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Interor and $\mathfrak{h}$ operators of the category of locales

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Abstract: We construct the concrete categories **I-Loc** and $\mathfrak{h}$-**Loc** over the category **Loc** of locales and we deduce that they are topological categories, where **I** and $\mathfrak{h}$ denote respectively the classes of interior and h operators of the category **Loc** of locales.

Keywords: Filter; Bases of filters; Ultrafilter; S-filter; Cover-neighborhood; $\mathfrak{G}$-neighborhood; Grothendieck topology; Convergence; Compactness; Frames; Locales.

MSC: 18A35; 18F10; 18F70; 54A20; 54D30.

1. Introduction

Kuratowski operators (closure, interior, exterior, boundary and others) have been used intensively in General(set-theoretic) Topology ([1–3]). For a topological space it is well-known that, for example, the associated closure and interior operators provide equivalent descriptions of the topology; but this is not always true in other categories, consequently it makes sense to define and study separately these operators. In this context, we study an interior operator I on the the coframe $\mathcal{S}_{\ell}(L)$ of sublocales of every object L in the category **Loc**.

On the other hand, a new topological operador $\mathfrak{h}$ was introduced by M. Suarez [4] in order to complete a Boolean algebra with all topological operators in General Topology. Following his ideas, we study an operator $\mathfrak{h}$ on the collection $\mathcal{S}^c_{\ell}(L)$ of all complemented sublocales of every object L in the category **Loc**.

The paper is organized as follows, we begin presenting, in §2, the basic concepts of Heyting algebras, Frames, locales, sublocales, images and preimages of sublocales for the morphisms of **Loc** and the notions of closed and open sublocales; these notions can be found in Picado and Pultr [5] and A. L. Suarez [6]. In §3, we present the concept of interior operator I on the category **Loc** and then we construct a topological category $(\mathbf{I\text{-}Loc},\ U)$, where $U : \mathbf{I\text{-}Loc} \to \mathbf{Loc}$ is a forgetful functor. Finally in §4 we present the notion of $\mathfrak{h}$ operator on the category **Loc** and discuss some of their properties for constructing the topological category $(\mathfrak{h}\text{-}\mathbf{Loc},\ U)$ associated to the forgetful functor $U : \mathfrak{h}\text{-}\mathbf{Loc} \to \mathbf{Loc}$.

2. Preliminaries

For a comprehensive account on the the categories of frames and locales we refer to Picado and Pultr [5] and A. L. Suarez [6], from whom we take the following notions:

- The adjunction $\Omega : \mathbf{Top} \to \mathbf{Frm}^{op}$, $\mathbf{pt} : \mathbf{Frm}^{op} \to \mathbf{Top}$ with $\Omega \dashv \mathbf{pt}$, connects the categories of frames with that of topological spaces. The functor Ω assigns to each space its lattice of opens, and **pt** assigns to a frame L the collection of the frame maps $f : L \to 2$, topologized by setting the opens to be exactly the sets of the form $\{f : L \to 2 \mid f(a) = 1\}$ for some $a \in L$.
- A frame L is *spatial* if for $a, b \in L$ whenever $a \nleq b$ there is some frame map $f : L \to 2$ such that $f(a) = 1 \neq f(b)$. Spatial frames are exactly those of the form $\Omega(X)$ for some space X.
- A space is *sober* if every irreducible closed set is the closure of a unique point. Sober spaces are exactly those of the form $\mathbf{pt}(L)$ for some frame L.
- The adjunction $\Omega \dashv \mathbf{pt}$ restricts to a dual equivalence of categories between spatial frames and sober spaces.
- The category of sober spaces is a full reflective subcategory of **Top**. For each space X we have a sobrification map $N : X \to \mathbf{pt}(\Omega(X))$ mapping each point $x \in X$ to the map $f_x : (X) \to 2$ defined as $f(U) = 1$ if and only if $x \in U$.

