Open Journal of
Mathematical Sciences

PSR Press

Article

Exploiting quadratic $\varphi(\delta_1, \delta_2)-$function inequalities on fuzzy Banach spaces based on general quadratic equations with $2k$-variables

Ly Van An

[1] Faculty of Mathematics Teacher Education, Tay Ninh University, Tay Ninh, Vietnam
* Correspondence: lyvanan145@gmail.com, lyvananvietnam@gmail.com

Communicated by: Waqas Nazeer
Received: 28 June 2023; Accepted: 11 October 2023; Published: 27 December 2023.

Abstract: In this manuscript, our primary focus revolves around extending the inequalities associated with the Quadratic $\varphi(\delta_1, \delta_2)-$function. Our approach involves leveraging the general quadratic functional equation encompassing 2k-variables within the context of the fuzzy Banach space. Our main contribution lies in the expansion of these inequalities, representing a significant result within this study.

Keywords: Generalized quadratic type $\varphi(\delta_1, \delta_2)-$functional inequality; Generalized quadratic type functional equations; Fuzzy Banach space; Fuzzy normed vector spaces.

MSC: 39B22, 39B82, 46S10.

1. Introduction

Let $\mathbf{X}$ and $\mathbf{Y}$ be fuzzy normed spaces defined over the same field $\mathbb{K}$, and let $f : \mathbf{X} \to \mathbf{Y}$ be a mapping. We denote the norms on $\mathbf{X}$ and $\mathbf{Y}$ as N and N, respectively. In this paper, we investigate the relationship between Quadratic-type functional equations and Quadratic $\varphi(\delta_1, \delta_2)$-function inequalities when $(\mathbf{X}, N)$ is a fuzzy normed space and $(\mathbf{Y}, N)$ is a fuzzy Banach space. Specifically, when $\mathbf{X}$ is a fuzzy normed space and $\mathbf{Y}$ is a fuzzy Banach space, we address the Hyers-Ulam stability of the following relationship between quadratic (μ_1, μ_2)-function inequalities and quadratic-type functional equations: :

$$\begin{aligned}
N\Bigg(&2kf\Big(\frac{\sum_{i=1}^{k}(x_i)+\sum_{i=1}^{k}y_i}{2k}\Big)+2kf\Big(\frac{\sum_{i=1}^{k}(x_i)-\sum_{i=1}^{k}y_i}{2k}\Big)-\sum_{i=1}^{k}f(x_i)-\sum_{i=1}^{k}f(y_i)\\
&-\delta_1\Big(f\Big(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i\Big)+f\Big(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\Big)-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)\Big)\\
&-\delta_2\Big(4kf\Big(\frac{\sum_{i=1}^{k}(x_i)+\sum_{i=1}^{k}y_i}{2k}\Big)+f\big(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\big)-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)\Big),t\Bigg)\\
&\geq\frac{t}{t+\psi(x_1,...,x_k,y_1,...,y_k)}
\end{aligned}\tag{1}$$

based on following Generalized Quadratic functional equations with $2k$-variable

$$f\Big(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i\Big)+f\Big(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\Big)=2\sum_{i=1}^{k}f(x_i)+2\sum_{k=1}^{k}f(y_i)\tag{2}$$

$$D=\left\{\varphi:\mathbb{R}\to\mathbb{R}:\varphi(\delta_1,\delta_2)=\sum_{j=1}^{2}\delta_j,\varphi(\delta_1,\delta_2)\neq\frac{1}{2},\delta_1,\delta_2\in\mathbb{R},\varphi(\delta_1,\delta_2)-\delta_1\neq 1\right\}\tag{3}$$

Note: Let k be a positive integer, and $h \in A$. The study of functional equation stability originated from a question of S.M. Ulam [1] concerning the stability of group homomorphisms.

Let $(\mathbb{G}, *)$ be a group, and let $(\mathbb{G}', \circ, d)$ be a metric group with metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : \mathbb{G} \to \mathbb{G}'$ satisfies

$$d\Big(f\big(x * y\big), f(x) \circ f(y)\Big) < \delta$$

for all $x, y \in \mathbb{G}$, then there is a homomorphism $h : \mathbb{G} \to \mathbb{G}'$ with

$$d\Big(f(x), h(x)\Big) < \epsilon$$

for all $x \in \mathbb{G}$? If the answer is affirmative, we would say that the equation of homomorphism $h(x * y) = h(y) \circ h(y)$ is stable.

The concept of stability for a functional equation arises when we replace a functional equation by an inequality that acts as a perturbation of the equation. Thus, the stability question of functional equations is: How do the solutions of the inequality differ from those of the given functional equation?

Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Th.M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvrut [5] by replacing the unbounded Cauchy difference with a general control function in the spirit of Th.M. Rassias' approach.

The stability problems of several functional equations have been extensively studied through the process of examining the works of mathematicians such as ([6–11]).

