theta\right)r^n\right\},$$

(ii) if $\delta\left(P,\theta\right) < 0$, then

$$\exp\left\{\left(1+\varepsilon\right)\delta\left(P,\theta\right)r^n\right\} \leq \left|f\left(re^{i\theta}\right)\right| \leq \exp\left\{\left(1-\varepsilon\right)\delta\left(P,\theta\right)r^n\right\}.$$

Lemma 5. [13] *Let $\varphi : [0, +\infty) \to \mathbb{R}$ and $\psi : [0, +\infty) \to \mathbb{R}$ be monotone nondecreasing functions such that $\varphi(r) \leq \psi(r)$ for all $r \notin (E_4 \cup [0,1])$, where E_4 is a set of finite logarithmic measure. Let $\alpha > 1$ be a given constant. Then there exists an $r_1 = r_1(\alpha) > 0$ such that $\varphi(r) \leq \psi(\alpha r)$ for all $r > r_1$.*

Lemma 6. [11,14] *Suppose that $k \geq 2$ and $A_0, A_1, ..., A_{k-1}, F$ $(F \not\equiv 0$ or $F \equiv 0)$ are meromorphic functions such that $\rho = \max\left\{\rho\left(A_j\right), \rho\left(F\right) : j = 0, 1, ...k-1\right\} < \infty$. Let f be a transcendental meromorphic solution whose poles are of uniformly bounded multiplicities of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = F. \tag{7}$$

Then $\rho_2\left(f\right) \leq \rho$.

Lemma 7. [15,16] *Let $A_j(z)(\not\equiv 0), j = 0, 1, \cdots, k-1, F(z) \not\equiv 0$ be finite order meromorphic functions.*

(i) If f is a meromorphic solution of Equation (7) with $\rho\left(f\right) = +\infty$, then f satisfies

$$\overline{\lambda}\left(f\right) = \lambda\left(f\right) = \rho\left(f\right) = +\infty.$$

(ii) If f is a meromorphic solution of Equation (7) with $\rho\left(f\right) = +\infty$ and $\rho_2\left(f\right) = \rho$, then f satisfies

$$\overline{\lambda}\left(f\right) = \lambda\left(f\right) = \rho\left(f\right) = +\infty,\ \overline{\lambda}_2\left(f\right) = \lambda_2\left(f\right) = \rho_2\left(f\right) = \rho.$$

Lemma 8. [17] *Let f be a meromorphic function of order $\rho\left(f\right) = \rho < \infty$. Then for any given $\varepsilon > 0$, there exists a set $E_5 \subset (1, +\infty)$ that has finite linear measure and finite logarithmic measure such that when $|z| = r \notin [0,1] \cup E_5$, $r \to +\infty$, we have $|f(z)| \leq \exp\left(r^{\rho+\varepsilon}\right)$.*

3. Proof of Theorem 3

First we prove that every transcendental meromorphic solution f of Equation (3) is of order $\rho\left(f\right) \geq n$. Assume that f is a transcendental meromorphic solution of Equation (3) with $\rho\left(f\right) < n$. We can rewrite Equation (3) in the form

$$\left(A_{k-1,1}\left(z\right) e^{P_{k-1}(z)} + A_{k-1,2}\left(z\right) e^{Q_{k-1}(z)}\right) f^{(k-1)} + \cdots + \left(A_{0,1}\left(z\right) e^{P_0(z)} + A_{0,2}\left(z\right) e^{Q_0(z)}\right) f = -f^{(k)}. \tag{8}$$

Since

$$\max\left\{\rho\left(A_{j,i}\right) : j = 0, 1, ..., k-1; i = 1, 2\right\} < n$$

and $\rho\left(f\right) < n$, then $A_{j,i} f^{(j)}$ $(j = 0, 1, ..., k-1; i = 1, 2)$ and $f^{(k)}$ are meromorphic functions of finite order with $\rho\left(A_{j,i} f^{(j)}\right) < n$ and $\rho\left(f^{(k)}\right) < n$. We have $A_{s,i} f^{(s)} \not\equiv 0\,(i = 1, 2)$. Indeed, if $A_{s,i} f^{(s)} \equiv 0$, it follows that $f^{(s)} \equiv 0$. Then f has to be a polynomial of degree less than s. This is a contradiction. Since $a_{s,n} \neq b_{s,n}$ and $a_{j,n} = c_j a_{s,n}, b_{j,n} = c_j' b_{s,n}, (0 < c_j < 1), (0 < c_j' < 1), (j \neq s), a_{j,n} = c_j a_{s,n}$ or $a_{j,n} = c_j a_{d,n} (0 < c_j < 1)$, then $a_{j,n} \neq b_{j,n}, a_{s,n} \neq a_{j,n}, b_{s,n} \neq b_{j,n}$ and therefore $\deg\left(P_s - P_j\right) = \deg\left(Q_s - Q_j\right) = n$. Thus, by (8) and Lemma 1, we find that $\rho\left(-f^{(k)}\right) = n$, this contradicts the fact $\rho\left(f^{(k)}\right) < n$. Consequently, every meromorphic solution f of Equation (3) is transcendental with order $\rho\left(f\right) \geq n$.

Assume that f is a transcendental meromorphic solution whose poles are of uniformly bounded multiplicities of Equation (3). By Lemma , there exist a set $E_1 \subset (1, +\infty)$ having finite logarithmic measure and a constant $B > 0$, such that for all z satisfying $|z| = r \notin [0,1] \diagdown E_1$, we have

$$\left|\frac{f^{(j)}\left(z\right)}{f\left(z\right)}\right| \leq B\left(T\left(2r, f\right)\right)^{k+1},\ j = 1, 2, ..., k,\ j \neq s. \tag{9}$$

By (3), it follows that the poles of f can only occur at the poles of $A_{j,i}(z)$ $(j = 0, 1, ..., k-1; i = 1, 2)$. Note that the poles of f are of uniformly bounded multiplicities. Hence

$$\lambda\left(\frac{1}{f}\right) \leq \max\left\{\rho\left(A_{j,i}\right) : j = 0, 1, ..., k-1; i = 1, 2\right\} < n.$$

By Hadamard factorization theorem, we know that f can be written as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions with

$$\lambda(d) = \rho(d) = \lambda\left(\frac{1}{f}\right) < n \le \rho(f) = \rho(g).$$

For each sufficiently large $|z| = r$, let $z_r = re^{i\theta_r}$ be a point satisfying $|g(z_r)| = M(r,g)$. By Lemma 3, there exist a constant $\delta_r\ (>0)$, a sequence $\{r_m\}_{m\in\mathbb{N}}$, $r_m \to +\infty$ and a set E_2 of finite logarithmic measure such that the estimation

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \le r_m^{2s} \tag{10}$$

holds for all z satisfying $|z| = r_m \notin E_2$, $r_m \to \infty$ and $\arg z = \theta \in [\theta_r - \delta_r,\ \theta_r + \delta_r]$.

Set $z = re^{i\theta}$, $a_{s,n} = |a_{s,n}|\, e^{i\theta_s}$, $a_{d,n} = \left|a_{d,n}\right| e^{i\theta_d}$, $b_{s,n} = |b_{s,n}|\, e^{i\varphi}$, $\theta_s, \theta_d, \varphi \in [0, 2\pi)$, $\theta_s \ne \theta_d$. Then

$$\begin{cases} \delta(P_s,\theta) = |a_{s,n}| \cos(n\theta + \theta_s), \\ \delta(Q_s,\theta) = |b_{s,n}| \cos(n\theta + \varphi), \\ \delta(P_d,\theta) = \left|a_{d,n}\right| \cos(n\theta + \theta_d). \end{cases} \tag{11}$$

Since $a_{j,n} = c_j a_{s,n}$, $b_{j,n} = c'_j b_{s,n}$, $(0 < c_j < 1)$, $(0 < c'_j < 1)$, $(j \ne s)$ and c_j, c'_j $(j = 0, 1, ..., k-1)$ are distinct numbers, then

$$\delta\left(P_j,\theta\right) = c_j \delta(P_s,\theta) \text{ or } \delta\left(P_j,\theta\right) = c_j \delta(P_d,\theta),\ \delta\left(Q_j,\theta\right) = c'_j \delta(Q_s,\theta). \tag{12}$$

Set $H_1 = \{\theta \in [0, 2\pi) : \delta(P_s,\theta) = 0, \delta(P_d,\theta) = 0\}$ and $H_2 = \{\theta \in [0, 2\pi) : \delta(P_s,\theta) = \delta(P_d,\theta)\}$. For any given $\theta \in [\theta_r - \delta_r,\ \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we have $\delta(P_s,\theta) \ne 0$, $\delta(P_d,\theta) \ne 0$ and

$$\delta(P_s,\theta) > \delta(P_d,\theta) \text{ or } \delta(P_s,\theta) < \delta(P_d,\theta).$$

I. $\delta(P_s,\theta) > \delta(P_d,\theta)$. Here we also divide our proof in three subcases: $(\varphi = \theta_s)$ or $(\varphi = \theta_d)$ or $(\varphi \ne \theta_s$ and $\varphi \ne \theta_d)$.

Case 1. $\delta(P_s,\theta) > \delta(P_d,\theta)$ **and (**$\varphi \ne \theta_s$ **and** $\varphi \ne \theta_d$**).**

Subcase 1.1. $\delta(P_s,\theta) > \delta(P_d,\theta) > 0, \delta(Q_s,\theta) > 0$. If $\delta(P_s,\theta) > 0$, $\delta(Q_s,\theta) > 0$, then we suppose $\delta(P_s,\theta) > \delta(Q_s,\theta)$ without loss of generality. Set $\delta_3 = \max\left\{\delta\left(P_j,\theta\right), \delta\left(Q_j,\theta\right); j \ne s\right\}$ and $H_3 = \{\theta \in [0, 2\pi) : \delta(Q_s,\theta) = 0, \delta(Q_s,\theta) = \delta(P_s,\theta)\}$, then $0 < \delta_3 < \delta(P_s,\theta)$. Thus, by Lemma , for any given

$$0 < 2\varepsilon < \min\left\{\frac{\delta(P_s,\theta) - \delta_3}{\delta(P_s,\theta) + \delta_3}, \frac{\delta(P_s,\theta) - \delta(Q_s,\theta)}{\delta(P_s,\theta) + \delta(Q_s,\theta)}\right\},$$

where

$$c = \max\left\{c_j, c'_j : j = 0, 1, ..., k-1, j \ne s\right\},\ c_s = 1,$$

there is a set $E_3 \subset [1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_3$, $r \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r,\ \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3)$, we have

$$\begin{aligned} \left|A_{s,1}(z)\, e^{P_s(z)} + A_{s,2}(z)\, e^{Q_s(z)}\right| &\ge \left|A_{s,1}(z)\, e^{P_s(z)}\right| - \left|A_{s,2}(z)\, e^{Q_s(z)}\right| \\ &\ge \exp\{(1-\varepsilon)\,\delta(P_s,\theta)\, r^n\} - \exp\{(1+\varepsilon)\,\delta(Q_s,\theta)\, r^n\} \\ &\ge \frac{1}{2}\exp\{(1-\varepsilon)\,\delta(P_s,\theta)\, r^n\}, \end{aligned} \tag{13}$$

$$\begin{aligned} \left|A_{j,1}(z)\, e^{P_j(z)} + A_{j,2}(z)\, e^{Q_j(z)}\right| &\le \left|A_{j,1}(z)\, e^{P_j(z)}\right| + \left|A_{j,2}(z)\, e^{Q_j(z)}\right| \\ &\le \exp\left\{(1+\varepsilon)\,\delta\left(P_j,\theta\right) r^n\right\} + \exp\left\{(1+\varepsilon)\,\delta\left(Q_j,\theta\right) r^n\right\} \\ &\le 2\exp\left\{(1+\varepsilon)\,\delta_3 r^n\right\},\ j = 0, 1, 2, ..., k-1,\ j \ne s. \end{aligned} \tag{14}$$