- The category of spatial frames is a full reflective subcategory of **Frm**. For each frame we have a spatialization map $\phi : L \to \Omega(\mathbf{pt}(L))$ which sends each $a \in L$ to $\{f : L \to 2 \mid f(a) = 1\}$.
- We call **Loc** the category $\mathbf{Frm}^{op}$, and we call its objects locales. Maps in the category of locales have a concrete description: they can be characterized as the right adjoints of frame maps (since frame maps preserve all joins, they always have right adjoints).
- A *sublocale* of a locale L is a subset $S \subseteq L$ such that it is closed under arbitrary meets, and such that $s \in S$ implies $x \to s \in S$ for every $x \in L$. This is equivalent to $S \subseteq L$ being a locale in the inherited order, and the subset inclusion being a map in **Loc**.
- Sublocales of L are closed under arbitrary intersections, and so the collection $\mathcal{S}_{\ell}(L)$ of all sublocales of L, ordered under set inclusion, is a complete lattice. The join of sublocales is (of course) not the union, but we have a very simple formula $\bigvee_i S_i = \{\bigvee M \mid M \subseteq \bigcup_i S_i\}$.
- In the coframe $\mathcal{S}_{\ell}(L)$ the bottom element is the sublocale $\{1\}$ and the top element is L.
- Let $f : L \to M$ be a localic map. The set-theoretic preimage $f^{-1}[T]$ of a sublocale $T \subseteq M$ is not necessarily a sublocale of L. To obtain a concept of a preimage suitable for our purposes we will, first, make the following observation: "Let $A \subseteq L$ be a subset closed under meets. Then $\{1\} \subseteq A$ and if $S_i \subseteq A$ for $i \in J$ then $\bigwedge_{i\in J} S_i \subseteq A$". Consequently there exists the largest sublocale contained in A. It will be denoted by A_{sloc}.
- The set-theoretic preimage $f^{-1}[T]$ of a sublocale T is closed under meets $\big($indeed, $f(1) = 1$, and if $x_i \in f^{-1}[T]$) then $f(x_i) \in T$, and hence $f(\bigwedge_{i\in J} x_i) = \bigwedge_{i\in J} f(x_i)$ belongs to T and $\bigwedge_{i\in J} x_i \in f^{-1}[T]$ $\big)$ and we have the sublocale $f_{-1}[T] := f^{-1}[T]_{sloc}$. It will be referred to as *the preimage* of T, and we shall sat that $f_{-1}[-]$ is *the preimage function* of f.
- For every localic map $f : L \to M$, the preimage function $f_{-1}[-]$ is a right Galois adjoint of the image function $f[-] : \mathcal{S}_{\ell}(L) \to \mathcal{S}_{\ell}(M)$.
- Embedded in $\mathcal{S}_{\ell}(L)$ we have the coframe of *closed sublocales* which is isomorphic to L^{op}. The closed sublocale $\mathfrak{c}(a) \subseteq L$ is defined to be $\uparrow a$ for $a \in L$.
- Embedded in $\mathcal{S}_{\ell}(L)$ we also have the frame of open sublocales which is isomorphic to L. The open sublocale is defined to be $\{a \to x \mid x \in L\}$ for $a \in L$.
- The sublocales $\mathfrak{o}(a)$ and $\mathfrak{c}(a)$ are complements of one another in the coframe $\mathcal{S}_{\ell}(L)$ for any element $a \in L$. Furthermore, open and closed sublocales generate the coframe $\mathcal{S}_{\ell}(L)$ in the sense that for each $S \in \mathcal{S}_{\ell}(L)$ we have $S = \bigcap\{\mathfrak{o}(x) \cup \mathfrak{c}(y) \mid S \subseteq \mathfrak{o}(x) \cup \mathfrak{c}(y)\}$.
- A pseudocomplement of an element a in a meet-semilattice L with 0 is the largest element b such that $b \wedge a = 0$, if it exists. It is usually denoted by $\neg a$. Recall that in a Heyting algebra H the pseudocomplement can be expressed as $\neg x = x \to 0$.

3. Interior Operators

We shall be conserned in this section with a version on locales of the interior operator studied in [7].