In 2020, we set up a general quadratic equation with $2k$ variables in the Non-Archimedean Banach space:

$$f\Big(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i\Big) + f\Big(\sum_{i=1}^{k} x_i - \sum_{i=1}^{k} y_i\Big) - 2\sum_{i=1}^{k} f(x_i) - 2\sum_{i=1}^{k} f(y_i). \tag{4}$$

Next, also in 2020, we built quadratic inequalities in the application of groups and rings:

$$\left\| f\Big(\sum_{j=1}^{n} x_j + \frac{1}{n}\sum_{j=1}^{n} x_{n+j}\Big) + f\Big(\sum_{j=1}^{n} x_j - \frac{1}{n}\sum_{j=1}^{n} x_{n+j}\Big) - 2\sum_{j=1}^{n} f\big(x_j\big) - 2\sum_{j=1}^{n} f\Big(\frac{x_{n+j}}{n}\Big)\right\|_{\mathbb{Y}} \le \epsilon, \tag{5}$$

for all $\epsilon \ge 0$, and

$$\left\| f\Big(\prod_{j=1}^{n} x_j + \frac{1}{n}\prod_{j=1}^{n} x_{n+j}\Big) + f\Big(\prod_{j=1}^{n} x_j - \frac{1}{n}\prod_{j=1}^{n} x_{n+j}\Big) - 2\prod_{j=1}^{n} f\big(x_j\big) - 2\prod_{j=1}^{n} f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}} \le \delta, \tag{6}$$

for all $\delta \ge 0$.

In 2021, we constructed quadratic inequality functional inequalities in non-Archimedean Banach spaces and Banach spaces:

$$\begin{aligned} &\left\| F\Big(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} + \sum_{j=1}^{k} x_j\Big) + F\Big(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} - \sum_{j=1}^{k} x_j\Big) - 2\sum_{j=1}^{k} F\big(\frac{x_{k+j}}{k}\big) - 2\sum_{j=1}^{k} F(x_j)\right\|_{\mathbf{X}_2} \\ \le &\left\| F\Big(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} + \frac{1}{k}\sum_{j=1}^{k} x_j\Big) + F\Big(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} - \frac{1}{k}\sum_{j=1}^{k} x_j\Big) - \frac{2}{k}\sum_{j=1}^{k} F\big(\frac{x_{k+j}}{k}\big) - \frac{2}{k}\sum_{j=1}^{k} F(x_j)\right\|_{\mathbf{X}_2} \end{aligned} \tag{7}$$

and

$$\Big\|F\Big(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\Big)+F\Big(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}-\frac{1}{k}\sum_{j=1}^{k}x_j\Big)-\frac{2}{k}\sum_{j=1}^{k}F\Big(\frac{x_{k+j}}{k}\Big)-\frac{2}{k}\sum_{j=1}^{k}F(x_j)\Big\|_{\mathbf{X}_2}$$

$$\leq\Big\|F\Big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\Big)+F\Big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}-\sum_{j=1}^{k}x_j\Big)-2\sum_{j=1}^{k}F\Big(\frac{x_{k+j}}{k}\Big)-2\sum_{j=1}^{k}F(x_j)\Big\|_{\mathbf{X}_2}, \tag{8}$$

Continuing into 2023, we generalized the stability of functional inequalities with $3k$ variables associated with Jordan-von Neumann-type additive functional equations:

$$\Big\|\sum_{j=1}^{k}f(x_j)+\sum_{j=1}^{k}f(y_j)+\sum_{j=1}^{k}f(z_j)\Big\|_{\mathbf{Y}}\leq\Big\|2kf\Big(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j}{2k}\Big)\Big\|_{\mathbf{Y}}, \tag{9}$$

and

$$\Big\|\sum_{j=1}^{k}f(x_j)+\sum_{j=1}^{k}f(y_j)+\sum_{j=1}^{k}f(z_j)\Big\|_{\mathbf{Y}}\leq\Big\|f\Big(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j\Big)\Big\|_{\mathbf{Y}}, \tag{10}$$

Finally, in 2023, we constructed a broadly derived fuzzy Banach algebra involving functional equations and general Cauchy-Jensen functional inequalities:

$$\Big\|\sum_{j=1}^{k}f(x_j)+\sum_{j=1}^{k}f(y_j)+f\Big(2k\sum_{j=1}^{k}z_j\Big)\Big\|\leq\Big\|2kf\Big(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\Big)\Big\|. \tag{11}$$

We aim to solve and prove the Hyers-Ulam-type stability for the functional equation (1.1), i.e., the functional equation with $2k$ variables. Under suitable assumptions on spaces $\mathbf{X}$ and $\mathbf{Y}$, we will demonstrate that mappings satisfying the functional equations (1.1). The results presented in this paper generalize those found in [12–16].

The study of quadratic functional inequalities in fuzzy Banach spaces is based on recent research and the works of mathematicians worldwide, with no restriction on the number of variables, in order to improve the inequality from finite dimensional to multidimensional space.

The paper is organized as follows:

In the preliminary section, we provide a reminder of some basic notations from [17–23], including fuzzy normed spaces, the Extended metric space theorem, and solutions of the Jensen function equation. In Section 3, we set up quadratic (μ_1, μ_2)-function inequalities (1.1) based on quadratic equation (1.2)

1.1. Condition for existence of solution of (1.1)

1.2. Establishing a solution for the quadratic $h(\mu_1, \mu_2)$-function inequality (1.1)

.

2. preliminaries

2.1. Fuzzy normed spaces.

Definition 1. Let X be a real vector space. Afunction $N : X \times R \to [0, 1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

1. (N1) $N(x, t) = 0$ for $t \leq 0$;

2. (N2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;

3. (N3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$
4. (N4) $N(x + y, s + t) \geq min\{N(x, s), N(y, t)\}$;
5. (N5) $N(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
6. (N6) for $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed vector space

1. Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n\to\infty} x_n = x$.
2. Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in N$ such that for all $n = n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

It is well-known that every convergent sequence in a fuzzy normedvector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space. We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X.

Definition 2. Let X be an algebra and (X, N) a fuzzy normed space.