From (3), we have

$$\left|A_{s,1}(z)e^{P_s(z)}+A_{s,2}(z)e^{Q_s(z)}\right|\leq\left|\frac{f}{f^{(s)}}\right|\left(\left|\frac{f^{(k)}}{f}\right|+\sum_{j=0,j\neq s}^{k-1}\left\{\left|A_{j,1}(z)e^{P_j(z)}+A_{j,2}(z)e^{Q_j(z)}\right|\left|\frac{f^{(j)}}{f}\right|\right\}\right). \tag{15}$$

By substituting (9), (10), (13), (14), into (15), for all z satisfying $|z|=r_m\notin[0,1]\cup(E_1\cup E_2\cup E_3)$, $r_m\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\ \theta_r+\delta_r]\setminus(H_1\cup H_2\cup H_3)$, we have

$$\frac{1}{2}\exp\left\{(1-\varepsilon)\delta(P_s,\theta)r_m^n\right\}\leq 2kBr_m^{2s}\ \left[T(2r_m,f)\right]^{k+1}\exp\left\{(1+\varepsilon)\delta_3 r_m^n\right\}$$

which gives

$$\exp\left\{(1-\varepsilon)\delta(P_s,\theta)r_m^n\right\}\leq 4kBr_m^{2s}\ \left[T(2r_m,f)\right]^{k+1}\exp\left\{(1+\varepsilon)\delta_3 r_m^n\right\}.$$

Since $0<2\varepsilon<\frac{\delta(P_s,\theta)-\delta_3}{\delta(P_s,\theta)+\delta_3}$, then we can get

$$\exp\left\{\frac{\delta(P_s,\theta)-\delta_3}{2}r_m^n\right\}\leq 4kBr_m^{2s}\ \left[T(2r_m,f)\right]^{k+1}. \tag{16}$$

By Lemma 5 and (16), we obtain

$$\rho(f)=\limsup_{r_m\to+\infty}\frac{\log T(r_m,f)}{\log r_m}=+\infty$$

and

$$\rho_2(f)=\limsup_{r_m\to+\infty}\frac{\log\log T(r_m,f)}{\log r_m}\geq n.$$

By Lemma 6 and Equation (3), we have $\rho_2(f)\leq n$, so $\rho_2(f)=n$.

Subcase 1.2. $\delta(P_s,\theta)>\delta(P_d,\theta)>0, \delta(Q_s,\theta)<0$. We have $\delta(Q_s,\theta)<\delta(P_s,\theta)$ and $\delta(Q_s,\theta)<\delta(Q_j,\theta)<0<\delta(P_s,\theta)$, Put

$$d=\max\left\{c_j: j=0,1,...,k-1, j\neq s\right\},\ d_s=1.$$

By Lemma 3, for any given ε $(0<\varepsilon<\frac{1}{2}\left(\frac{1-d}{1+d}\right))$, there is a set $E_3\subset[1,+\infty)$ having finite logarithmic measure such that for all z satisfying $|z|=r\notin[0,1]\cup E_3, r\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\ \theta_r+\delta_r]\setminus(H_1\cup H_2\cup H_3)$, where $H_1=\{\theta\in[0,2\pi):\delta(P_s,\theta)=0,\delta(P_d,\theta)=0\}$, $H_2=\{\theta\in[0,2\pi):\delta(P_s,\theta)=\delta(P_d,\theta)\}$ and $H_3=\{\theta\in[0,2\pi):\delta(Q_s,\theta)=0,\delta(Q_s,\theta)=\delta(P_s,\theta)\}$ are finite sets, we have

$$\begin{aligned}\left|A_{s,1}(z)e^{P_s(z)}+A_{s,2}(z)e^{Q_s(z)}\right|&\geq\left|A_{s,1}(z)e^{P_s(z)}\right|-\left|A_{s,2}(z)e^{Q_s(z)}\right|\\&\geq\exp\left\{(1-\varepsilon)\delta(P_s,\theta)r^n\right\}-\exp\left\{(1-\varepsilon)\delta(Q_s,\theta)r^n\right\}\\&\geq\frac{1}{2}\exp\left\{(1-\varepsilon)\delta(P_s,\theta)r^n\right\},\end{aligned} \tag{17}$$

$$\begin{aligned}\left|A_{j,1}(z)e^{P_j(z)}+A_{j,2}(z)e^{Q_j(z)}\right|&\leq\left|A_{j,1}(z)e^{P_j(z)}\right|+\left|A_{j,2}(z)e^{Q_j(z)}\right|\\&\leq\exp\left\{(1+\varepsilon)\delta(P_j,\theta)r^n\right\}+\exp\left\{(1-\varepsilon)\delta(Q_j,\theta)r^n\right\}\\&\leq 2\exp\left\{(1+\varepsilon)d\,\delta(P_s,\theta)r^n\right\},\ j=0,1,2,...,k-1, j\neq s.\end{aligned} \tag{18}$$

By substituting (9), (10), (17), (18) into (15), for all z satisfying $|z|=r_m\notin[0,1]\cup(E_1\cup E_2\cup E_3)$, $r_m\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\ \theta_r+\delta_r]\setminus(H_1\cup H_2\cup H_3)$, we have

$$\exp\left\{(1-\varepsilon)\delta(P_s,\theta)r_m^n\right\}\leq 4kBr_m^{2s}\ \left[T(2r_m,f)\right]^{k+1}\exp\left\{(1+\varepsilon)d\,\delta(P_s,\theta)r_m^n\right\}.$$

Since $0<\varepsilon<\frac{1}{2}\left(\frac{1-d}{1+d}\right)$, then the last inequalities leads to

$$\exp\left\{\frac{(1-d)}{2}\delta\left(P_s,\theta\right)r_m^n\right\}\leq 4kBr_m^{2s}\ \left[T\left(2r_m,f\right)\right]^{k+1}. \tag{19}$$

By Lemma 5 and (19), we obtain

$$\rho\left(f\right)=\limsup_{r_m\to+\infty}\frac{\log T\left(r_m,f\right)}{\log r_m}=+\infty$$

and

$$\rho_2\left(f\right)=\limsup_{r_m\to+\infty}\frac{\log\log T\left(r_m,f\right)}{\log r_m}\geq n.$$

In addition, by Lemma 6 and from Equation (3), we have $\rho_2\left(f\right)\leq n$, so $\rho_2\left(f\right)=n$.

Subcase 1.3. $\delta\left(P_s,\theta\right)>0>\delta\left(P_d,\theta\right),\delta\left(Q_s,\theta\right)>0$. We suppose $\delta\left(P_s,\theta\right)>\delta\left(Q_s,\theta\right)$ without loss of generality. By Lemma 3, for any given ε ($0<\varepsilon<\min\left\{\frac{1}{2}\left(\frac{1-\nu}{1+\nu}\right),\frac{1}{2}\left(\frac{\delta(P_s,\theta)-\delta(Q_s,\theta)}{\delta(P_s,\theta)+\delta(Q_s,\theta)}\right)\right\}$), where

$$\nu=\max\left\{c_j,c_j':j=0,1,...,k-1,j\neq s,j\neq d\right\},\ \nu_s=1,$$

there is a set $E_3\subset[1,+\infty)$ having finite logarithmic measure such that for all z satisfying $|z|=r\notin[0,1]\cup E_3$, $r\to+\infty$ and

$$\arg z=\theta\in[\theta_r-\delta_r,\ \theta_r+\delta_r]\diagdown(H_1\cup H_2\cup H_3),$$

where $H_1=\{\theta\in[0,2\pi):\delta\left(P_s,\theta\right)=0,\delta\left(P_d,\theta\right)=0\}$, $H_2=\{\theta\in[0,2\pi):\delta\left(P_s,\theta\right)=\delta\left(P_d,\theta\right)\}$ and $H_3=\{\theta\in[0,2\pi):\delta\left(Q_s,\theta\right)=0,\delta\left(Q_s,\theta\right)=\delta\left(P_s,\theta\right)\}$ are finite sets, we have

$$\begin{aligned}\left|A_{s,1}\left(z\right)e^{P_s(z)}+A_{s,2}\left(z\right)e^{Q_s(z)}\right|&\geq\left|A_{s,1}\left(z\right)e^{P_s(z)}\right|-\left|A_{s,2}\left(z\right)e^{Q_s(z)}\right|\\&\geq\exp\left\{(1-\varepsilon)\,\delta\left(P_s,\theta\right)r^n\right\}-\exp\left\{(1+\varepsilon)\,\delta\left(Q_s,\theta\right)r^n\right\}\\&\geq\frac{1}{2}\exp\left\{(1-\varepsilon)\,\delta\left(P_s,\theta\right)r^n\right\},\end{aligned}\tag{20}$$

$$\begin{aligned}\left|A_{j,1}\left(z\right)e^{P_j(z)}+A_{j,2}\left(z\right)e^{Q_j(z)}\right|&\leq\left|A_{j,1}\left(z\right)e^{P_j(z)}\right|+\left|A_{j,2}\left(z\right)e^{Q_j(z)}\right|\\&\leq\exp\left\{(1+\varepsilon)\,c_j\delta\left(P_s,\theta\right)r^n\right\}+\exp\left\{(1+\varepsilon)\,c_j'\delta\left(Q_s,\theta\right)r^n\right\}\\&\leq 2\exp\left\{(1+\varepsilon)\,\nu\delta\left(P_s,\theta\right)r^n\right\},\ j=0,1,2,...,k-1,\ j\neq s.\end{aligned}\tag{21}$$

By substituting (9), (10), (20) and (21), into (15), for all z satisfying $|z|=r_m\notin[0,1]\cup(E_1\cup E_2\cup E_3)$, $r_m\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\ \theta_r+\delta_r]\diagdown(H_1\cup H_2\cup H_3)$, we have

$$\exp\left\{(1-\varepsilon)\,\delta\left(P_s,\theta\right)r_m^n\right\}\leq 4kBr_m^{2s}\ \left[T\left(2r_m,f\right)\right]^{k+1}\exp\left\{(1+\varepsilon)\,\nu\delta\left(P_s,\theta\right)r_m^n\right\}.$$

Since $0<\varepsilon<\frac{1}{2}\left(\frac{1-\nu}{1+\nu}\right)$, then

$$\exp\left\{\frac{(1-\nu)}{2}\delta\left(P_s,\theta\right)r_m^n\right\}\leq 4kBr_m^{2s}\left[T\left(2r_m,f\right)\right]^{k+1}. \tag{22}$$

By Lemma 5 and (22), we obtain

$$\rho\left(f\right)=\limsup_{r_m\to+\infty}\frac{\log T\left(r_m,f\right)}{\log r_m}=+\infty$$

and

$$\rho_2\left(f\right)=\limsup_{r_m\to+\infty}\frac{\log\log T\left(r_m,f\right)}{\log r_m}\geq n.$$

In addition, then by Lemma 6 and from Equation (3), we have $\rho_2\left(f\right)\leq n$, so $\rho_2\left(f\right)=n$.