Before stating the next definition, we need to observe that since for localic maps $f : L \to M$ and $g : M \to N$:

- the preimage function $f_{-1}[-]$ is a right Galois adjoint of the image function $f[-] : \mathcal{S}_{\ell}(L) \to \mathcal{S}_{\ell}(M)$;
- $g[-] \circ f[-] = (g \circ f)[-]$.

Therefore $g_{-1}[-] \circ f_{-1}[-] = (g \circ f)_{-1}[-]$ because given two adjunctions the composite functors yield an adjunction.

Definition 1. An interior operator I of the category **Loc** is given by a family $I = (i_L)_{L \in \mathbf{Loc}}$ of maps $i_L : \mathcal{S}_{\ell}(L) \to \mathcal{S}_{\ell}(L)$ such that

(I_1) (Contraction) $i_L(S) \subseteq S$ for all $S \in \mathcal{S}_{\ell}(L)$;
(I_2) (Monotonicity) If $S \subseteq T$ in $\mathcal{S}_{\ell}(L)$, then $i_L(S) \subseteq i_L(T)$
(I_3) (Upper bound) $i_L(L) = L$.

Definition 2. An I-space is a pair (L, i_L) where L is an object of **Loc** and i_L is an interior operator on L.

Definition 3. A morphism $f : L \to M$ of **Loc** is said to be I-continuous if

$$f_{-1}\left[i_M(T)\right] \subseteq i_L\left(f_{-1}[T]\right) \tag{1}$$

for all $T \in \mathcal{S}_{\ell}(M)$. Where $f_{-1}[-]$ is the preimage of $f[-]$.

Proposition 1. *Let $f : L \to M$ and $g : M \to N$ be two I-continuous morphisms of* **Loc** *then $g \centerdot f$ is an I-continuous morphism of* **Loc**.

Proof. Since $g : M \to N$ is I-continuous, we have

$$g_{-1}\left[i_N(S)\right] \subseteq i_M\left(g_{-1}[S]\right)$$

for all $S \in \mathcal{S}_{\ell}(N)$, it fallows that

$$f_{-1}\Big[g_{-1}\left[(i_N(S)\right]\Big] \subseteq f_{-1}\Big[i_M\left(g_{-1}[S]\right)\Big];$$

now, by the I-continuity of f,

$$f_{-1}\Big[i_M\left(g_{-1}[S]\right)\Big] \subseteq i_L\Big(f_{-1}\left[g_{-1}[S]\right]\Big),$$

therefore

$$f_{-1}\Big[g_{-1}\left[i_N(S)\right]\Big] \subseteq i_L\Big(f_{-1}\left[g_{-1}[S]\right]\Big),$$

that is to say

$$(g \centerdot f)_{-1}\left[i_N(S)\right] \subseteq i_L\Big((g \centerdot f)_{-1}[S]\Big)$$

■

As a consequence we obtain;

Definition 4. The category **I-Loc** of I-spaces comprises the following data:

1. **Objects**: Pairs (L, i_L) where L is an object of **Loc** and i_L is an interior operator on L.
2. **Morphisms**: Morphisms of **Loc** which are I-continuous.

3.1. The lattice structure of all interior operators

For the category **Loc** we consider the collection

$$Int(\mathbf{Loc})$$

of all interior operators on **Loc**. It is ordered by

$$I \leqslant J \Leftrightarrow i_L(S) \subseteq j_L(S), \text{ for all } S \in \mathcal{S}_{\ell} \text{ and all } L \text{ object of } \mathbf{Loc}.$$

This way $Int(\mathbf{Loc})$ inherits a lattice structure from $\mathcal{S}_{\ell}$:

Proposition 2. *Every family $(I_\lambda)_{\lambda \in \Lambda}$ in $Int(\mathbf{Loc})$ has a join $\bigvee_{\lambda \in \Lambda} I_\lambda$ and a meet $\bigwedge_{\lambda \in \Lambda} I_\lambda$ in $Int(\mathbf{Loc})$. The discrete interior operator*

$$I_D = (i_{D_L})_{L \in \mathbf{Loc}} \text{ with } i_{D_L}(S) = S \text{ for all } S \in \mathcal{S}_{\ell}$$

is the largest element in $Int(\mathbf{Loc})$, and the trivial interior operator

$$I_T = (i_{T_L})_{L \in \mathbf{Loc}} \text{ with } i_{T_L}(S) = \begin{cases} \{1\} & \text{for all } S \in \mathcal{S}_{\ell},\ S \neq L \\ L & \text{if } S = L \end{cases}$$

is the least one.

