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Laplace transform method for logistic growth in a population and predator models with fractional order

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Abstract: In this paper, we develop a new application of the Laplace transform method (LTM) using the series expansion of the dependent variable for solving fractional logistic growth models in a population as well as fractional prey-predator models. The fractional derivatives are described in the Caputo sense. To illustrate the reliability of the method some examples are provided. The results reveal that the technique introduced here is very effective and convenient for solving fractional-order nonlinear differential equations.

Keywords: Laplace transform method; Fractional power series; Caputo fractional derivative; Fractional differential equations.

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1. Introduction

Several techniques have been utilized to obtain approximate solutions to fractional differential equations, such as ADM [1–5], VIM [6–8], HPM [9–11], and HAM [12]. Recently, more applications of fractional-order models with Caputo fractional derivatives for different areas were presented. For instance, the authors in [13] have studied a fractional model of the phytoplankton-toxic Phytoplankton-Zooplankton system with convergence analysis. Dubey et al. [14] successfully modified the computational scheme and convergence for the fractional-order hepatitis E virus model. The authors in [34] have studied a fractional model of the atmospheric dynamics of carbon dioxide gas. The approximate analytical solution of the fractional-order biochemistry reaction model and its stability analysis were provided by Dubey et al. [15]. The authors in [16] have investigated the fractional order model of transmission dynamics of HIV/AIDS with the effect of weak CD4+T cells. Dubey et al. [17] successfully constructed an efficient computational solution for the time-fractional modified Degasperis-Procesi equation arising in the propagation of nonlinear dispersive waves. The authors in [18,19] have presented a numerical solution to the time-fractional three-species food chain model arising in mathematical ecology.

In the literature, the logistic growth model was introduced by Pierre Verhulst [20], it has useful applications to humans' population and population biology to describe animal populations. On the other hand, the Predator–Prey model was introduced by Alfred Lotka and Vito Volterra [21,22], and is used to describe the behavior of biological systems in which two species interact, one as a predator and the other as prey. These models have gained interest from many researchers (see [23–31]).

Laplace transform is a powerful technique for solving linear differential and integral equations. It allows us to transform differential equations into polynomial equations and then by solving these polynomial equations we can obtain the unknown function by using the Inverse Laplace transform. For more details, see [32]. However, the Laplace transform cannot be used alone for the nonlinear differential equations. Therefore, for the nonlinear case, we use the series expansion of the dependent variable to get the solution.

The present work suggests LTM to a couple of fractional nonlinear biological models that consist of a fractional logistic growth model in a population and a fractional prey-predator model. First, we study the fractional logistic growth model in a population [33], then

$$D^{\alpha}u\left(t\right)=u\left(t\right)-u^{2}(t)\,,\;\;0<\alpha\leq 1, \tag{1}$$

subject to the initial condition

$$u\left(0\right)=u_0, \tag{2}$$

where D^{α} is the fractional differential operator (Caputo derivative) of order $0<\alpha\leq 1$, and if we set $\alpha=1$, then the analytical solution of the logistic equation (1) is given by

$$u\left(t\right)=\frac{1}{1+(\frac{1}{u_0}-1)e^{-t}}. \tag{3}$$

Next, we study the fractional Predator-Prey models: fractional Lotka-Volterra systems as an interacting species model defined by [33]

$$D^{\alpha}u\left(t\right)=u-f\left(u,v\right)\ ,\ \ 0<\alpha\leq 1 \tag{4}$$

$$D^{\alpha}v\left(t\right)=\beta\left[g\left(u,v\right)-v\right]\ ,\ \ 0<\alpha\leq 1 \tag{5}$$

subject to

$$u\left(t\right)=u_0, \qquad v\left(t\right)=v_0 \tag{6}$$

where D^{α} is the fractional Caputo derivative of order $0<\alpha\leq 1$, $u{=}u(t)$ is the prey population and $v{=}v(t)$ that of the predator at time t, f and g are nonlinear functions of u and v. Here β is some positive constant.

This paper is organized as follows. In Section 2, some definitions of fractional calculus. In Section 3, an analysis of the method is presented. In Section 4, the approximate solution for the proposed examples is obtained. Finally, a conclusion is drawn in Section 5.

2. Preliminaries

In this section, we give some basic definitions and properties of fractional calculus and Laplace transform, which are used later in this paper.

Definition 1. The Laplace transform for the function $f(t)$, $t\geq 0$, is defined by [32]

$$\mathcal{L}\left(f\left(t\right)\right)=F\left(s\right)=\int_0^{\infty}f\left(t\right)e^{-st}dt. \tag{7}$$

Definition 2. The Mittag-Leffler functions is defined by [32]

$$E_{\alpha}\left(t\right)=\sum_{n=0}^{\infty}\frac{t^n}{\Gamma\left(\alpha n+1\right)}\ , \tag{8}$$

and the generalized Mittag-Leffler function

$$E_{\alpha,\beta}\left(t\right)=\sum_{n=0}^{\infty}\frac{t^n}{\Gamma\left(\alpha n+\beta\right)}\ , \tag{9}$$

For all $\alpha,\ \beta>0$.