Subcase 1.4. $\delta\left(P_s,\theta\right) > 0 > \delta\left(P_d,\theta\right), \delta\left(Q_s,\theta\right) < 0$. We have $\delta\left(P_s,\theta\right) > \delta\left(P_j,\theta\right) > 0, \delta\left(Q_s,\theta\right) < \delta\left(Q_j,\theta\right) < 0$, then $\delta\left(Q_s,\theta\right) < \delta\left(P_s,\theta\right)$. Put

$$\nu' = \max\left\{c_j : j = 0,1,...,k-1, j \neq s, j \neq d\right\}, \nu'_s = 1.$$

By Lemma 3, for any given $0 < \varepsilon < \frac{1}{2}\left(\frac{1-\nu'}{1+\nu'}\right)$, there is a set $E_3 \subset [1,+\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_3, r \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r,\ \theta_r + \delta_r] \diagdown (H_1 \cup H_2 \cup H_3)$, where $H_1 = \{\theta \in [0,2\pi) : \delta\left(P_s,\theta\right) = 0, \delta\left(P_d,\theta\right) = 0\}$, $H_2 = \{\theta \in [0,2\pi) : \delta\left(P_s,\theta\right) = \delta\left(P_d,\theta\right)\}$ and $H_3 = \{\theta \in [0,2\pi) : \delta\left(Q_s,\theta\right) = 0, \delta\left(Q_s,\theta\right) = \delta\left(P_s,\theta\right)\}$ are finite sets, we have

$$\begin{aligned}\left|A_{s,1}(z)\, e^{P_s(z)} + A_{s,2}(z)\, e^{Q_s(z)}\right| &\geq \left|A_{s,1}(z)\, e^{P_s(z)}\right| - \left|A_{s,2}(z)\, e^{Q_s(z)}\right| \\ &\geq \exp\left\{(1-\varepsilon)\,\delta\left(P_s,\theta\right) r^n\right\} - \exp\left\{(1-\varepsilon)\,\delta\left(Q_s,\theta\right) r^n\right\} \\ &\geq \frac{1}{2}\exp\left\{(1-\varepsilon)\,\delta\left(P_s,\theta\right) r^n\right\}, \end{aligned} \tag{23}$$

$$\begin{aligned}\left|A_{j,1}(z)\, e^{P_j(z)} + A_{j,2}(z)\, e^{Q_j(z)}\right| &\leq \left|A_{j,1}(z)\, e^{P_j(z)}\right| + \left|A_{j,2}(z)\, e^{Q_j(z)}\right| \\ &\leq \exp\left\{(1+\varepsilon)\,\delta\left(P_j,\theta\right) r^n\right\} + \exp\left\{(1-\varepsilon)\,\delta\left(Q_j,\theta\right) r^n\right\} \\ &\leq 2\exp\left\{(1+\varepsilon)\,\nu'\,\delta\left(P_s,\theta\right) r^n\right\},\ j = 0,1,2,...,k-1, j \neq s. \end{aligned} \tag{24}$$

By substituting (9), (10), (23), (24) into (15), for all z satisfying $|z| = r_m \notin [0,1] \cup (E_1 \cup E_2 \cup E_3), r_m \to +\infty$ and

$$\arg z = \theta \in [\theta_r - \delta_r,\ \theta_r + \delta_r] \diagdown (H_1 \cup H_2 \cup H_3),$$

we have

$$\exp\left\{(1-\varepsilon)\,\delta\left(P_s,\theta\right) r_m^n\right\} \leq 4kBr_m^{2k}\left[T\left(2r_m,f\right)\right]^{k+1}\exp\left\{(1+\varepsilon)\,\nu'\,\delta\left(P_s,\theta\right) r_m^n\right\}.$$

Since $0 < \varepsilon < \frac{1}{2}\left(\frac{1-\nu'}{1+\nu'}\right)$, then

$$\exp\left\{\frac{(1-\nu')}{2}\delta\left(P_s,\theta\right) r_m^n\right\} \leq 4kBr_m^{2k}\ \left[T\left(2r_m,f\right)\right]^{k+1}. \tag{25}$$

By Lemma 5 and (25) we obtain

$$\rho(f) = \limsup_{r_m \to +\infty}\frac{\log T\left(r_m,f\right)}{\log r_m} = +\infty$$

and

$$\rho_2(f) = \limsup_{r_m \to +\infty}\frac{\log\log T\left(r_m,f\right)}{\log r_m} \geq n.$$

In addition, by Lemma 6 and from Equation (3), we have $\rho_2(f) \leq n$, so $\rho_2(f) = n$.

Subcase 1.5. $0 > \delta\left(P_s,\theta\right) > \delta\left(P_d,\theta\right), \delta\left(Q_s,\theta\right) > 0$. We have $\delta\left(P_d,\theta\right) < \delta\left(P_s,\theta\right) < 0 < \delta\left(Q_s,\theta\right)$. Put

$$d' = \max\left\{c'_j : j = 0,1,...,k-1, j \neq s\right\},\ d'_s = 1.$$

By Lemma 3, for any given ε ($0 < \varepsilon < \frac{1}{2}\left(\frac{1-d'}{1+d'}\right)$), there is a set $E_3 \subset [1,+\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_3, r \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r,\ \theta_r + \delta_r] \diagdown (H_1 \cup H_2 \cup H_3)$, where $H_1 = \{\theta \in [0,2\pi) : \delta\left(P_s,\theta\right) = 0, \delta\left(P_d,\theta\right) = 0\}$, $H_2 = \{\theta \in$

$[0,2\pi):\delta(P_s,\theta)=\delta(P_d,\theta)\}$ and $H_3=\{\theta\in[0,2\pi):\delta(Q_s,\theta)=0,\delta(Q_s,\theta)=\delta(P_s,\theta)\}$ are finite sets, we have

$$\begin{aligned}\left|A_{s,1}(z)e^{P_s(z)}+A_{s,2}(z)e^{Q_s(z)}\right| &\geq \left|A_{s,2}(z)e^{Q_s(z)}\right|-\left|A_{s,1}(z)e^{P_s(z)}\right|\\ &\geq \exp\{(1-\varepsilon)\delta(Q_s,\theta)r^n\}-\exp\{(1-\varepsilon)\delta(P_s,\theta)r^n\}\\ &\geq \frac{1}{2}\exp\{(1-\varepsilon)\delta(Q_s,\theta)r^n\},\end{aligned} \tag{26}$$

$$\begin{aligned}\left|A_{j,1}(z)e^{P_j(z)}+A_{j,2}(z)e^{Q_j(z)}\right| &\leq \left|A_{j,1}(z)e^{P_j(z)}\right|+\left|A_{j,2}(z)e^{Q_j(z)}\right|\\ &\leq \exp\left\{(1-\varepsilon)\delta\left(P_j,\theta\right)r^n\right\}+\exp\left\{(1+\varepsilon)\delta\left(Q_j,\theta\right)r^n\right\}\\ &\leq 2\exp\left\{(1+\varepsilon)d'\delta(Q_s,\theta)r^n\right\},\ j=0,1,2,...,k-1,\ j\neq s.\end{aligned} \tag{27}$$

By using a similar proof as that of subcase 1.2, since $0<\varepsilon<\frac{1}{2}\left(\frac{1-d'}{1+d'}\right)$, we can obtain for all z satisfying $|z|=r_m\notin[0,1]\cup(E_1\cup E_2\cup E_3)$, $r_m\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\ \theta_r+\delta_r]$

$$\exp\left\{\frac{(1-d')}{2}\delta(Q_s,\theta)r_m^n\right\}\leq 4kBr_m^{2s}\ [T(2r_m,f)]^{k+1}. \tag{28}$$

So, by Lemma 5 and (28) we obtain

$$\rho(f)=\limsup_{r_m\to+\infty}\frac{\log T(r_m,f)}{\log r_m}=+\infty$$

and

$$\rho_2(f)=\limsup_{r_m\to+\infty}\frac{\log\log T(r_m,f)}{\log r_m}\geq n.$$

In addition, by Lemma 6 and from Equation (3), we have $\rho_2(f)\leq n$, so $\rho_2(f)=n$.

Subcase 1.6. $0>\delta(P_s,\theta)>\delta(P_d,\theta),\delta(Q_s,\theta)<0$. Set

$$\lambda=\min\left\{c_j,c_j':j=0,1,...,k-1,j\neq s,j\neq d\right\}.$$

By Lemma 3, for any given $0<\varepsilon<1$, there is a set $E_3\subset[1,+\infty)$ having finite logarithmic measure such that for all z satisfying $|z|=r\notin[0,1]\cup E_3, r\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\ \theta_r+\delta_r]\setminus(H_1\cup H_2\cup H_3)$, where $H_1=\{\theta\in[0,2\pi):\delta(P_s,\theta)=0,\delta(P_d,\theta)=0\}$, $H_2=\{\theta\in[0,2\pi):\delta(P_s,\theta)=\delta(P_d,\theta)\}$ and $H_3=\{\theta\in[0,2\pi):\delta(Q_s,\theta)=0,\delta(Q_s,\theta)=\delta(P_s,\theta)\}$ are finite sets, we have

$$\begin{aligned}\left|A_{j,1}(z)e^{P_j(z)}+A_{j,2}(z)e^{Q_j(z)}\right| &\leq \left|A_{j,1}(z)e^{P_j(z)}\right|+\left|A_{j,2}(z)e^{Q_j(z)}\right|\\ &\leq \exp\left\{(1-\varepsilon)\delta\left(P_j,\theta\right)r^n\right\}+\exp\left\{(1-\varepsilon)\delta\left(Q_j,\theta\right)r^n\right\}\\ &\leq 2\exp\{(1-\varepsilon)\lambda\delta(P_s,\theta)r^n\},\ j=0,1,2,...,k-1.\end{aligned} \tag{29}$$

From (3), we have

$$1\leq\left|\frac{f}{f^{(k)}}\right|\sum_{j=0}^{k-1}\left\{\left|A_{j,1}(z)e^{P_j(z)}+A_{j,2}(z)e^{Q_j(z)}\right|\left|\frac{f^{(j)}}{f}\right|\right\}. \tag{30}$$

By substituting (9), (10), (29) into (30), for all z satisfying $|z|=r_m\notin[0,1]\cup(E_1\cup E_2\cup E_3)$, $r_m\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\ \theta_r+\delta_r]\setminus(H_1\cup H_2\cup H_3)$, we have

$$1\leq 2kBr_m^{2k}[T(2r_m,f)]^{k+1}\exp\{(1-\varepsilon)\lambda\delta(P_s,\theta)r_m^n\}$$

which gives

$$\exp\{(\varepsilon-1)\lambda\delta(P_s,\theta)r_m^n\}\leq 2kBr_m^{2k}[T(2r_m,f)]^{k+1}. \tag{31}$$

Since $0<\varepsilon<1$ and $\lambda\delta\left(P_{s},\theta\right)<0$, then by Lemma 2.5 and (31), we obtain

$$\rho\left(f\right)=\limsup_{r_{m}\rightarrow+\infty}\frac{\log T\left(r_{m},f\right)}{\log r_{m}}=+\infty$$

and

$$\rho_{2}\left(f\right)=\limsup_{r_{m}\rightarrow+\infty}\frac{\log\log T\left(r_{m},f\right)}{\log r_{m}}\geq n.$$

In addition, by Lemma 6 and from Equation (3), we have $\rho_{2}\left(f\right)\leq n$, so $\rho_{2}\left(f\right)=n$.