1. The fuzzy normed space (X, N) is called a fuzzy normed algebra if
$$N(xy, st) \geq N(x, s) \cdot N(y, t)$$
for all $x, y \in X$ and all positive real numbers s and t.
2. A complete fuzzy normed algebra is called a fuzzy Banach algebra.

Definition 3. Let (X, N_X) and (Y, N) be fuzzy normed algebras. Then a multiplicative $\mathbb{R}-$linear mapping $H : (X, N_X) \to (Y, N)$ is called a fuzzy algebra homomorphism.

EXAMPLE

Let $(X, \|\cdot\|)$ be a normed algebra. Let

$$N(x,t) = \begin{cases} \frac{t}{t+||x||} & t > 0 \\ 0 & t \leq 0 \; x \in X \end{cases}$$

Then $N(x, t)$ is a fuzzy norm on X and $(X, N(x, t))$ is a fuzzy normed algebra.

Definition 4. Let (X, N_X) and (Y, N) be fuzzy normed algebras. Then a multiplicative $\mathbb{R}-$linear mapping $H : (X, N_X) \to (Y, N)$ is called a fuzzy algebra homomorphism.

2.2. Extended metric space theorem.

Theorem 1. *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n, J^{n+1}) = \infty$$

for all nonegative integers n or there exists a positive integer n_0 such that

1. *$d(J^n, J^{n+1}) < \infty, \forall n \geq n_0$;*
2. *The sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
3. *y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^n, J^{n+1}) < \infty\}$;*
4. *$d(y, y^*) \leq \frac{1}{1-l} d(y, Jy) \; \forall y \in Y$*

2.3. Solutions of the equation.

The functional equation
$$f(x+y)+f(x-y)=2f(x)+2f(y)$$
is called the Qquadratic equation. In particular, every solution of the quadratic equation is said to be a quadratic mapping.

2.4. Solutions of the inequalities.

The solution of the quadratic function inequalities is called the quadratic mapping.

3. S

etting up quadratic (μ_1,μ_2)-function inequalities (1.1) based on quadratic equation (1.2)

3.1. Condition for existence of solution of (1.1)

In this section, assume that $\mathbf{X}$ and $\mathbf{Y}$ be a fuzzy normed vector spaces. Under this setting, we can show that the mappings satisfying (1.1) is quadratic and $h \in A$.

Lemma 1. *Suppose that $f:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfying*
$$f(0)=0 \tag{12}$$
and
$$\begin{aligned}&2kf\Big(\frac{\sum_{i=1}^{k}(x_i)+\sum_{i=1}^{k}y_i}{2k}\Big)+2kf\Big(\frac{\sum_{i=1}^{k}(x_i)-\sum_{i=1}^{k}y_i}{2k}\Big)-\sum_{i=1}^{k}f(x_i)-\sum_{i=1}^{k}f(y_i)\\&=\delta_1\Big(f\Big(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i\Big)+f\Big(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\Big)-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)\Big)+\\&\delta_2\Big(4kf\Big(\frac{\sum_{i=1}^{k}(x_i)+\sum_{i=1}^{k}y_i}{2k}\Big)+g(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i)-2\sum_{i=1}^{k}g(x_i)-2\sum_{i=1}^{k}g(y_i)\Big)\end{aligned}\tag{13}$$

For all $x_i,y_i\in\mathbf{X}, i=1\to k$ then mapping $f:\mathbf{X}\to\mathbf{Y}$ is quadratic.

Proof. I replacing $(x_1,...,x_k,y_1,...,y_k)$ by$(x,...,x,x,...,x)$ in (32), we have
$$\delta_1\Big(f(2kx)-4kf(x)\Big)=0 \tag{14}$$
So
$$f(2kx)=4kf(x) \tag{15}$$
For all $x\in\mathbf{X}$
Therefore (32) I have
$$\begin{aligned}&\frac{1}{2}\Big(4kf\Big(\frac{\sum_{i=1}^{k}(x_i)+\sum_{i=1}^{k}y_i}{2k}\Big)+4kf\Big(\frac{\sum_{i=1}^{k}(x_i)-\sum_{i=1}^{k}y_i}{2k}\Big)-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)\Big)\\&=\delta_1\Big(f\Big(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i\Big)+f\Big(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\Big)-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)\Big)+\\&\delta_2\Big(4kf\Big(\frac{\sum_{i=1}^{k}(x_i)+\sum_{i=1}^{k}y_i}{2k}\Big)+f(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i)-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)\Big)\end{aligned}\tag{16}$$
so from (14) and (16) I have

$$\frac{1}{2}\Big(f\Big(\sum_{i=1}^{k}(x_i)+\sum_{i=1}^{k}y_i\Big)+f\Big(\sum_{i=1}^{k}(x_i)-\sum_{i=1}^{k}y_i\Big)-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)\Big)$$
$$=\delta_1\Big(f\Big(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i\Big)+f\Big(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\Big)-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)\Big)+$$
$$\delta_2\Big(f\Big(\sum_{i=1}^{k}(x_i)+\sum_{i=1}^{k}y_i\Big)+f(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i)-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)\Big) \tag{17}$$

Thus from (17) I have

$$\frac{1}{2}f\Big(\sum_{i=1}^{k}(x_i)+\sum_{i=1}^{k}y_i\Big)+f\Big(\sum_{i=1}^{k}(x_i)-\sum_{i=1}^{k}y_i\Big)-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)\Big)$$
$$=\varphi(\delta_1,\delta_2)\Big(f\Big(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i\Big)+f\Big(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\Big)-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)\Big) \tag{18}$$

So the from hypothesis (3) and (18) infer that

$$f\Big(\sum_{i=1}^{k}(x_i)+\sum_{i=1}^{k}y_i\Big)+f\Big(\sum_{i=1}^{k}(x_i)-\sum_{i=1}^{k}y_i\Big)-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)=0 \tag{19}$$

For all $x_1,...,x_k,y_1,...,y_k\in\mathbf{X}$. Hence f is quadratic mapping as we expected.
□

3.2. Establishing a solution for the quadratic $\varphi(\delta_1,\delta_2)$-function inequality (1.1)

In this section, assume that $(\mathbf{X},N)$ is a fuzzy normed space and $(\mathbf{Y},N)$ is a fuzzy Banach space. Under this setting, we can show that the mappings satisfying (1.1) is quadratic and $h\in A$.