Definition 3. The Riemann-Liouville fractional integral operator $\left(I^{\alpha}\right)$, of a function $f\left(t\right)\in C_{\mu}$, $n\in\mathbb{N}$. [32]

$$I^{\alpha}f\left(t\right)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_0^t\left(t-\tau\right)^{\alpha\,-\,1}f\left(\tau\right)d\tau, & \alpha>0\\ f\left(t\right), \ \alpha=0\end{cases} \tag{10}$$

Some properties of the operator $\left(I^{\alpha}\right)$, which are needed here, are as follows [32]:

$I^{\alpha}\left[af\left(t\right)+b\,g(t)\right]=aI^{\alpha}f\left(t\right)+bI^{\alpha}\,g\left(t\right)$,
$I^{\alpha}I^{\beta}f\left(t\right)=I^{\alpha+\beta}f\left(t\right)$,
$I^{\alpha}I^{\beta}f\left(t\right)=I^{\beta}I^{\alpha}f\left(t\right)$,
$I^{\alpha}t^{v}=\frac{\Gamma(v+1)}{\Gamma(v+\alpha+1)}t^{v+\alpha}$.

Definition 4. The Caputo fractional differential operator of $f(t)$ is defined by [34]

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \tag{11}$$

for $n-1 < \alpha \leq n$.

Lemma 1. *For $\alpha, \beta > 0$, $a \in R$ and $s^{\alpha} > |a|$ we have the following inverse Laplace transform formula [32]*

$$\mathcal{L}\left[\frac{s^{\alpha-\beta}}{s^{\alpha}+a}\right] = t^{\beta-1} E_{\alpha,\beta}(-at^{\alpha}). \tag{12}$$

3. Analysis of the Method

In this section, we will explain how to solve some nonlinear fractional differential equations by the proposed method.

3.1. LTM for fractional logistic growth model

We consider the following model equation of the form:

$$D^{\alpha} u(t) = u(t) - f(u), \quad u(0) = u_0, \tag{13}$$

where f is a nonlinear function of u, D^{α} is Caputo derivative of order $0 < \alpha \leq 1$. We look for the solution u satisfying (13). Therefore, we assume the solution, u of (13) has an infinite series representation of the Mittag-Leffler function form

$$u(t) = E_{\alpha}(at^{\alpha}) = \sum_{n=0}^{\infty} a_n \frac{t^{\alpha n}}{\Gamma(\alpha n+1)}, \tag{14}$$

and it satisfies the required conditions for the existence of the Laplace transform, defined in equation (7). Taking the Laplace transform to both sides of the differential equation in (13), we obtain:

$$\frac{s\mathcal{L}([u(t)]) - u(0)}{s^{1-\alpha}} = \mathcal{L}(u(t)) - \mathcal{L}([f(u)]), \tag{15}$$

solving equation (15) for $\mathcal{L}(u(t))$, we obtain

$$\mathcal{L}(u(t)) = \frac{u_0 s^{\alpha-1}}{s^{\alpha}-1} - \frac{\mathcal{L}([f(u)])}{s^{\alpha}-1}. \tag{16}$$

Therefore, assuming the inverse Laplace transform $\mathcal{L}^{-1}$ exists and applying it to (16), we end up with

$$u(t) = u_0 E_{\alpha}(t^{\alpha}) - \mathcal{L}^{-1}\left[\frac{\mathcal{L}([f(u)])}{s^{\alpha}-1}\right]. \tag{17}$$

3.2. LTM for fractional prey-predator model

In this part, we study the system of nonlinear fractional differential equations of the form in equations (4)-(5) with initial condition (6).

We look for the solutions (u, v) satisfying (4)-(5). We assume the solutions u and v of the system (4)-(5) have the following infinite series expansions of the form:

$$u(t) = E_{\alpha}(at^{\alpha}) = \sum_{n=0}^{\infty} a_n \frac{t^{\alpha n}}{\Gamma(\alpha n+1)}, \qquad v(t) = E_{\alpha}(bt^{\alpha}) = \sum_{n=0}^{\infty} b_n \frac{t^{\alpha n}}{\Gamma(\alpha n+1)}. \tag{18}$$