Case 2. $\delta\left(P_{s},\theta\right)>\delta\left(P_{d},\theta\right)$ and $\varphi=\theta_{s}$

Subcase 2.1. $\delta\left(P_{s},\theta\right)>\delta\left(P_{d},\theta\right)>0$. Because of $a_{s,n}\neq b_{s,n}$, we suppose $\left|a_{s,n}\right|<\left|b_{s,n}\right|$ without loss of generality. In this case, by (3.4) and (3.5) we have $\delta\left(Q_{s},\theta\right)>\delta\left(Q_{j},\theta\right)>0$, $\delta\left(Q_{s},\theta\right)>\delta\left(P_{s},\theta\right)>\delta\left(P_{j},\theta\right)>0$. Put

$$c=\max\left\{c_{j},c_{j}^{\prime}:j=0,1,...,k-1,j\neq s,j\neq d\right\},c_{s}=1.$$

Then, $0<c<1$. By Lemma 3, for any given ε with

$$0<\varepsilon<\min\left\{\frac{1}{2}\left(\frac{1-c}{1+c}\right),\frac{1}{2}\left(\frac{\delta\left(Q_{s},\theta\right)-\delta\left(P_{s},\theta\right)}{\delta\left(Q_{s},\theta\right)+\delta\left(P_{s},\theta\right)}\right)\right\},$$

there is a set $E_{3}\subset\left[1,+\infty\right)$ having finite logarithmic measure such that for all z satisfying $\left|z\right|=r\notin\left[0,1\right]\cup E_{3},r\rightarrow+\infty$ and

$$\arg z=\ \theta\in\ \left[\theta_{r}-\delta_{r},\ \theta_{r}+\delta_{r}\right]\diagdown\left(H_{1}\cup H_{2}\cup H_{3}\right),$$

where $H_{1}=\left\{\theta\in\left[0,2\pi\right):\delta\left(P_{s},\theta\right)=0,\delta\left(P_{d},\theta\right)=0\right\}$, $H_{2}=\left\{\theta\in\left[0,2\pi\right):\delta\left(P_{s},\theta\right)=\delta\left(P_{d},\theta\right)\right\}$ and $H_{3}=\left\{\theta\in\left[0,2\pi\right):\delta\left(Q_{s},\theta\right)=0\right\}$ are finite sets, we have

$$\begin{aligned}\left|A_{s,1}\left(z\right)e^{P_{s}\left(z\right)}+A_{s,2}\left(z\right)e^{Q_{s}\left(z\right)}\right|&\geq\left|A_{s,2}\left(z\right)e^{Q_{s}\left(z\right)}\right|-\left|A_{s,1}\left(z\right)e^{P_{s}\left(z\right)}\right|\\&\geq\exp\left\{\left(1-\varepsilon\right)\delta\left(Q_{s},\theta\right)r^{n}\right\}-\exp\left\{\left(1+\varepsilon\right)\delta\left(P_{s},\theta\right)r^{n}\right\}\\&\geq\frac{1}{2}\exp\left\{\left(1-\varepsilon\right)\delta\left(Q_{s},\theta\right)r^{n}\right\},\end{aligned}\tag{32}$$

$$\begin{aligned}\left|A_{j,1}\left(z\right)e^{P_{j}\left(z\right)}+A_{j,2}\left(z\right)e^{Q_{j}\left(z\right)}\right|&\leq\left|A_{j,1}\left(z\right)e^{P_{j}\left(z\right)}\right|+\left|A_{j,2}\left(z\right)e^{Q_{j}\left(z\right)}\right|\\&\leq\exp\left\{\left(1+\varepsilon\right)c\delta\left(P_{s},\theta\right)r^{n}\right\}+\exp\left\{\left(1+\varepsilon\right)c\delta\left(Q_{s},\theta\right)r^{n}\right\}\\&\leq2\exp\left\{\left(1+\varepsilon\right)c\delta\left(Q_{s},\theta\right)r^{n}\right\},\ j=0,1,2,...,k-1,\ j\neq s.\end{aligned}\tag{33}$$

By substituting (9), (10), (32), (33) into (15), since $0<\varepsilon<\frac{1}{2}\left(\frac{1-c}{1+c}\right)$, for all z satisfying $\left|z\right|=r_{m}\notin\left[0,1\right]\cup\left(E_{1}\cup E_{2}\cup E_{3}\right)$, $r_{m}\rightarrow+\infty$ and $\arg z=\theta\in\left[\theta_{r}-\delta_{r},\ \theta_{r}+\delta_{r}\right]\diagdown\left(H_{1}\cup H_{2}\cup H_{3}\right)$, we obtain

$$\exp\left\{\frac{\left(1-c\right)}{2}\delta\left(Q_{s},\theta\right)r_{m}^{n}\right\}\leq4kBr_{m}^{2s}\ \left[T\left(2r_{m},f\right)\right]^{k+1}.\tag{34}$$

Thus, by Lemma 5 and (34) we get

$$\rho\left(f\right)=\limsup_{r_{m}\rightarrow+\infty}\frac{\log T\left(r_{m},f\right)}{\log r_{m}}=+\infty$$

and

$$\rho_{2}\left(f\right)=\limsup_{r_{m}\rightarrow+\infty}\frac{\log\log T\left(r_{m},f\right)}{\log r_{m}}\geq n.$$

In addition, by Lemma 6 and from Equation (3), we have $\rho_{2}\left(f\right)\leq n$, so $\rho_{2}\left(f\right)=n$.

Subcase 2.2. $\delta(P_s,\theta) > 0 > \delta(P_d,\theta)$. Because of $a_{s,n} \neq b_{s,n}$, we suppose $|a_{s,n}| < |b_{s,n}|$ without loss of generality. In this case, by (11) and (12) we have $\delta(Q_s,\theta) > \delta(Q_j,\theta) > 0$, $\delta(Q_s,\theta) > \delta(P_s,\theta) > \delta(P_j,\theta) > 0$. Put

$$c = \max\left\{c_j, c_j' : j = 0,1,...,k-1, j \neq s, j \neq d\right\}, c_s = 1.$$

Using the same reasoning as in **Subcase 2.1** , we can also obtain $\rho(f) = +\infty$ and $\rho_2(f) = n$.

Subcase 2.3. $0 > \delta(P_s,\theta) > \delta(P_d,\theta)$. We have $\delta(Q_s,\theta) < \delta(Q_j,\theta) < \delta(P_s,\theta) < 0$, $\delta(P_s,\theta) < \delta(P_j,\theta) < 0$. Put

$$\lambda = \min\left\{c_j, c_j' : j = 0,1,...,k-1, j \neq s, j \neq d\right\}.$$

By Lemma 3, for any given $0 < \varepsilon < 1$, there is a set $E_3 \subset [1,+\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_3, r \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r,\ \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3)$, where $H_1 = \{\theta \in [0,2\pi) : \delta(P_s,\theta) = 0, \delta(P_d,\theta) = 0\}$, $H_2 = \{\theta \in [0,2\pi) : \delta(P_s,\theta) = \delta(P_d,\theta)\}$ and $H_3 = \{\theta \in [0,2\pi) : \delta(Q_s,\theta) = 0\}$ are finite sets, we have

$$\begin{aligned} \left|A_{j,1}(z)e^{P_j(z)} + A_{j,2}(z)e^{Q_j(z)}\right| &\leq \left|A_{j,1}(z)e^{P_j(z)}\right| + \left|A_{j,2}(z)e^{Q_j(z)}\right| \\ &\leq \exp\left\{(1-\varepsilon)\delta(P_j,\theta)r^n\right\} + \exp\left\{(1-\varepsilon)\delta(Q_j,\theta)r^n\right\} \\ &\leq 2\exp\left\{(1-\varepsilon)\lambda\delta(P_s,\theta)r^n\right\},\ j = 0,1,2,...,k-1. \end{aligned} \tag{35}$$

By substituting (9), (10) and (35) into (30), for all z satisfying $|z| = r_m \notin [0,1] \cup (E_1 \cup E_2 \cup E_3)$, $r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r,\ \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3)$, we have

$$1 \leq 2kBr_m^{2k}\left[T(2r_m,f)\right]^{k+1}\exp\left\{(1-\varepsilon)\lambda\delta(P_s,\theta)r_m^n\right\}$$

which gives

$$\exp\left\{(\varepsilon-1)\lambda\delta(P_s,\theta)r_m^n\right\} \leq 2kBr_m^{2k}\left[T(2r_m,f)\right]^{k+1}. \tag{36}$$

Since $0 < \varepsilon < 1$ and $\lambda\delta(P_s,\theta) < 0$, then by Lemma 5 and (36) we obtain

$$\rho(f) = \limsup_{r_m\to+\infty}\frac{\log T(r_m,f)}{\log r_m} = +\infty$$

and

$$\rho_2(f) = \limsup_{r_m\to+\infty}\frac{\log\log T(r_m,f)}{\log r_m} \geq n.$$

In addition, by Lemma 6 and from Equation (3), we have $\rho_2(f) \leq n$, so $\rho_2(f) = n$.