Theorem 2. *Suppose that $\psi:\mathbf{X}^{2k}\to[0,\infty)$ be a function such that there exists an $L<1$*

$$\psi\Big(x_1,...,x_k,y_1,...,y_k\Big)\le\frac{L}{4k}\psi(2kx_1,...,2kx_1,2ky_1,...,2ky_k) \tag{20}$$

for all $x_j,y_j\in\mathbf{X}$ for $j=1\to k$.
Let $f:\mathbf{X}\to\mathbf{Y}$ be a mapping sattisfying

$$f(0)=0 \tag{21}$$

and

$$N\Bigg(2kf\Big(\frac{\sum_{i=1}^{k}(x_i)+\sum_{i=1}^{k}y_i}{2k}\Big)+2kf\Big(\frac{\sum_{i=1}^{k}(x_i)-\sum_{i=1}^{k}y_i}{2k}\Big)-\sum_{i=1}^{k}f(x_i)-\sum_{i=1}^{k}f(y_i)$$
$$-\delta_1\Big(f\Big(\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}y_i\Big)+f\Big(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i\Big)-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)\Big)$$
$$-\delta_2\Big(4kf\Big(\frac{\sum_{i=1}^{k}(x_i)+\sum_{i=1}^{k}y_i}{2k}\Big)+f(\sum_{i=1}^{k}x_i-\sum_{i=1}^{k}y_i)-2\sum_{i=1}^{k}f(x_i)-2\sum_{i=1}^{k}f(y_i)\Big),t\Bigg)$$
$$\ge\frac{t}{t+\psi(x_1,...,x_k,y_1,...,y_k)} \tag{22}$$

for all $x_j,y_j\in\mathbf{X}$ for $j=1\to k$, for all $t>0$
. Then

$$G(x)=N-\lim_{n\to\infty}(4k)^nf(\frac{1}{(2k)^n}x) \tag{23}$$

exists each $x\in\mathbf{X}$ and defines a quadratic mapping $G:\mathbf{X}\to\mathbf{Y}$

such that

$$N\Big(f(x)-G(x),t\Big)\geq\frac{(1-\varphi(\delta_1,\delta_2)+\delta_1)(1-L)t}{(1-\varphi(\delta_1,\delta_2)+\delta_1)(1-L)t+\psi(x,...,x,x,...,x)} \tag{24}$$

for all $x\in\mathbf{X}$ *and* $t>0$.

Proof. I replacing $(x_1,...,x_k,y_1,...,y_k)$ by $(x,0,...,,0,...,0)$ in (22), I have

$$(1-\varphi(\delta_1,\delta_2)+\delta_1)N\Big(4kf(\frac{x}{2k})-f(x)\Big),t\Big)\geq\frac{t}{t+\varphi(x,0,...,0,0,...,0)} \tag{25}$$

for all $x\in\mathbf{X}$. Now we consider the set

$$\mathbb{M}:=\Big\{h:\mathbf{X}\to\mathbf{Y},h(0)=0\Big\}$$

, and introduce the generalized metric on $\mathbb{M}$ as follows:

$$\begin{aligned}d(g,h):=inf\Big\{\beta\in\mathbb{R}_+:N\Big(g(x)-h(x),\beta t\Big)\\ \geq\frac{t}{t+\varphi(x,0,...,0,0,...,0)},\forall x\in\mathbf{X},\forall t>0\Big\},\end{aligned} \tag{26}$$

where, as usual, $inf\phi=+\infty$. That has been proven by mathematicians $(\mathbb{M},d)$ is complete see[24]

Now we cosider the linear mapping $T:\mathbb{M}\to\mathbb{M}$ such that

$$Tg(x):=4kg(\frac{x}{2k})$$

for all $x\in\mathbf{X}$. Let $g,h\in\mathbb{M}$ be given such that $d(g,h)=\epsilon$ then

$$N\Big(g(x)-h(x),\epsilon t\Big)\geq\frac{t}{t+\varphi(x,...,x,x,...,x)},\forall x\in\mathbf{X},\forall t>0.$$

Hence

$$\begin{aligned}N\Big(g(x)-h(x),\epsilon Lt\Big)&=N\Big(4kg(\frac{x}{2k})-4kh(\frac{x}{2k}),L\epsilon t\Big)\\ &=N\Big(g(\frac{x}{2k})-h(\frac{x}{2k}),\frac{L}{4k}\epsilon t\Big)\\ &\geq\frac{\frac{L}{4k}}{,\frac{L}{4k}+\varphi(\frac{x}{2k},...,\frac{x}{2k},\frac{x}{2k},...,\frac{x}{2k})}\geq\\ &\geq\frac{\frac{L}{4k}}{\frac{L}{4k}+\frac{L}{4k}\varphi(x,0...,0,0,...,0)}\\ &=\frac{t}{t+\varphi(x,0,...,0,0,...,0)},\forall x\in\mathbf{X},\forall t>0.\end{aligned} \tag{27}$$