Applying the Laplace transform to the system (4)-(5) and using (6), we get

$$\frac{s\mathcal{L}([u(t)]) - u(0)}{s^{1-\alpha}} = \mathcal{L}(u(t)) - \mathcal{L}([f(u,v)]), \tag{19}$$

$$\frac{s\mathcal{L}([v(t)]) - v(0)}{s^{1-\alpha}} = \mathcal{L}(g(u,v)) - \mathcal{L}([v(t)]), \tag{20}$$

solving equation (19) and (20) for $\mathcal{L}\left(u\left(t\right)\right)$ and $\mathcal{L}\left(v\left(t\right)\right)$, respectively, we obtain

$$\mathcal{L}\left(u\left(t\right)\right)=\frac{u_0 s^{\alpha-1}}{s^{\alpha}-1}-\frac{\mathcal{L}\left(\left[f\left(u,v\right)\right]\right)}{s^{\alpha}-1}, \tag{21}$$

$$\mathcal{L}\left(v\left(t\right)\right)=\frac{v_0 s^{\alpha-1}}{s^{\alpha}+\beta}+\beta\frac{\mathcal{L}\left(\left[g\left(u,v\right)\right]\right)}{s^{\alpha}+\beta}. \tag{22}$$

If we assume the inverse Laplace transforms exist and apply them to the system (21)-(22), we find

$$u\left(t\right)=u_0 E_{\alpha}\left(t^{\alpha}\right)-\mathcal{L}^{-1}\left(\frac{\mathcal{L}\left(\left[f\left(u,v\right)\right]\right)}{s^{\alpha}-1}\right), \tag{23}$$

$$v\left(t\right)=v_0 E_{\alpha}\left(-\beta t^{\alpha}\right)+\beta\mathcal{L}^{-1}\left(\frac{\mathcal{L}\left(\left[g\left(u,v\right)\right]\right)}{s^{\alpha}+\beta}\right), \tag{24}$$

which are the desired solutions of the initial value problem (4)-(5).

4. Applications

In this section, we propose some examples that demonstrate the performance and effectiveness of this method for solving the fractional logistic growth model in a population and a fractional prey-predator model.

Example 1. We consider the initial value problem (13). For numerical computation we take $u_0=2$, and we let $f\left(u\right)=u^2$ as in (1) so that one has

$$\begin{aligned} f\left(u\right)=\left(\sum_{n=0}^{\infty}a_n\frac{t^{\alpha n}}{\Gamma\left(\alpha n+1\right)}\right)^2 &= {a_0}^2+\frac{2a_0a_1}{\Gamma(\alpha+1)}t^{\alpha}+\left[\frac{{a_1}^2}{(\Gamma(\alpha+1))^2}+\frac{2a_0a_2}{\Gamma(2\alpha+1)}\right]t^{2\alpha}\\ &+\left[\frac{2a_0a_3}{\Gamma(3\alpha+1)}+\frac{2a_1a_2}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right]t^{3\alpha}+\dots\,. \end{aligned} \tag{25}$$

We then obtain

$$\begin{aligned} \mathcal{L}\left(\left[f(u)\right]\right)=\frac{{a_0}^2}{s}+\frac{2a_0a_1}{s^{\alpha+1}}+\left[\frac{{a_1}^2}{(\Gamma(\alpha+1))^2}+\frac{2a_0a_2}{\Gamma(2\alpha+1)}\right]\frac{\Gamma(2\alpha+1)}{s^{2\alpha+1}}\\ +\left[\frac{2a_0a_3}{\Gamma(3\alpha+1)}+\frac{2a_1a_2}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right]\frac{\Gamma(3\alpha+1)}{s^{3\alpha+1}}+\cdots. \end{aligned} \tag{26}$$

Using (16), we get

$$\begin{aligned} u\left(t\right)=u_0\,E_{\alpha}\left(t^{\alpha}\right)-\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}-1}\left(\frac{{a_0}^2}{s}+\frac{2a_0a_1}{s^{\alpha+1}}+\left[\frac{{a_1}^2}{\left(\Gamma\left(\alpha+1\right)\right)^2}+\frac{2a_0a_2}{\Gamma\left(2\alpha+1\right)}\right]\frac{\Gamma\left(2\alpha+1\right)}{s^{2\alpha+1}}\right.\right.\\ \left.\left.+\left[\frac{2a_0a_3}{\Gamma\left(3\alpha+1\right)}+\frac{2a_1a_2}{\Gamma\left(\alpha+1\right)\Gamma\left(2\alpha+1\right)}\right]\frac{\Gamma\left(3\alpha+1\right)}{s^{3\alpha+1}}+\dots\right)\right]. \end{aligned} \tag{27}$$