Case 3. $\delta(P_s,\theta) > \delta(P_d,\theta)$ and $\varphi = \theta_d$

Subcase 3.1. $\delta(P_s,\theta) > \delta(P_d,\theta) > 0$. Because of $a_{d,n} \neq b_{s,n}$, we suppose $|a_{d,n}| < |b_{s,n}|$ without loss of generality. In this case, by (11) and (12), we have $\delta(Q_s,\theta) > \delta(Q_j,\theta) > 0$, $\delta(Q_s,\theta) > \delta(P_s,\theta) > \delta(P_j,\theta) > 0$. Then, $0 < c < 1$. By Lemma 3, for any given ε with

$$0 < \varepsilon < \min\frac{1}{2}\left\{\left(\frac{1-c}{1+c}\right), \frac{1}{2}\left(\frac{\delta(Q_s,\theta) - \delta(P_s,\theta)}{\delta(Q_s,\theta) + \delta(P_s,\theta)}\right)\right\},$$

where

$$c = \max\left\{c_j, c_j' : j = 0,1,...,k-1, j \neq s, j \neq d\right\}, c_s = 1,$$

there is a set $E_3 \subset [1,+\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_3, r \to +\infty$ and

$$\arg z = \ \theta \in \ [\theta_r - \delta_r,\ \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3),$$

where $H_1 = \{\theta \in [0,2\pi) : \delta(P_s,\theta) = 0, \delta(P_d,\theta) = 0\}$, $H_2 = \{\theta \in [0,2\pi) : \delta(P_s,\theta) = \delta(P_d,\theta)\}$ and $H_3 = \{\theta \in [0,2\pi) : \delta(Q_s,\theta) = 0\}$ are finite sets, we have (32) and (33) hold. By substituting (9), (10), (32), (33) into (15), we obtain (34) for all z satisfying $|z| = r_m \notin [0,1] \cup (E_1 \cup E_2 \cup E_3)$, $r_m \to +\infty$ and

$\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3)$. Since $0 < \varepsilon < \frac{1}{2}\left(\frac{1-c}{1+c}\right)$, then by Lemma 5 and (34) we obtain

$$\rho(f) = \limsup_{r_m \to +\infty} \frac{\log T(r_m, f)}{\log r_m} = +\infty$$

and

$$\rho_2(f) = \limsup_{r_m \to +\infty} \frac{\log\log T(r_m, f)}{\log r_m} \geq n.$$

In addition, by Lemma 6 and from Equation (3), we have $\rho_2(f) \leq n$, so $\rho_2(f) = n$.

Subcase 3.2. $\delta(P_s, \theta) > 0 > \delta(P_d, \theta)$. Because of $a_{d,n} \neq b_{s,n}$, we suppose $|a_{d,n}| < |b_{s,n}|$ without loss of generality. In this case, by (11) and (12), we have $\delta(Q_s, \theta) < \delta(Q_j, \theta) < 0$, $\delta(Q_s, \theta) < \delta(P_d, \theta) < 0 < \delta(P_s, \theta)$. Then, $0 < c < 1$. By Lemma 3, for any given ε $\left(0 < \varepsilon < \frac{1}{2}\left(\frac{1-c}{1+c}\right)\right)$, where

$$c = \max\left\{c_j, c_j' : j = 0, 1, ..., k-1, j \neq s, j \neq d\right\}, c_s = 1,$$

there is a set $E_3 \subset [1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_3, r \to +\infty$ and

$$\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3),$$

where $H_1 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = 0, \delta(P_d, \theta) = 0\}$, $H_2 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = \delta(P_d, \theta)\}$ and $H_3 = \{\theta \in [0, 2\pi) : \delta(Q_s, \theta) = 0\}$ are finite sets, we have

$$\begin{aligned}\left|A_{s,1}(z) e^{P_s(z)} + A_{s,2}(z) e^{Q_s(z)}\right| &\geq \left|A_{s,2}(z) e^{P_s(z)}\right| - \left|A_{s,1}(z) e^{Q_s(z)}\right| \\ &\geq \exp\{(1-\varepsilon)\delta(P_s, \theta) r^n\} - \exp\{(1-\varepsilon)\delta(Q_s, \theta) r^n\} \\ &\geq \frac{1}{2}\exp\{(1-\varepsilon)\delta(P_s, \theta) r^n\}, \end{aligned} \tag{37}$$

$$\begin{aligned}\left|A_{j,1}(z) e^{P_j(z)} + A_{j,2}(z) e^{Q_j(z)}\right| &\leq \left|A_{j,1}(z) e^{P_j(z)}\right| + \left|A_{j,2}(z) e^{Q_j(z)}\right| \\ &\leq \exp\{(1+\varepsilon) c\delta(P_s, \theta) r^n\} + \exp\{(1-\varepsilon) c\delta(Q_s, \theta) r^n\} \\ &\leq 2\exp\{(1+\varepsilon) c\delta(P_s, \theta) r^n\}, \ j = 0, 1, 2, ..., k-1, \ j \neq s. \end{aligned} \tag{38}$$

By substituting (9), (10), (37) and (38) into (15), by $0 < \varepsilon < \frac{1}{2}\left(\frac{1-c}{1+c}\right)$, for all z satisfying $|z| = r_m \notin [0,1] \cup (E_1 \cup E_2 \cup E_3)$, $r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3)$, we obtain

$$\exp\left\{\frac{(1-c)}{2}\delta(P_s, \theta) r_m^n\right\} \leq 4kBr_m^{2s}\ [T(2r_m, f)]^{k+1}. \tag{39}$$

Therefore, by Lemma 5 and (39) we obtain

$$\rho(f) = \limsup_{r_m \to +\infty} \frac{\log T(r_m, f)}{\log r_m} = +\infty$$

and

$$\rho_2(f) = \limsup_{r_m \to +\infty} \frac{\log\log T(r_m, f)}{\log r_m} \geq n.$$

In addition, by Lemma 6 and from Equation (3), we have $\rho_2(f) \leq n$, so $\rho_2(f) = n$.

Subcase 3.3. $0 > \delta(P_s, \theta) > \delta(P_d, \theta)$. Because of $a_{d,n} \neq b_{s,n}$, we suppose $|a_{d,n}| < |b_{s,n}|$ without loss of generality. In this case, by (11) and (12), we have $\delta(Q_s, \theta) < \delta(Q_j, \theta) < 0$, $\delta(Q_s, \theta) < \delta(P_d, \theta) < \delta(P_s, \theta) < 0$. Put

$$c' = \min\left\{c_j, c_j' : j = 0, 1, ..., k-1\right\}.$$

By Lemma 3, for any given $0 < \varepsilon < 1$, there is a set $E_3 \subset [1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_3, r \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3)$, where $H_1 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = 0, \delta(P_d, \theta) = 0\}$, $H_2 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = \delta(P_d, \theta)\}$ and $H_3 = \{\theta \in [0, 2\pi) : \delta(Q_s, \theta) = 0\}$ are finite sets, we have

$$\begin{aligned} \left|A_{j,1}(z) e^{P_j(z)} + A_{j,2}(z) e^{Q_j(z)}\right| &\leq \left|A_{j,1}(z) e^{P_j(z)}\right| + \left|A_{j,2}(z) e^{Q_j(z)}\right| \\ &\leq \exp\left\{(1-\varepsilon)\delta\left(P_j, \theta\right) r^n\right\} + \exp\left\{(1-\varepsilon)\delta\left(Q_j, \theta\right) r^n\right\} \\ &\leq 2\exp\left\{(1-\varepsilon) c' \delta(P_s, \theta) r^n\right\}, \ j = 0, 1, 2, ..., k-1. \end{aligned} \tag{40}$$

By substituting (9), (10) and (40) into (30), for all z satisfying $|z| = r_m \notin [0, 1] \cup (E_1 \cup E_2 \cup E_3)$, $r_m \to +\infty$ and

$$\arg z = \theta \in [\theta_r - \delta_r, \ \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3),$$

we obtain

$$1 \leq 2kBr_m^{2k} [T(2r_m, f)]^{k+1} \exp\left\{(1-\varepsilon) c' \delta(P_s, \theta) r_m^n\right\}$$

which gives

$$\exp\left\{(\varepsilon - 1) c' \delta(P_s, \theta) r_m^n\right\} \leq 2kr_m^{2k} B [T(2r_m, f)]^{k+1}. \tag{41}$$

By Lemma 5 and (41) we obtain

$$\rho(f) = \limsup_{r_m \to +\infty} \frac{\log^+ T(r_m, f)}{\log r_m} = +\infty$$

and

$$\rho_2(f) = \limsup_{r_m \to +\infty} \frac{\log_2^+ T(r_m, f)}{\log r_m} \geq n.$$

In addition, by Lemma 6 and from Equation (3), we have $\rho_2(f) \leq n$, so $\rho_2(f) = n$.

II. $\delta(P_s, \theta) < \delta(P_d, \theta)$. Here we also divide our proof in three subcases: $(\varphi = \theta_s)$ or $(\varphi = \theta_d)$ or $(\varphi \neq \theta_s$ and $\varphi \neq \theta_d)$. Using the same reasoning as in **I**, we can also obtain $\rho_2(f) \leq n$, so $\rho_2(f) = n$.

4. Proof of Theorem 4

First, we show that (4) can possess at most one exceptional transcendental meromorphic solution f_0 of finite order. In fact, if f_* is another transcendental meromorphic solution of finite order of Equation (4), then $f_0 - f_*$ is of finite order. But $f_0 - f_*$ is a transcendental meromorphic solution of the corresponding homogeneous equation of (4). This contradicts Theorem 3. We assume that f is an infinite order meromorphic solution of (4) whose poles are of uniformly bounded multiplicities. By Lemma 7 and Lemma 8, we have $\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$ and $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) \leq n$.

Now, we prove that $\rho_2(f) \geq n$. By Lemma 3, there exists a set $E_1 \subset (1, +\infty)$ having finite logarithmic measure and a constant $B > 0$ such that for all z satisfying $|z| = r \notin [0, 1] \setminus [E_1$, we have (9). Set

$$\rho_1 = \max\left\{\rho(F), \rho\left(A_{j,i}(z)\right) : j = 0, 1, ..., k-1; i = 1, 2\right\}.$$

By (4), it follows that the poles of f can only occur at the poles of F and $A_{j,i}(z), j = 0, 1, ..., k-1; i = 1, 2$. Note that the poles of f are of uniformly bounded multiplicities. Hence

$$\lambda\left(\frac{1}{f}\right) \leq \max\left\{\rho\left(A_{j,i}(z)\right) : j = 0, 1, ..., k-1; i = 1, 2\right\} = \rho_1.$$

By Hadamard factorization theorem, we know that f can be written as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions with

$$\lambda(d) = \rho(d) = \lambda\left(\frac{1}{f}\right) \leq \rho_1 < \rho(f) = \rho(g) = +\infty.$$

For each sufficiently large $|z| = r$, let $z_r = re^{i\theta_r}$ be a point satisfying $|g(z_r)| = M(r, g)$. By Lemma 3, there exist a constant δ_r (> 0), a sequence $\{r_m\}_{m \in \mathbb{N}}$, $r_m \to +\infty$ and a set E_2 of finite logarithmic measure such that the estimation (10) holds for all z satisfying $|z| = r_m \notin E_2$, $r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \ \theta_r + \delta_r]$. Since $|g(z)|$

is continuous in $|z| = r$, then there exists a constant $r(>0)$ such that for all z satisfying $|z| = r$ sufficiently large and $\arg z = \theta \in [\theta_r - \delta_r,\ \theta_r + \delta_r]$, we have

$$\frac{1}{2}|g(z_r)| < |g(z)| < \frac{3}{2}|g(z_r)|. \tag{42}$$

On the other hand, by Lemma 8, for a given $\varepsilon\,(0 < \varepsilon < n - \rho_1)$, there exists a set $E_5 \subset (1, +\infty)$ that has finite linear measure and finite logarithmic measure such that when $|z| = r \notin [0,1] \cup E_5, r \to +\infty$, we have

$$|F(z)| \leq \exp\left\{r^{\rho_1+\varepsilon}\right\}, |d(z)| \leq \exp\left\{r^{\rho_1+\varepsilon}\right\}. \tag{43}$$