. So $d(g,h)=\epsilon$ implies that $d(Tg,Th)\leq L\cdot\epsilon$. This means that

$$d(Tg,Th)\leq Ld(g,h)$$

for all $g,h\in\mathbb{M}$. It folows from (25) that

$$N\Big(f(x)-4kf(\frac{x}{2k}),\frac{t}{1-(\varphi(\delta_1,\delta_2)-\delta_1)}\geq\frac{t}{t+\varphi(x,0,...,0,0,...,0)} \tag{28}$$

for all $x\in\mathbb{X}$. So $d(f,Tf)\leq\frac{1}{\varphi(\delta_1,\delta_2)-\delta_1}$. By Theorem 1.2, there exists a mapping $A:\mathbf{X}\to\mathbf{Y}$ satisfying the fllowing:

1. A is a fixed point of T, ie.,

$$A(\frac{1}{2k}) = \frac{1}{4k}A(x) \tag{29}$$

for all $x \in \mathbf{X}$. The mapping A is a unique fixed point T in the set

$$\mathbb{Q} = \Big\{ g \in \mathbb{M} : d(f,g) < \infty \Big\}$$

This implies that A is a unique mapping satisfying (29) such that there exists a $\beta \in (0,\infty)$ satisfying.

$$N\Big(f(x) - G(x), \beta t\Big) \geq \frac{t}{t + \varphi(x, 0, ..., 0, 0, ..., 0)}, \forall x \in \mathbf{X}.$$

2. $d(T^l f, H) \to 0$ as $l \to \infty$. This implies equality

$$N - \lim_{l\to\infty} \frac{1}{(4k)^l} f\Big((2k)^l x\Big) = G(x)$$

for all $x \in \mathbf{X}$

3. $d(f,G) \leq \frac{1}{1-L} d(f,Tf)$.

4. Which implies the inequality

$$d(f,G) \leq \frac{1}{(1 - \varphi(\delta_1,\delta_2) + \delta_1)(1-L)}$$

. This implies that the inequality (24) holds

By (22)

$$\begin{aligned} &N\Bigg((4k)^l\Big(2kf\Big(\frac{\sum_{i=1}^k (x_i) + \sum_{i=1}^k y_i}{(2k)^l}\Big) + 2kf\Big(\frac{\sum_{i=1}^k (x_i) - \sum_{i=1}^k y_i}{(2k)^l}\Big) - \sum_{i=1}^k f(\frac{x_i}{(2k)^l}) - \sum_{i=1}^k f(\frac{y_i}{(2k)^l}) \\ &- (4k)^k\delta_1\Big(f\Big(\frac{\sum_{i=1}^k x_i + \sum_{i=1}^k y_i}{(2k)^l}\Big) + f\Big(\frac{\sum_{i=1}^k x_i - \sum_{i=1}^k y_i}{(2k)^l}\Big) - 2\sum_{i=1}^k f(\frac{x_i}{(2k)^l}) - 2\sum_{i=1}^k f(\frac{y_i}{(2k)^l})\Big) \\ &- (4k)^k\delta_2\Big(4kf\Big(\frac{\sum_{i=1}^k (x_i) + \sum_{i=1}^k y_i}{(2k)^l}\Big) + f(\frac{\sum_{i=1}^k x_i - \sum_{i=1}^k y_i}{(2k)^l}) - 2\sum_{i=1}^k f(\frac{x_i}{(2k)^l}) \\ &- 2\sum_{i=1}^k f(\frac{y_i}{(2k)^l})\Big), (4k)^l t\Bigg) \\ &\geq \frac{t}{t + \psi(\frac{x_1}{(2k)^l}, ..., \frac{x_k}{(2k)^l}, \frac{y_1}{(2k)^l}, ..., \frac{y_k}{(2k)^l})} \end{aligned} \tag{30}$$

for all $x_j, y_j \in \mathbf{X}$ for $j = 1 \to k$, for all $t > 0$ and for all $l \in \mathbb{N}$. So

$$N\Bigg((4k)^l\Big(2kf\Big(\frac{\sum_{i=1}^k (x_i) + \sum_{i=1}^k y_i}{(2k)^l}\Big) + 2kf\Big(\frac{\sum_{i=1}^k (x_i) - \sum_{i=1}^k y_i}{(2k)^l}\Big) - \sum_{i=1}^k f(\frac{x_i}{(2k)^l}) - \sum_{i=1}^k f(\frac{y_i}{(2k)^l})$$
$$- (4k)^k\delta_1\Big(f\Big(\frac{\sum_{i=1}^k x_i + \sum_{i=1}^k y_i}{(2k)^l}\Big) + f\Big(\frac{\sum_{i=1}^k x_i - \sum_{i=1}^k y_i}{(2k)^l}\Big) - 2\sum_{i=1}^k f(\frac{x_i}{(2k)^l}) - 2\sum_{i=1}^k f(\frac{y_i}{(2k)^l})\Big)$$
$$- (4k)^k\delta_2\Big(4kf\Big(\frac{\sum_{i=1}^k (x_i) + \sum_{i=1}^k y_i}{(2k)^l}\Big) + f(\frac{\sum_{i=1}^k x_i - \sum_{i=1}^k y_i}{(2k)^l}) - 2\sum_{i=1}^k f(\frac{x_i}{(2k)^l})$$
$$- 2\sum_{i=1}^k f(\frac{y_i}{(2k)^l})\Big), (4k)^l t\Bigg)$$
$$\geq \frac{\frac{t}{(4k)^l}}{\frac{t}{(4k)^l} + \frac{L}{(4k)^l}\psi(x_1, ..., x_k, y_1, ..., y_k)} \tag{31}$$