Proof. For $\Lambda \neq \emptyset$, let $\widehat{I} = \bigvee_{\lambda \in \Lambda} I_\lambda$, then

$$\widehat{i_L} = \bigvee_{\lambda \in \Lambda} i_{\lambda_L},$$

for all L object of **Loc**, satisfies

- $\widehat{i_L}(S) \subseteq S$, because $i_{\lambda_L}(S) \subseteq S$ for all $S \in \mathcal{S}_{\ell}$ and for all $\lambda \in \Lambda$.
- If $S \leqslant T$ in $\mathcal{S}_{\ell}$ then $i_{\lambda_L}(S) \subseteq i_{\lambda_L}(T)$ for all $S \in \mathcal{S}_{\ell}$ and for all $\lambda \in \Lambda$, therefore $\widehat{i_S}(S) \subseteq \widehat{i_L}(T)$.

- Since $i_{\lambda_L}(L) = L$ for all $\lambda \in \Lambda$, we have that $\widehat{i_L}(L) = L$.

Similarly $\bigwedge_{\lambda \in \Lambda} I_\lambda$, I_D and I_T are interior operators. ■

Corollary 1. *For every object L of* **Loc**

$$Int(L) = \{i_L \mid i_L \ \text{ is an interior operator on } \ L\}$$

is a complete lattice.

3.2. Initial interior operators

Let **I-Loc** be the ctegory of I-spaces. Let (M, i_M) be an object of **I-Loc** and let L be an object of **Loc**. For each morphism $f : L \to M$ in **Loc** we define on L the operotor

$$i_{L_f} := f_{-1}[-] \centerdot i_M \centerdot f[-]. \tag{2}$$

Proposition 3. *The operator (2) is an interior operator on L for which the morphism f is I-continuous.*

Proof.

(I_1) (Contraction) $i_{L_f}(S) = f_{-1}\left[i_M\left(f[S]\right)\right] \subseteq f_{-1}\left[f[S]\right] \subseteq S$ for all $S \in \mathcal{S}_{\ell}$;

(I_2) (Monotonicity) $S \subseteq T$ in $\mathcal{S}_{\ell}$, implies $f[S] \subseteq f[T]$, then $i_M\left(f[S] \subseteq i_M\left(f[T]\right)\right.$, consequently $f_{-1}\left[i_M\left(f[S]\right)\right] \subseteq f_{-1}\left[i_M\left(f[T]\right)\right]$;

(I_3) (Upper bound) $i_{L_f}(L) = f_{-1}\left[i_M\left(f[L]\right)\right] = L$.

Finally,

$$f_{-1}\left[i_M(T)\right] \subseteq f_{-1}\left[i_M\left(f\left[[f_{-1}[T]\right)\right)\right] \subseteq i_{L_f}\left(f_{-1}[T]\right),$$

for all $T \in \mathcal{S}_{\ell}$. ■

It is clear that i_{L_f} is the coarsest interior operator on L for which the morphism f is I-continuous; more precisaly

Proposition 4. *Let (L, i_L) and (M, i_M) be objects of* **I-Loc***, and let N be an object of* **Loc***. For each morphism $g : N \to L$ in* **Loc** *and for $f : (L, i_{L_f}) \to (M, i_N)$ an I-continuous morphism, g is I-continuous if and only if $f \centerdot g$ is I-continuous.*

Proof. Suppose that $g \centerdot f$ is I-continuous, i. e.