Since $|g(z)| = M(r, g) \geq 1$, from (43), we obtain

$$\left|\frac{F(z)}{f(z)}\right| = \left|\frac{d(z)\,F(z)}{g(z)}\right| \leq \frac{|d(z)\,F(z)|}{M(r,g)} \leq \exp\left\{r^{\rho_1+\varepsilon}\right\}\exp\left\{r^{\rho_1+\varepsilon}\right\} = \exp\left\{2r^{\rho_1+\varepsilon}\right\} \tag{44}$$

for $|z| = r \notin [0,1] \cup E_5$, $r \to +\infty$. Set $\nu = \min\{\delta_r, \lambda_r\}$. Suppose that $A_{j,i}(z)$, $P_j(z)$, $Q_j(z)$ $a_{j,n}, b_{j,n}, (j = 0, 1, ..., k-1; i = 1, 2)$ satisfy the hypotheses of Theorem 3. Set $z = re^{i\theta}$, $a_{s,n} = |a_{s,n}|\,e^{i\theta_s}$, $a_{d,n} = |a_{d,n}|\,e^{i\theta_d}$, $b_{s,n} = |b_{s,n}|\,e^{i\varphi}$, $\theta_s, \theta_d, \varphi \in [0, 2\pi)$, $\theta_s \neq \theta_d$,. For any given $\theta \in [\theta_r - \delta_r,\ \theta_r + \delta_r] \diagdown (H_1 \cup H_2)$, we have $\delta(P_s, \theta) \neq 0, \delta(P_d, \theta) \neq 0$ and

$$\delta(P_s, \theta) > \delta(P_d, \theta) \text{ or } \delta(P_s, \theta) < \delta(P_d, \theta).$$

1. $\delta(P_s, \theta) > \delta(P_d, \theta)$. Here we also divide our proof in three cases: $(\varphi = \theta_s)$ or $(\varphi = \theta_d)$ or $(\varphi \neq \theta_s$ and $\varphi \neq \theta_d)$.

   Case 1. $\delta(P_s, \theta) > \delta(P_d, \theta)$ and $(\varphi \neq \theta_s$ and $\varphi \neq \theta_d)$**.**

   Subcase 1.1. $\delta(P_s, \theta) > \delta(P_d, \theta) > 0, \delta(Q_s, \theta) > 0$. If $\delta(P_s, \theta) > 0$, $\delta(Q_s, \theta) > 0$, then we suppose $\delta(P_s, \theta) > \delta(Q_s, \theta)$ without loss of generality. Set $\delta_3 = \max\left\{\delta(P_j, \theta), \delta(Q_j, \theta); j \neq s\right\}$ and $H_3 = \{\theta \in [0, 2\pi) : \delta(Q_s, \theta) = 0, \delta(Q_s, \theta) = \delta(P_s, \theta)\}$, then $0 < \delta_3 < \delta(P_s, \theta)$. Thus by Lemma 3, for any given ε with

$$0 < \varepsilon < \min\left\{\frac{1}{2}\left(\frac{\delta(P_s,\theta) - \delta_3}{\delta(P_s,\theta) + \delta_3}\right), \frac{1}{2}\left(\frac{\delta(P_s,\theta) - \delta(Q_s,\theta)}{\delta(P_s,\theta) + \delta(Q_s,\theta)}\right), n - \rho_1\right\},$$

   where

$$c = \max\left\{c_j, c_j' : j = 0, 1, ..., k-1, j \neq s\right\}, c_s = 1,$$

   there is a set $E_3 \subset [1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_3, r \to +\infty$ and

$$\arg z = \theta \in [\theta_r - \delta_r,\ \theta_r + \delta_r] \diagdown (H_1 \cup H_2 \cup H_3),$$

   we have (13) and (14) hold. From (4), we can write

$$\left|A_{s,1}(z)\,e^{P_s(z)} + A_{s,2}(z)\,e^{Q_s(z)}\right| \leq \left|\frac{f}{f^{(s)}}\right|\left(\left|\frac{F(z)}{f}\right| + \left|\frac{f^{(k)}}{f}\right| + \sum_{j=0, j\neq s}^{k-1}\left\{\left|A_{j,1}(z)\,e^{P_j(z)} + A_{j,2}(z)\,e^{Q_j(z)}\right|\left|\frac{f^{(j)}}{f}\right|\right\}\right). \tag{45}$$

   By substituting (9), (10), (13), (14) and (44) into (45), for all z satisfying $|z| = r_m \notin [0,1] \cup (E_1 \cup E_2 \cup E_3)$, $r_m \to +\infty$ and

$$\arg z = \theta \in [\theta_r - \delta_r,\ \theta_r + \delta_r] \diagdown (H_1 \cup H_2 \cup H_3),$$

   we have

$$\exp\left\{(1-\varepsilon)\,\delta(P_s,\theta)\,r_m^n\right\} \leq 4\,(k+1)\,Br_m^{2s}\left[T(2r_m, f)\right]^{k+1}\exp\left\{2r^{\rho_1+\varepsilon}\right\}\exp\left\{(1+\varepsilon)\,\delta_3 r_m^n\right\}. \tag{46}$$

   Since $0 < \varepsilon < \min\left\{\frac{1}{2}\left(\frac{\delta(P_s,\theta)-\delta_3}{\delta(P_s,\theta)+\delta_3}\right), n - \rho_1\right\}$, then by Lemma 5 and (46), we obtain

$$\rho(f) = \limsup_{r_m \to +\infty} \frac{\log T(r_m, f)}{\log r_m} = +\infty$$

and

$$\rho_2(f) = \limsup_{r_m \to +\infty} \frac{\log \log T(r_m, f)}{\log r_m} \geq n.$$

By Lemma 6 and from (4), we have $\rho_2(f) \leq n$, so $\rho_2(f) = n$.

Subcase 1.2. $\delta(P_s,\theta) > \delta(P_d,\theta) > 0, \delta(Q_s,\theta) < 0$. We have $\delta(Q_s,\theta) < \delta(P_s,\theta)$ and $\delta(Q_s,\theta) < \delta(Q_j,\theta) < 0 < \delta(P_s,\theta)$. Put

$$d = \max\{c_j : j = 0, 1, ..., k-1, j \neq s\}, d_s = 1$$

By Lemma 3, for any given ε $(0 < \varepsilon < \min\left\{\frac{1}{2}\left(\frac{1-d}{1+d}\right), n - \rho_1\right\})$, there is a set $E_3 \subset [1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_3, r \to +\infty$ and

$$\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3),$$

where $H_1 = \{\theta \in [0, 2\pi) : \delta(P_s,\theta) = 0, \delta(P_d,\theta) = 0\}$, $H_2 = \{\theta \in [0, 2\pi) : \delta(P_s,\theta) = \delta(P_d,\theta)\}$ and $H_3 = \{\theta \in [0, 2\pi) : \delta(Q_s,\theta) = 0, \delta(Q_s,\theta) = \delta(P_s,\theta)\}$ are finite sets, we have (17) and (18) hold. By substituting (9), (10), (17), (18) and (44) into (45) for all z satisfying $|z| = r_m \notin [0, 1] \cup (E_1 \cup E_2 \cup E_3)$, $r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3)$, we get

$$\exp\{(1-\varepsilon)\delta(P_s,\theta) r_m^n\} \leq 4(k+1) B r_m^{2k} [T(2r_m, f)]^{k+1} \exp\{2r^{\rho_1+\varepsilon}\} \exp\{(1+\varepsilon) d\, \delta(P_s,\theta) r_m^n\}. \quad (47)$$

Since $0 < \varepsilon < \min\left\{\frac{1}{2}\left(\frac{1-d}{1+d}\right), n - \rho_1\right\}$, by Lemma 5 and (47), we obtain

$$\rho(f) = \limsup_{r_m \to +\infty} \frac{\log T(r_m, f)}{\log r_m} = +\infty$$

and

$$\rho_2(f) = \limsup_{r_m \to +\infty} \frac{\log \log T(r_m, f)}{\log r_m} \geq n.$$

In addition, by Lemma 6 and from (4), we have $\rho_2(f) \leq n$, so $\rho_2(f) = n$.

Subcase 1.3. $\delta(P_s,\theta) > 0 > \delta(P_d,\theta), \delta(Q_s,\theta) > 0$. We suppose $\delta(P_s,\theta) > \delta(Q_s,\theta)$ without loss of generality. By Lemma 3, for any given ε $(0 < \varepsilon < \min\left\{\frac{1}{2}\left(\frac{1-\nu}{1+\nu}\right), \frac{1}{2}\left(\frac{\delta(P_s,\theta)-\delta(Q_s,\theta)}{\delta(P_s,\theta)+\delta(Q_s,\theta)}\right), n - \rho_1\right\})$, where

$$\nu = \max\left\{c_j, c_j' : j = 0, 1, ..., k-1, j \neq s, j \neq d\right\}, \nu_s = 1,$$

there is a set $E_3 \subset [1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_3, r \to +\infty$ and

$$\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3),$$

where $H_1 = \{\theta \in [0, 2\pi) : \delta(P_s,\theta) = 0, \delta(P_d,\theta) = 0\}$, $H_2 = \{\theta \in [0, 2\pi) : \delta(P_s,\theta) = \delta(P_d,\theta)\}$ and $H_3 = \{\theta \in [0, 2\pi) : \delta(Q_s,\theta) = 0, \delta(Q_s,\theta) = \delta(P_s,\theta)\}$ are finite sets, we have (20) and (21) hold. By substituting (9), (10), (20), (21) and (44) into (45) for all z satisfying $|z| = r_m \notin [0, 1] \cup (E_1 \cup E_2 \cup E_3)$, $r_m \to +\infty$ and

$$\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3),$$

we get

$$\exp\{(1-\varepsilon)\delta(P_s,\theta) r_m^n\} \leq 4(k+1) B r_m^{2s} [T(2r_m, f)]^{k+1} \exp\{2r^{\rho_1+\varepsilon}\} \exp\{(1+\varepsilon) \nu \delta(P_s,\theta) r_m^n\}. \quad (48)$$

Since $0 < \varepsilon < \min\left\{\frac{1}{2}\left(\frac{1-\nu}{1+\nu}\right), n - \rho_1\right\}$, then by Lemma 5 and (48), we obtain

$$\rho(f) = \limsup_{r_m \to +\infty} \frac{\log T(r_m, f)}{\log r_m} = +\infty$$

and

$$\rho_2(f) = \limsup_{r_m \to +\infty} \frac{\log \log T(r_m, f)}{\log r_m} \geq n.$$

In addition, then by Lemma 6 and from (4), we have $\rho_2(f) \leq n$, so $\rho_2(f) = n$.