for all $x_j, y_j \in \mathbf{X}$ for $j = 1 \to k$, for all $t > 0$ and for all $l \in \mathbb{N}$. So Since

$$\lim_{n\to\infty} \frac{(4k)^n t}{(4k)^n t + (4k)^n L^n \psi(x_1, ..., x_k, y_1, ..., y_k)} = 1$$

for all $x_j, y_j \in \mathbb{X}$ for all $j \to k$, $\forall t > 0, q \in \mathbb{R}$,. So

$$2kG\Big(\frac{\sum_{i=1}^k (x_i) + \sum_{i=1}^k y_i}{2k}\Big) + 2kG\Big(\frac{\sum_{i=1}^k (x_i) - \sum_{i=1}^k y_i}{2k}\Big) - \sum_{i=1}^k G(x_i) - \sum_{i=1}^k f(y_i)$$
$$= \delta_1\Big(G\Big(\sum_{i=1}^k x_i + \sum_{i=1}^k y_i\Big) + G\Big(\sum_{i=1}^k x_i - \sum_{i=1}^k y_i\Big) - 2\sum_{i=1}^k f(x_i) - 2\sum_{i=1}^k G(y_i)\Big) +$$
$$\delta_2\Big(4kG\Big(\frac{\sum_{i=1}^k (x_i) + \sum_{i=1}^k y_i}{2k}\Big) + G(\sum_{i=1}^k x_i - \sum_{i=1}^k y_i) - 2\sum_{i=1}^k G(x_i) - 2\sum_{i=1}^k G(y_i)\Big) \tag{32}$$

For all $x_i, y_i \in \mathbf{X}, i = 1 \to k$. By Lemma 3.1 then mapping $G : \mathbf{X} \to \mathbf{Y}$ is quadratic □

Theorem 3. *Suppose that $\psi : \mathbf{X}^{2k} \to [0, \infty)$ be a function such that there exists an $L < 1$*

$$\psi\Big(x_1, ..., x_k, y_1, ..., y_k\Big) \leq 4k\psi(\frac{x_1}{2k}, ..., \frac{x_k}{2k}, \frac{y_1}{2k}, ..., \frac{y_k}{2k}) \tag{33}$$

for all $x_j, y_j \in \mathbf{X}$ for $j = 1 \to k$.

Let $f : \mathbf{X} \to \mathbf{Y}$ be a mapping sattisfying

$$f(0) = 0 \tag{34}$$

and

$$N\Bigg(2kf\Big(\frac{\sum_{i=1}^k (x_i) + \sum_{i=1}^k y_i}{2k}\Big) + 2kf\Big(\frac{\sum_{i=1}^k (x_i) - \sum_{i=1}^k y_i}{2k}\Big) - \sum_{i=1}^k f(x_i) - \sum_{i=1}^k f(y_i)$$
$$- \delta_1\Big(f\Big(\sum_{i=1}^k x_i + \sum_{i=1}^k y_i\Big) + f\Big(\sum_{i=1}^k x_i - \sum_{i=1}^k y_i\Big) - 2\sum_{i=1}^k f(x_i) - 2\sum_{i=1}^k f(y_i)\Big)$$
$$- \delta_2\Big(4kf\Big(\frac{\sum_{i=1}^k (x_i) + \sum_{i=1}^k y_i}{2k}\Big) + f(\sum_{i=1}^k x_i - \sum_{i=1}^k y_i) - 2\sum_{i=1}^k f(x_i) - 2\sum_{i=1}^k f(y_i)\Big), t\Bigg)$$
$$\geq \frac{t}{t + \psi(x_1, ..., x_k, y_1, ..., y_k)} \tag{35}$$

for all $x_j, y_j \in \mathbf{X}$ *for* $j = 1 \to k$, *for all* $t > 0$. *Then*

$$G(x) = N - \lim_{n\to\infty} \frac{1}{(4k)^n} f((2k)^n x) \tag{36}$$

exists each $x \in \mathbf{X}$ *and defines a quadratic mapping* $G : \mathbf{X} \to \mathbf{Y}$
such that

$$N\Big(f(x) - G(x), t\Big) \geq \frac{(1-\varphi(\delta_1,\delta_2)+\delta_1)(1-L)t}{(1-\varphi(\delta_1,\delta_2)+\delta_1)(1-L)t + L\psi(x,...,x,0,...,0)} \tag{37}$$

for all $x \in \mathbf{X}$ *and* $t > 0$.