$$(f \centerdot g)_{-1}\left[i_M(T)\right] \subseteq i_N\left((f \centerdot g)_{-1}[T]\right)$$

for all $T \in \mathbf{S}(\mathbf{N})$. Then, for all $S \in \mathcal{S}_{\ell}$, we have

$$\begin{aligned} g_{-1}\left[i_{L_f}(S)\right] &= g_{-1}\left[f_{-1} \centerdot i_M \centerdot f[S]\right] = (f \centerdot g)_{-1}\left[i_M\left(f[S]\right)\right] \\ &\subseteq i_N\left((f \centerdot g)_{-1}\left[f[S]\right]\right) = i_N\left(g_{-1} \centerdot f_{-1} \centerdot f[S]\right) \\ &\subseteq i_N\left(g_{-1}[S]\right), \end{aligned}$$

i.e. g is I-continuous. ■

As a consequence of Corollary 1, Proposition 3 and Proposition 4 (cf. [8] or [9]), we obtain

Theorem 2. *The forgetful functor U :* **I-Loc** $\to$ **Loc** *is topological, i.e. the concrete category* $\big($**I-Loc**, $U\big)$ *is topological.*

3.3. Open subobjects

Definition 5. An sublocale S of a locale L is called I-open $\big($in $L\big)$ if it is isomorphic to its I-interior, that is: if $i_L(S) = S$.

The I-continuity condition (1) implies that I-openness is preserve by inverse images:

Proposition 5. *Let $f : L \to M$ be a morphism in* **Loc**. *If T is I-open in M, then $f_{-1}(T)$ is I-open in L.*

Proof. If $T = i_M(T)$ then $f_{-1}[T] = f_{-1}[i_M(T)] \subseteq i_L(f_{-1}[T])$, therefore $i_L(f_{-1}[T]) = f_{-1}[T]$. ∎

4. $\mathfrak{h}$ Operators

In this section we shall be conserned with a weak categorical version of a topological function studied by M, Suarez M. in [4]. For that purpose we will use the colection $\mathcal{S}^c_{\ell}(L)$ of all complemented sublocales of a locale L (See P, T. Johnston [10], for example).

Definition 6. An $\mathfrak{h}$ operator of the category **Loc** is given by a family $\mathfrak{h} = (h_L)_{L \in \mathbf{Loc}}$ of maps $h_L : \mathcal{S}^c_{\ell}(L) \to \mathcal{S}^c_{\ell}(L)$ such that

(h_1) $S \cap h_L(S) \subseteq S$, for all $S \in \mathcal{S}^c_{\ell}(L)$;
(h_2) If $S \subseteq T$ then $S \cap h_L(S) \subseteq T \cap h_L(T)$, for all $S, T \in \mathcal{S}^c_{\ell}(L)$;
(h_3) $h_L(L) = L$.

Definition 7. An $\mathfrak{h}$-space is a pair (L, h_L) where L is an object of **Loc** and h_L is an $\mathfrak{h}$ operator on L.

Definition 8. A morphism $f : L \to M$ of **Loc** is said to be $\mathfrak{h}$-continuous if

$$f_{-1}[T \cap h_M(T)] \subseteq f_{-1}[T] \cap h_L(f_{-1}[T]) \tag{3}$$

for all $T \in \mathcal{S}^c_{\ell}(M)$. Where $f_{-1}[-]$ is the inverse image of $f[-]$.

Proposition 6. *Let $f : L \to M$ and $g : M \to N$ be two $\mathfrak{h}$-continuous morphisms of* **Loc** *then $g \centerdot f$ is an $\mathfrak{h}$-continuous morphism of* **Loc**.

Proof. Since $g : M \to N$ is I-continuous, we have

$$g_{-1}[V \cap h_N(V)] \subseteq g_{-1}[V] \cap h_M(g_{-1}[V])$$

for all $V \in \mathcal{S}^c_{\ell}(N)$, it fallows that

$$f_{-1}\big[g_{-1}[V \cap h_N(V)]\big] \subseteq f_{-1}\big[g_{-1}[V] \cap h_M(g_{-1}[V])\big]$$

now, by the $\mathfrak{h}$-continuity of f,

$$f_{-1}\big[g_{-1}[V] \cap h_M(g_{-1}[V])\big] \subseteq f_{-1}\big[g_{-1}[V]\big] \cap h_L\left(f_{-1}\big[g_{-1}[V]\big]\right)$$

therefore

$$(g \centerdot f)_{-1}[V \cap h_N(V)] \subseteq (g \centerdot f)_{-1} \cap h_L((g \centerdot f)_{-1}[V]).$$

This complete the proof. ∎

As a consequence we obtain

Definition 9. The category $\mathfrak{h}$-**Loc** of $\mathfrak{h}$-spaces comprises the following data:

1. **Objects**: Pairs (L, h_L) where L is an object of **Loc** and h_L is an $\mathfrak{h}$-operator on L.
2. **Morphisms**: Morphisms of **Loc** which are $\mathfrak{h}$-continuous.