Subcase 1.4. $\delta(P_s, \theta) > 0 > \delta(P_d, \theta), \delta(Q_s, \theta) < 0$. We have $\delta(P_s, \theta) > \delta(P_j, \theta) > 0, \delta(Q_s, \theta) < \delta(Q_j, \theta) < 0$, then $\delta(Q_s, \theta) < \delta(P_s, \theta)$. Put

$$\nu' = \max\{c_j : j = 0, 1, ..., k-1, j \neq s, j \neq d\}, \nu'_s = 1.$$

By Lemma 3, for any given ε $\left(0 < \varepsilon < \min\left\{\frac{1}{2}\left(\frac{1-\nu'}{1+\nu'}\right), n - \rho_1\right\}\right)$, there is a set $E_3 \subset [1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_3, r \to +\infty$ and

$$\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3),$$

where $H_1 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = 0, \delta(P_d, \theta) = 0\}$, $H_2 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = \delta(P_d, \theta)\}$ and $H_3 = \{\theta \in [0, 2\pi) : \delta(Q_s, \theta) = 0, \delta(Q_s, \theta) = \delta(P_s, \theta)\}$ are finite sets, we have (3.16) and (3.17) hold. By substituting (9), (10), (23), (24) and (44) into (45) for all z satisfying $|z| = r_m \notin [0, 1] \cup (E_1 \cup E_2 \cup E_3)$, $r_m \to +\infty$ and

$$\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3),$$

we have

$$\exp\{(1-\varepsilon)\delta(P_s, \theta) r_m^n\} \leq 4(k+1) B r_m^{2k} [T(2r_m, f)]^{k+1} \exp\{2r^{\rho_1+\varepsilon}\} \exp\{(1+\varepsilon)\nu' \delta(P_s, \theta) r_m^n\}. \quad (49)$$

Since $0 < \varepsilon < \min\left\{\frac{1}{2}\left(\frac{1-\nu'}{1+\nu'}\right), n - \rho_1\right\}$, then by Lemma 5 and (49), we obtain

$$\rho(f) = \limsup_{r_m \to +\infty} \frac{\log T(r_m, f)}{\log r_m} = +\infty$$

and

$$\rho_2(f) = \limsup_{r_m \to +\infty} \frac{\log \log T(r_m, f)}{\log r_m} \geq n.$$

In addition, by Lemma 6 and from (4), we have $\rho_2(f) \leq n$, so $\rho_2(f) = n$.

Subcase 1.5. $0 > \delta(P_s, \theta) > \delta(P_d, \theta), \delta(Q_s, \theta) > 0$. We have $\delta(P_d, \theta) < \delta(P_s, \theta) < 0 < \delta(Q_s, \theta)$. Put

$$d' = \max\left\{c'_j : j = 0, 1, ..., k-1, j \neq s\right\}, d'_s = 1.$$

By Lemma 3, for any given ε $\left(0 < \varepsilon < \min\left\{\frac{1}{2}\left(\frac{1-d'}{1+d'}\right), n - \rho_1\right\}\right)$, there is a set $E_3 \subset [1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_3$, $r \to +\infty$ and

$$\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3),$$

where $H_1 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = 0, \delta(P_d, \theta) = 0\}$, $H_2 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = \delta(P_d, \theta)\}$ and $H_3 = \{\theta \in [0, 2\pi) : \delta(Q_s, \theta) = 0, \delta(Q_s, \theta) = \delta(P_s, \theta)\}$ are finite sets, we have (26) and (27) hold. Using a similar proof as that of **Subcase 1.5** of Theorem 3, by (45), we can obtain for all z satisfying $|z| = r_m \notin [0, 1] \cup (E_1 \cup E_2 \cup E_3)$, $r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$

$$\exp\{(1-\varepsilon)\delta(Q_s, \theta) r_m^n\} \leq 4(k+1) B r_m^{2s} [T(2r_m, f)]^{k+1} \exp\{2r^{\rho_1+\varepsilon}\} \exp\{(1+\varepsilon) d' \delta(Q_s, \theta) r_m^n\}. \quad (50)$$

Since $0 < \varepsilon < \min\left\{\frac{1}{2}\left(\frac{1-d'}{1+d'}\right), n - \rho_1\right\}$, by Lemma 5 and (50), we obtain

$$\rho(f) = \limsup_{r_m \to +\infty} \frac{\log T(r_m, f)}{\log r_m} = +\infty$$

and

$$\rho_2(f) = \limsup_{r_m \to +\infty} \frac{\log \log T(r_m, f)}{\log r_m} \geq n.$$

In addition, by Lemma 6 and from (4), we have $\rho_2(f) \leq n$, so $\rho_2(f) = n$.

Subcase 1.6. $0 > \delta(P_s, \theta) > \delta(P_d, \theta), \delta(Q_s, \theta) < 0$. Set

$$\lambda = \min\left\{c_j, c_j' : j = 0, 1, ..., k-1, j \neq s, j \neq d\right\}.$$

By Lemma 3, for any given ε $(0 < \varepsilon < \min\left\{\frac{1}{2}, n - \rho_1\right\})$, there is a set $E_3 \subset [1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_3$, $r \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3)$, where $H_1 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = 0, \delta(P_d, \theta) = 0\}$, $H_2 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = \delta(P_d, \theta)\}$ and $H_3 = \{\theta \in [0, 2\pi) : \delta(Q_s, \theta) = 0, \delta(Q_s, \theta) = \delta(P_s, \theta)\}$ are finite sets, we have (29) holds. From (4), we have

$$1 \leq \left|\frac{f}{f^{(k)}}\right| \left(\left|\frac{F(z)}{f(z)}\right| + \sum_{j=0}^{k-1} \left\{\left|A_{j,1}(z) e^{P_j(z)} + A_{j,2}(z) e^{Q_j(z)}\right| \left|\frac{f^{(j)}}{f}\right|\right\}\right). \tag{51}$$

By substituting (9), (10), (29) and (44) into (51) for all z satisfying $|z| = r_m \notin [0,1] \cup (E_1 \cup E_2 \cup E_3)$, $r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3)$, we get

$$1 \leq 2(k+1) B r_m^{2k} [T(2r_m, f)]^{k+1} \exp\left\{2r^{\rho_1 + \varepsilon}\right\} \exp\left\{(1-\varepsilon) \lambda \delta(P_s, \theta) r_m^n\right\}$$

which gives

$$\exp\left\{(\varepsilon - 1) \lambda \delta(P_s, \theta) r_m^n\right\} \leq 2(k+1) B r_m^{2k} \exp\left\{2r^{\rho_1 + \varepsilon}\right\} [T(2r_m, f)]^{k+1}. \tag{52}$$

Since $0 < \varepsilon < \min\left\{\frac{1}{2}, n - \rho_1\right\}$ and $\delta(P_s, \theta) < 0$, by Lemma 5 and (52), we obtain

$$\rho(f) = \limsup_{r_m \to +\infty} \frac{\log T(r_m, f)}{\log r_m} = +\infty$$

and

$$\rho_2(f) = \limsup_{r_m \to +\infty} \frac{\log \log T(r_m, f)}{\log r_m} \geq n.$$

In addition, by Lemma 6 and from (4), we have $\rho_2(f) \leq n$, so $\rho_2(f) = n$.

Case 2. $\delta(P_s, \theta) > \delta(P_d, \theta)$ and $\varphi = \theta_s$.

Subcase 2.1. $\delta(P_s, \theta) > \delta(P_d, \theta) > 0$. Because of $a_{s,n} \neq b_{s,n}$, we suppose $|a_{s,n}| < |b_{s,n}|$ without loss of generality. In this case, by (11) and (12), we have $\delta(Q_s, \theta) > \delta(Q_j, \theta) > 0$, $\delta(Q_s, \theta) > \delta(P_s, \theta) > \delta(P_j, \theta) > 0$. Put

$$c = \max\left\{c_j, c_j' : j = 0, 1, ..., k-1, j \neq s, j \neq d\right\}, c_s = 1.$$

Then, $0 < c < 1$. By Lemma 3, for any given ε with

$$0 < \varepsilon < \min\left\{\frac{1}{2}\left(\frac{1-c}{1+c}\right), \frac{1}{2}\left(\frac{\delta(Q_s, \theta) - \delta(P_s, \theta)}{\delta(Q_s, \theta) + \delta(P_s, \theta)}\right), n - \rho_1\right\},$$

there is a set $E_3 \subset [1, +\infty)$ having finite logar measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_3, r \to +\infty$ and

$$\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3),$$

where $H_1 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = 0, \delta(P_d, \theta) = 0\}$, $H_2 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = \delta(P_d, \theta)\}$ and $H_3 = \{\theta \in [0, 2\pi) : \delta(Q_s, \theta) = 0\}$ are finite sets, we have (32) and (33) hold. By substituting (9), (10), (32), (33) and (44) into (45), we obtain for all z satisfying $|z| = r_m \notin [0,1] \cup (E_1 \cup E_2 \cup E_3)$, $r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3)$

$$\exp\left\{(1-\varepsilon) \delta(Q_s, \theta) r_m^n\right\} \leq 4(k+1) B r_m^{2s} [T(2r_m, f)]^{k+1} \exp\left\{2r^{\rho_1 + \varepsilon}\right\} \exp\left\{(1+\varepsilon) c \delta(Q_s, \theta) r_m^n\right\}. \tag{53}$$

Since $0<\varepsilon<\min\left\{\frac{1}{2}\left(\frac{1-c}{1+c}\right),n-\rho_1\right\}$, then by Lemma 5 and (53) we obtain

$$\rho\left(f\right)=\limsup_{r_m\to+\infty}\frac{\log T\left(r_m,f\right)}{\log r_m}=+\infty$$

and

$$\rho_2\left(f\right)=\limsup_{r_m\to+\infty}\frac{\log\log T\left(r_m,f\right)}{\log r_m}\geq n.$$

In addition, by Lemma 6 and from (4), we have $\rho_2\left(f\right)\leq n$, so $\rho_2\left(f\right)=n$.

Subcase 2.2. $\delta\left(P_s,\theta\right)>0>\delta\left(P_d,\theta\right)$. Because of $a_{s,n}\neq b_{s,n}$, we suppose $|a_{s,n}|<|b_{s,n}|$ without loss of generality. In this case, by (11) and (12) we have $\delta\left(Q_s,\theta\right)>\delta\left(Q_j,\theta\right)>0$, $\delta\left(Q_s,\theta\right)>\delta\left(P_s,\theta\right)>\delta\left(P_j,\theta\right)>0$. Put

$$c=\max\left\{c_j,c_j':j=0,1,...,k-1,j\neq s,j\neq d\right\},c_s=1.$$

Using the same reasoning as in **Subcase 2.1** , we can also obtain $\rho\left(f\right)=+\infty$ and $\rho_2\left(f\right)=n$.