Proof. Suppose that $(\mathbb{M}, d)$ be the generalized metric space defined in the proof of theorem 3.2
From (25) I have

$$\frac{t}{t+\varphi(2kx,...,2kx,0,...,0)} \leq N\Big(f(x) - \frac{1}{4k}f((2k)x), \frac{t}{4k(1-\varphi(\delta_1,\delta_2)+\delta_1)}\Big) \tag{38}$$

for all $x \in \mathbf{X}$. and for all $t > 0$.
And thus

$$\begin{aligned} N\Big(f(x) - \frac{1}{4k}f((2k)x), t\Big) &\geq \frac{4k(1-\varphi(\delta_1,\delta_2)+\delta_1)t}{4k(1-\varphi(\delta_1,\delta_2)+\delta_1)t + 4kL\psi(x,...,x,0,...,0)} \\ &= \frac{(1-\varphi(\delta_1,\delta_2)+\delta_1)t}{(1-\varphi(\delta_1,\delta_2)+\delta_1)t + L\psi(x,...,x,0,...,0)} \end{aligned} \tag{39}$$

Now we cosider the linear mapping $T : \mathbb{M} \to \mathbb{M}$ such that

$$Tg(x) := \frac{1}{4k}g(2kx)$$

for all $x \in \mathbf{X}$. So $d(f, Tf) \leq \frac{L}{1-\varphi(\delta_1,\delta_2)+\delta_1}$. Thus

$$d(f, A) \leq \frac{L}{(1-\varphi(\delta_1,\delta_2)+\delta_1)(1-L)}.$$

which implies that the inequality (37) Satisfied. The rest of the prooft is similar to the proof of Theorem3.2 . □

From the above theorems we have the following corollary:

Corollary 1. *Suppose* $\theta \geq 0$ *and let* p *be a real number with* $0 < p < 2$.*Let* $\mathbf{X}$ *be a normed vector space with norm* $\|\cdot\|$
Let $f : \mathbf{X} \to \mathbf{Y}$ *be a mapping sattisfying*

$$f(0) = 0 \tag{40}$$

$$\begin{aligned} &N\Bigg(2kf\Big(\frac{\sum_{i=1}^k (x_i) + \sum_{i=1}^k y_i}{2k}\Big) + 2kf\Big(\frac{\sum_{i=1}^k (x_i) - \sum_{i=1}^k y_i}{2k}\Big) - \sum_{i=1}^k f(x_i) - \sum_{i=1}^k f(y_i) \\ &\quad - \delta_1\Big(f\Big(\sum_{i=1}^k x_i + \sum_{i=1}^k y_i\Big) + f\Big(\sum_{i=1}^k x_i - \sum_{i=1}^k y_i\Big) - 2\sum_{i=1}^k f(x_i) - 2\sum_{i=1}^k f(y_i)\Big) \\ &\quad - \delta_2\Big(4kf\Big(\frac{\sum_{i=1}^k (x_i) + \sum_{i=1}^k y_i}{2k}\Big) + f(\sum_{i=1}^k x_i - \sum_{i=1}^k y_i) - 2\sum_{i=1}^k f(x_i) - 2\sum_{i=1}^k f(y_i)\Big), t\Bigg) \\ &\quad \geq \frac{t}{t+\theta\Big(\sum_{i=1}^k \|x_i\|^p + \sum_{i=1}^k \|y_i\|^p\Big)} \end{aligned} \tag{41}$$

for all $x_j, y_j \in \mathbf{X}$ *for* $j = 1 \to k$, *for all* $t > 0$.

Then

$$G(x) = N - \lim_{n\to\infty} \frac{1}{(4k)^n} f((2k)^n x) \tag{42}$$

exists each $x \in \mathbf{X}$ *and defines a quadratic mapping* $G : \mathbf{X} \to \mathbf{Y}$
such that

$$N\Big(f(x) - G(x), t\Big) \geq \frac{\big(1 - \varphi(\delta_1, \delta_2) + \delta_1\big)\big(4k - (2k)^p\big)t}{\big(1 - \varphi(\delta_1, \delta_2) + \delta_1\big)\big(4k - (2k)^p\big)t + \theta \sum_{i=1}^k \|2kx_i\|^p} \tag{43}$$

for all $x \in \mathbf{X}$ *and* $t > 0$.

Corollary 2. *Suppose* $\theta \geq 0$ *and let* p *be a real number with* $p > 2$.*Let* $\mathbf{X}$ *be a normed vector space with norm* $\|\cdot\|$ *Let* $f : \mathbf{X} \to \mathbf{Y}$ *be a mapping sattisfying*

$$f(0) = 0 \tag{44}$$

$$\begin{aligned}
N\Bigg(&2kf\Big(\frac{\sum_{i=1}^k (x_i) + \sum_{i=1}^k y_i}{2k}\Big) + 2kf\Big(\frac{\sum_{i=1}^k (x_i) - \sum_{i=1}^k y_i}{2k}\Big) - \sum_{i=1}^k f(x_i) - \sum_{i=1}^k f(y_i) \\
&- \delta_1\Big(f\Big(\sum_{i=1}^k x_i + \sum_{i=1}^k y_i\Big) + f\Big(\sum_{i=1}^k x_i - \sum_{i=1}^k y_i\Big) - 2\sum_{i=1}^k f(x_i) - 2\sum_{i=1}^k f(y_i)\Big) \\
&- \delta_2\Big(4kf\Big(\frac{\sum_{i=1}^k (x_i) + \sum_{i=1}^k y_i}{2k}\Big) + f\big(\sum_{i=1}^k x_i - \sum_{i=1}^k y_i\big) - 2\sum_{i=1}^k f(x_i) - 2\sum_{i=1}^k f(y_i)\Big), t\Bigg) \\
&\geq \frac{t}{t + \theta\Big(\sum_{i=1}^k \|x_i\|^p + \sum_{i=1}^k \|y_i\|^p\Big)}
\end{aligned} \tag{45}$$

for all $x_j, y_j \in \mathbf{X}$ *for* $j = 1 \to k$, *for all* $t > 0$.
Then

$$G(x) = N - \lim_{n\to\infty} (4k)^n f\big(\frac{1}{(2k)^n} x\big) \tag{46}$$

exists each $x \in \mathbf{X}$ *and defines a quadratic mapping* $G : \mathbf{X} \to \mathbf{Y}$
such that

$$N\Big(f(x) - G(x), t\Big) \geq \frac{\big(1 - \varphi(\delta_1, \delta_2) + \delta_1\big)\big((2k)^p - 4k\big)t}{\big(1 - \varphi(\delta_1, \delta_2) + \delta_1\big)\big((2k)^p - 4k\big)t + \theta \sum_{i=1}^k \|2kx_i\|^p} \tag{47}$$

for all $x \in \mathbf{X}$ *and* $t > 0$.