4.1. The lattice structure of all $\mathfrak{h}$ operators

For the category **Loc** we consider the collection

$$\mathfrak{h}(\mathbf{Loc})$$

of all $\mathfrak{h}$ operators on **Loc**. It is ordered by

$$\mathfrak{h} \leqslant \mathfrak{h}' \Leftrightarrow h_L(S) \subseteq h'_L(S), \quad \text{for all } S \in \mathcal{S}^c_{\boldsymbol{\ell}}(L) \text{ and all } L \text{ object of } \mathbf{Loc}.$$

This way $\mathfrak{h}(\mathbf{Loc})$ inherits a lattice structure from $\mathcal{S}^c_{\boldsymbol{\ell}}$.

Proposition 7. *Every family $(\mathfrak{h}_\lambda)_{\lambda \in \Lambda}$ in $\mathfrak{h}(\mathbf{Loc})$ has a join $\bigvee_{\lambda \in \Lambda} \mathfrak{h}_\lambda$ and a meet $\bigwedge_{\lambda \in \Lambda} \mathfrak{h}_\lambda$ in $Int(\mathbf{Loc})$. The discrete $\mathfrak{h}$ operator*

$$\mathfrak{h}_D = (h_{DL})_{L \in \mathbf{Loc}} \;\; \textit{with} \;\; h_{DL}(S) = S \;\; \textit{for all} \;\; S \in \mathcal{S}^c_{\boldsymbol{\ell}}(L)$$

is the largest element in $\mathfrak{h}(\mathbf{Loc})$, and the trivial $\mathfrak{h}$ operator

$$\mathfrak{h}_T = (h_{TL})_{L \in \mathbf{Loc}} \;\; \textit{with} \;\; h_{TL}(S) = \begin{cases} \{1\} & \textit{for all } \; S \in \mathcal{S}^c_{\boldsymbol{\ell}}(L), \;\; S \neq L \\ L & \textit{if } \; S = L \end{cases}$$

is the least one.

Proof. For $\Lambda \neq \emptyset$, let $\widehat{\mathfrak{h}} = \bigvee_{\lambda \in \Lambda} \mathfrak{h}_\lambda$, then

$$\widehat{h}_L = \bigvee_{\lambda \in \Lambda} h_{\lambda L},$$

for all L object of **Loc**, satisfies

- $S \cap \widehat{h}_L(S) \subseteq S$, because $S \cap h_{\lambda S}(L) \subseteq S$, for all $S \in \mathcal{S}^c_{\boldsymbol{\ell}}(L)$ and for all $\lambda \in \Lambda$.
- If $S \subseteq T$ then $S \cap \widehat{h}_L(S) \subseteq T \cap \widehat{h}_L(T)$, since $S \cup h_{\lambda L}(S) \subseteq T \cup h_{\lambda L}(T)$, for all $S, T \in \mathcal{S}^c_{\boldsymbol{\ell}}(L)$ and for all $\lambda \in \Lambda$.
- $L \cap \widehat{h}_L(L) = L$, because $L \cap h_{\lambda L}(L) = L$ for all $\lambda \in \Lambda$.

Similarly $\bigwedge_{\lambda \in \Lambda} \mathfrak{h}_\lambda$, $\mathfrak{h}_D$ and $\mathfrak{h}_T$ are $\mathfrak{h}$ operators. ■

Corollary 3. *For every object L of* **Loc**

$$\mathfrak{h}(L) = \{h_L \mid h_L \;\; \textit{is an } \mathfrak{h} \textit{ operator on } \; L\}$$

is a complete lattice.