Subcase 2.3. $0>\delta\left(P_s,\theta\right)>\delta\left(P_d,\theta\right)$. We have $\delta\left(Q_s,\theta\right)<\delta\left(Q_j,\theta\right)<\delta\left(P_s,\theta\right)<0$, $\delta\left(P_s,\theta\right)<\delta\left(P_j,\theta\right)<0$. Put

$$\lambda=\min\left\{c_j,c_j':j=0,1,...,k-1,j\neq s,j\neq d\right\}.$$

By Lemma 3, for any given ε ($0<\varepsilon<\min\left\{\frac{1}{2},n-\rho_1\right\}$), there is a set $E_3\subset[1,+\infty)$ having finite logarithmic measure such that for all z satisfying $|z|=r\notin[0,1]\cup E_3,r\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\ \theta_r+\delta_r]\diagdown\left(H_1\cup H_2\cup H_3\right)$, where $H_1=\{\theta\in[0,2\pi):\delta\left(P_s,\theta\right)=0,\delta\left(P_d,\theta\right)=0\}$, $H_2=\{\theta\in[0,2\pi):\delta\left(P_s,\theta\right)=\delta\left(P_d,\theta\right)\}$ and $H_3=\{\theta\in[0,2\pi):\delta\left(Q_s,\theta\right)=0\}$ are finite sets, we have (35) holds. By (9), (10), (35), (44) and (51), we have for all z satisfying $|z|=r_m\notin[0,1]\cup\left(E_1\cup E_2\cup E_3\right),r_m\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\ \theta_r+\delta_r]\diagdown(H_1\cup H_2\cup H_3)$,

$$1\leq 2\left(k+1\right)Br_m^{2k}\left[T\left(2r_m,f\right)\right]^{k+1}\exp\left(2r^{\rho_1+\varepsilon}\right)\exp\left\{\left(1-\varepsilon\right)\lambda\delta\left(P_s,\theta\right)r_m^n\right\},$$

which gives

$$\exp\left\{\left(\varepsilon-1\right)\lambda\delta\left(P_s,\theta\right)r_m^n\right\}\leq 2\left(k+1\right)Bkr_m^{2k}\exp\left(2r^{\rho_1+\varepsilon}\right)\left[T\left(2r_m,f\right)\right]^{k+1}. \tag{54}$$

Since $0<\varepsilon<n-\rho_1$ and $\delta\left(P_s,\theta\right)<0$, then by Lemma 5 and (54), we obtain

$$\rho\left(f\right)=\limsup_{r_m\to+\infty}\frac{\log T\left(r_m,f\right)}{\log r_m}=+\infty$$

and

$$\rho_2\left(f\right)=\limsup_{r_m\to+\infty}\frac{\log\log T\left(r_m,f\right)}{\log r_m}\geq n.$$

In addition, by Lemma 6 and from (4), we have $\rho_2\left(f\right)\leq n$, so $\rho_2\left(f\right)=n$.

Case 3. $\delta\left(P_s,\theta\right)>\delta\left(P_d,\theta\right)$ and $\varphi=\theta_d$.

Subcase 3.1. $\delta\left(P_s,\theta\right)>\delta\left(P_d,\theta\right)>0$. Because of $a_{d,n}\neq b_{s,n}$, we suppose $|a_{d,n}|<|b_{s,n}|$ without loss of generality. In this case, by (11) and (12), we have $\delta\left(Q_s,\theta\right)>\delta\left(Q_j,\theta\right)>0$, $\delta\left(Q_s,\theta\right)>\delta\left(P_s,\theta\right)>\delta\left(P_j,\theta\right)>0$. Then, $0<c<1$. By Lemma 3, for any given ε with

$$0<\varepsilon<\min\left\{\frac{1}{2}\left(\frac{1-c}{1+c}\right),\frac{1}{2}\left(\frac{\delta\left(Q_s,\theta\right)-\delta\left(P_s,\theta\right)}{\delta\left(Q_s,\theta\right)+\delta\left(P_s,\theta\right)}\right),n-\rho_1\right\},$$

where

$$c=\max\left\{c_j,c_j':j=0,1,...,k-1,j\neq s,j\neq d\right\},c_s=1,$$

there is a set $E_3\subset[1,+\infty)$ having finite logarithmic measure such that for all z satisfying $|z|=r\notin[0,1]\cup E_3,r\to+\infty$ and

$$\arg z=\theta\in[\theta_r-\delta_r,\ \theta_r+\delta_r]\diagdown\left(H_1\cup H_2\cup H_3\right),$$

where $H_1 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = 0, \delta(P_d, \theta) = 0\}$, $H_2 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = \delta(P_d, \theta)\}$ and $H_3 = \{\theta \in [0, 2\pi) : \delta(Q_s, \theta) = 0\}$ are finite sets, we have (32) and (33) hold. By substituting (9), (10), (32), (33) and (44) into (45), we obtain

$$\exp\{(1-\varepsilon)\delta(Q_s,\theta)r_m^n\} \le 4(k+1)Br_m^{2s}[T(2r_m,f)]^{k+1}\exp\left(2r^{\rho_1+\varepsilon}\right)\exp\{(1+\varepsilon)c\delta(Q_s,\theta)r_m^n\} \tag{55}$$

for all z satisfying $|z| = r_m \notin [0,1] \cup (E_1 \cup E_2 \cup E_3)$, $r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3)$. Since $0 < \varepsilon < \min\left\{\frac{1}{2}\left(\frac{1-c}{1+c}\right), n - \rho_1\right\}$, then by Lemma 5 and (55), we obtain

$$\rho(f) = \limsup_{r_m \to +\infty} \frac{\log T(r_m, f)}{\log r_m} = +\infty$$

and

$$\rho_2(f) = \limsup_{r_m \to +\infty} \frac{\log\log T(r_m, f)}{\log r_m} \ge n.$$

In addition, by Lemma 6 and from (4), we have $\rho_2(f) \le n$, so $\rho_2(f) = n$.

Subcase 3.2. $\delta(P_s,\theta) > 0 > \delta(P_d,\theta)$. Because of $a_{d,n} \ne b_{s,n}$, we suppose $|a_{d,n}| < |b_{s,n}|$ without loss of generality. In this case, by (11) and (12), we have $\delta(Q_s,\theta) < \delta(Q_j,\theta) < 0$, $\delta(Q_s,\theta) < \delta(P_d,\theta) < 0 < \delta(P_s,\theta)$. Then, $0 < c < 1$. By Lemma 3, for any given ε $\left(0 < \varepsilon < \min\left\{\frac{1}{2}\left(\frac{1-c}{1+c}\right), n-\rho_1\right\}\right)$, where

$$c = \max\left\{c_j, c_j' : j = 0, 1, ..., k-1, j \ne s, j \ne d\right\}, c_s = 1,$$

there is a set $E_3 \subset [1, +\infty)$ having finite logar measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_3, r \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H_1 \cup H_2 \cup H_3$, where $H_1 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = 0, \delta(P_d, \theta) = 0\}$, $H_2 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = \delta(P_d, \theta)\}$ and $H_3 = \{\theta \in [0, 2\pi) : \delta(Q_s, \theta) = 0\}$ are finite sets, we have (37) and (38) hold. Substituting (9), (10), (37), (38) and (44) into (45), we obtain for all z satisfying $|z| = r_m \notin [0,1] \cup (E_1 \cup E_2 \cup E_3)$, $r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3)$

$$\exp\{(1-\varepsilon)\delta(P_s,\theta)r_m^n\} \le 4(k+1)Br_m^{2s}[T(2r_m,f)]^{k+1}\exp\{(1+\varepsilon)c\delta(P_s,\theta)r_m^n\}. \tag{56}$$

Since $0 < \varepsilon < \min\left\{\frac{1}{2}\left(\frac{1-c}{1+c}\right), n - \rho_1\right\}$, then by Lemma 5 and (56), we obtain

$$\rho(f) = \limsup_{r_m \to +\infty} \frac{\log T(r_m, f)}{\log r_m} = +\infty$$

and

$$\rho_2(f) = \limsup_{r_m \to +\infty} \frac{\log\log T(r_m, f)}{\log r_m} \ge n.$$

In addition, by Lemma 6 and from (4), we have $\rho_2(f) \le n$, so $\rho_2(f) = n$.

Subcase 3.3. $0 > \delta(P_s,\theta) > \delta(P_d,\theta)$. Because of $a_{d,n} \ne b_{s,n}$, we suppose $|a_{d,n}| < |b_{s,n}|$ without loss of generality. In this case, by (11) and (12), we have $\delta(Q_s,\theta) < \delta(Q_j,\theta) < 0$, $\delta(Q_s,\theta) < \delta(P_d,\theta) < \delta(P_s,\theta) < 0$. Put

$$c' = \min\left\{c_j, c_j' : j = 0, 1, ..., k-1\right\}.$$

By Lemma 3, for any given ε $(0 < \varepsilon < \min\left\{\frac{1}{2}, n - \rho_1\right\})$, there is a set $E_3 \subset [1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_3, r \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3)$, where $H_1 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = 0, \delta(P_d, \theta) = 0\}$, $H_2 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = \delta(P_d, \theta)\}$ and $H_3 = \{\theta \in [0, 2\pi) : \delta(Q_s, \theta) = 0\}$ are finite sets, we have (40) holds. By substituting (9), (10), (40) and (44) into (51), we obtain for all z satisfying $|z| = r_m \notin [0,1] \cup (E_1 \cup E_2 \cup E_3)$, $r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2 \cup H_3)$

$$1 \le 2(k+1)Br_m^{2k}[T(2r_m,f)]^{k+1}\exp\left(2r^{\rho_1+\varepsilon}\right)\exp\{(1-\varepsilon)c'\delta(P_s,\theta)r_m^n\}$$

which gives

$$\exp\{(\varepsilon-1)c'\delta(P_s,\theta)r_m^n\} \le 2(k+1)Br_m^{2k}[T(2r_m,f)]^{k+1}\exp\left(2r^{\rho_1+\varepsilon}\right). \tag{57}$$

By Lemma 5 and (57), we obtain

$$\rho(f) = \limsup_{r_m \to +\infty} \frac{\log T(r_m, f)}{\log r_m} = +\infty$$

and

$$\rho_2(f) = \limsup_{r_m \to +\infty} \frac{\log \log T(r_m, f)}{\log r_m} \geq n.$$

In addition, by Lemma 6 and from (4), we have $\rho_2(f) \leq n$, so $\rho_2(f) = n$.

II. $\delta(P_s, \theta) < \delta(P_d, \theta)$. Here we also divide our proof in three subcases: $(\varphi = \theta_s)$ or $(\varphi = \theta_d)$ or $(\varphi \neq \theta_s$ and $\varphi \neq \theta_d)$. Using the same reasoning as in **I**, we can obtain $\rho_2(f) \leq n$, so $\rho_2(f) = n$.

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