4. Conclusion

In this paper, we have successfully established a $\varphi(\delta_1, \delta_2)$-functional inequality on a fuzzy Banach space with an unlimited number of variables.

References

[1] Ulam, S. M. (1960). A collection of Mathematical problems, volume 8. Interscience Publishers. New York.

[2] Hyers, D. H. (1941). On the stability of the functional equation. *Proceedings of the National Academy of the United States of America*, 27(4), 222. https://doi.org/10.1073/pnas.27.4.222

[3] Aoki, T. (1950). On the Stability of the Linear Transformation in Banach Space. *Journal of the Mathematical Society of Japan*, 2, 64-66.

[4] Rassias, Th. M. (1978). On the stability of the linear mapping in Banach spaces. *Proceedings of the American Mathematical Society*, *72*(2), 297-300.

[5] Gavrut, P. (1994). A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. *Journal of Mathematical Analysis and Applications*, *184*(3), 431-436.

[6] Kim, K. J. H. (2002). The generalized Hyers-Ulam-Rassias stability of a cubic functional equation. *J. Math. Anal. Appl., 274*, 867-878.

[7] Lee, S., Im, S., & Hwang, I. (2005). Quartic functional equations. *J. Math. Anal. Appl., 307*, 387-394.

[8] Gilányi, A. (2001). Eine zur Parallelogrammgleichung äquivalente Ungleichung. *Aequationes Math., 62*, 303-309.

[9] Rätz, J. (2003). On inequalities associated with the Jordan-von Neumann functional equation. *Aequationes mathematicae, 66*(1), 191-200. https://doi.org/10-1007/s00010-0032684-8

[10] Park, C., Cho, Y., & Han, M. (2007). Stability of functional inequalities associated with Jordan-von Neumann type additive functional equations. *J. Inequal. Appl., 2007*, Art. ID 41820.

[11] Fechner, W. (2006). Stability of a functional inequalities associated with the Jordan-von Neumann functional equation. *Aequationes Math., 71*, 149-161.

[12] An, L. V. (2020). Generalized Hyers-Ulam type stability of the type functional equation with $2k$-variable in Non-Archimedean space. *Vol. 8, Issue 4, 2020 (Oct-Dec)*. KY PUBLICATIONS. http://www.bomsr.com.

[13] An, L. V. (2020). Generalized Hyers-Ulam type stability of the additive functional equation inequalities with $2n$-variables on an approximate group and ring homomorphism. *Asia Mathematika, Volume: 4, Issue: 2*. www.asiamath.org.

[14] An, L. V. (2021). Generalized Hyers-Ulam-Rassias type stability of the $2k$-variable quadratic functional inequalities in non-Archimedean Banach spaces and Banach spaces. *Asia Mathematika, Volume: 5, Issue: 3*. www.asiamath.org.

[15] An, L. V. (2021). Generalized Hyers-Ulam Type Stability of the $2k$-Variables Quadraticß-Functional Inequalities AndFunction in γ-Homogeneous Normed Space. *Int. J. Math. And Appl., 9(3)(2021), 81–93*. http://ijmaa.in/ApplicationsâĂćISSN:2347-1557âĂćInternationalJournalofMathematicsAnditsApplications.

[16] An, L. V. (2021). Generalized Stability of Functional Inequalities with 3k-Variables Associated for Jordan-von Neumann-Type Additive Functional Equation. *Ly Van An*. DOI:10.4236/oalib.

[17] Bag, T., & Samanta, S. K. (2003). Finite-dimensional fuzzy normed linear spaces. *J. Fuzzy Math., 11*, 687-705.

[18] Akhavan-Malayeri, A. K., & Moslehian, M. S. (2008). Fuzzy versions of Hyers-Ulam-Rassias theorem. *Fuzzy Sets and Systems, 159*, 720-729.

[19] Mirmostafaee, A. K., & Moslehian, M. S. (2008). Fuzzy approximately cubic mappings. *Inform. Sci., 178*, 3791-3798.

[20] Mirmostafaee, A. K., Mirzavaziri, M., & Moslehian, M. S. (2008). Fuzzy stability of the Jensen functional equation. *Fuzzy Sets and Systems, 159*, 730-738.

[21] Katsaras, A. K. (1984). Fuzzy topological vector spaces II. *Fuzzy Sets and Systems, 12*(1984), 143-154.

[22] Cadariu, L., & Radu, V. (2003). The fixed point alternative and the stability of functional equations. *Fixed Point Theory, 4*, 91-96.

[23] Diaz, J., & Margolis, B. (1968). A fixed point theorem of the alternative for contractions on a generalized complete metric space. *Bull. Amer. Math. Soc., 74*, 305-309.

[24] Mihet, D., & Radu, V. (2008). On the stability of the additive Cauchy functional equation in random normed spaces. Journal of mathematical Analysis and Applications, 343(1), 567-572.