4.2. Initial $\mathfrak{h}$ operators

Let $\mathfrak{h}$-**Loc** be the category of $\mathfrak{h}$-spaces. Let (M, h_M) be an object of $\mathfrak{h}$-**Loc** and let L be an object of **Loc**. For each morphism $f : L \to M$ in **Loc** we define on L the operotor

$$h_{L_f} := f_{-1}[-] \centerdot h_M \centerdot f[-]. \tag{4}$$

Proposition 8. *The operator (4) is an $\mathfrak{h}$ operator on L for which the morphism f is $\mathfrak{h}$-continuous.*

Proof.

(h_1) $S \cap h_{L_f}(S) = f_{-1}\Big[f[S] \cap h_M\big[f[S]\big]\Big] \subseteq f_{-1}\big[f[S]\big] \subseteq S$,
for all $S \in \mathcal{S}^c_{\boldsymbol{\ell}}(L)$.

(h_2) $S \subseteq T$ in $\mathcal{S}^c_{\boldsymbol{\ell}}(L)$, implies $f[S] \subseteq f[T]$, then
$f[S] \cap h_M\big(f[S]\big) \subseteq f[T] \cap h_M\big(f[T]\big)$, therefore
$f_{-1}\Big[f[S] \cap h_M\big(f[S]\big)\Big] \subseteq f_{-1}\Big[f[T] \cap h_M\big(f[T]\big)\Big]$,
consequently $S \cap h_{L_f}(S) \subseteq T \cap h_{L_f}(T)$, for all $S, T \in \mathcal{S}^c_{\boldsymbol{\ell}}(L)$;

(h_3) $L \cap h_{L_f}(L) = f_{-1}\Big[f[L] \cap h_M\big[f[L]\big]\Big] = L$.

■

It is clear that $h_{L_f}(L)$ is the coarsest $\mathfrak{h}$ operator on L for which the morphism f is $\mathfrak{h}$-continuous; more precisaly

Proposition 9. *Let (L, h_L) and (M, h_M) be objects of* $\mathfrak{h}$-**Loc**,*and let N be an object of* **Loc**. *For each morphism $g : N \to L$ in* **Loc** *and for $f : (L, h_{L_f}) \to (M, h_N)$ an $\mathfrak{h}$-continuous morphism, g is $\mathfrak{h}$-continuous if and only if $f \centerdot g$ is $\mathfrak{h}$-continuous.*

Proof. Suppose that $g \centerdot f$ is I-continuous, i. e.

$$(f \centerdot g)_{-1}\left[T \cap h_M(T)\right] \subseteq T \cap h_N\big((f \centerdot g)_{-1}[T]\big)$$

for all $T \in T \in \mathcal{S}^c_{\boldsymbol{\ell}}(N)$. Then, for all $S \in T \in \mathcal{S}^c_{\boldsymbol{\ell}}(L)$, we have

$$\begin{aligned} g_{-1}\Big[S \cap \big(h_{L_f}(S)\big)\Big] &= g_{-1}\Big[f_{-1}\big[f[S] \cap h_M(f[S])\big)\Big] = (f \centerdot g)_{-1}\big[f[S] \cap h_M(f[S])\big] \\ &\subseteq f \centerdot g)_{-1}[f[S]] \cap \Big(h_N\big((f \centerdot g)_{-1}[f[S]]\big) \\ &= (f \centerdot g)_{-1}\big[f[S]\big] \cap h_N\big(g_{-1} \centerdot f_{-1} \centerdot f[S]\big) \\ &\subseteq g_{-1}[S] \cap h_N\big(g_{-1}[S]\big), \end{aligned}$$

i.e. g is I-continuous. ∎

As a consequence of Corollary 3, Proposition 8 and Proposition 9 (cf. [8] or [9]), we obtain

Theorem 4. *The forgetful functor $U : \mathfrak{h}$-**Loc** $\to$ **Loc** is topological, i.e. the concrete category $\big(\mathfrak{h}$-**Loc**$,\ U\big)$ is topological.*

Conflicts of Interest: The author declares no conflict of interest.

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