Applying the inverse Laplace transform to this equation and using Lemma 1, we get

$$u\left(t\right)=u_0\,E_{\alpha}\left(t^{\alpha}\right)-{a_0}^2t^{\alpha}E_{\alpha,\alpha+1}\left(t^{\alpha}\right)-2a_0a_1t^{2\alpha}E_{\alpha,2\alpha+1}\left(t^{\alpha}\right)-\left(\frac{{a_1}^2\Gamma\left(2\alpha+1\right)}{\left(\Gamma\left(\alpha+1\right)\right)^2}+2a_0a_2\right)t^{3\alpha}E_{\alpha,3\alpha+1}\left(t^{\alpha}\right)+\dots. \tag{28}$$

Now, use (14) and equating coefficients of power t, yields

$$\left\{\begin{array}{c} a_0=u_0=2,\qquad a_1=u_0-{a_0}^2\Longrightarrow a_1=u_0-{a_0}^2=-2,\\ a_2=u_0-{a_0}^2-2a_0a_1\Longrightarrow a_2=6\,,\\ a_3=u_0-{a_0}^2-2a_0a_1-\left(\frac{{a_1}^2\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}+2a_0a_2\right)\Longrightarrow a_3=-18-4\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}, \end{array}\right. \tag{29}$$

Finally, we can be obtained the solution $u(t)$ from (13) as follows:

$$u(t) = \sum_{n=0}^{\infty} a_n \frac{t^{\alpha n}}{\Gamma(\alpha n+1)} = a_0 + \frac{a_1}{\Gamma(\alpha+1)} t^{\alpha} + \frac{a_2}{\Gamma(2\alpha+1)} t^{2\alpha} + \frac{a_3}{\Gamma(3\alpha+1)} t^{3\alpha} + \ldots, \tag{30}$$

$$u(t) = 2 - \frac{2}{\Gamma(\alpha+1)} t^{\alpha} + \frac{6}{\Gamma(2\alpha+1)} t^{2\alpha} - \frac{\left(18 + 4\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\right)}{\Gamma(3\alpha+1)} t^{3\alpha} + \ldots \tag{31}$$

which is the exact solution obtained in equation (3) in the closed-form. If we set $\alpha = 1$, the solution is precisely the same as the ones we obtained in [35,36].

Example 2. We now solve the problem (4)-(5) with initial data $u(0) = 1.3$, $v(0) = 0.6$. We proceed as in section 3.2. We take $\beta = 1$, $f(u, v) = g(u, v) = uv$ in (4)-(5) so that we have

$$\begin{aligned} f(u, v) = g(u, v) &= \left(\sum_{n=0}^{\infty} a_n \frac{t^{\alpha n}}{\Gamma(\alpha n+1)}\right)\left(\sum_{n=0}^{\infty} b_n \frac{t^{\alpha n}}{\Gamma(\alpha n+1)}\right) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \left(a_0 b_2 + \frac{a_1 b_1 \Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2} + a_2 b_0\right) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots \end{aligned} \tag{32}$$

and the corresponding Laplace transforms of these functions become

$$\mathcal{L}(f(u, v)) = \mathcal{L}(g(u, v)) = \frac{a_0 b_0}{s} + \frac{(a_0 b_1 + a_1 b_0)}{s^{\alpha+1}} + \left(a_0 b_2 + \frac{a_1 b_1 \Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2} + a_2 b_0\right) \frac{1}{s^{2\alpha+1}} + \ldots. \tag{33}$$

Using (23)-(24), one gets

$$u(t) = u_0 E_{\alpha}(t^{\alpha}) - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}-1}\left(\frac{a_0 b_0}{s} + \frac{(a_0 b_1 + a_1 b_0)}{s^{\alpha+1}} + \left(a_0 b_2 + \frac{a_1 b_1 \Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2} + a_2 b_0\right) \frac{1}{s^{2\alpha+1}} + \ldots\right)\right], \tag{34}$$

$$v(t) = v_0 E_{\alpha}(-t^{\alpha}) + \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}+1}\left(\frac{a_0 b_0}{s} + \frac{(a_0 b_1 + a_1 b_0)}{s^{\alpha+1}} + \left(a_0 b_2 + \frac{a_1 b_1 \Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2} + a_2 b_0\right) \frac{1}{s^{2\alpha+1}} + \ldots\right)\right]. \tag{35}$$

Applying the inverse Laplace transform to these equations and using Lemma 1, we get

$$\begin{aligned} u(t) &= u_0 E_{\alpha}(t^{\alpha}) - a_0 b_0 t^{\alpha} E_{\alpha,\alpha+1}(t^{\alpha}) \\ &- (a_0 b_1 + a_1 b_0) t^{2\alpha} E_{\alpha,2\alpha+1}(t^{\alpha}) - \left(a_0 b_2 + \frac{a_1 b_1 \Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2} + a_2 b_0\right) t^{3\alpha} E_{\alpha,3\alpha+1}(t^{\alpha}), \end{aligned} \tag{36}$$

$$\begin{aligned} v(t) &= v_0 E_{\alpha}(-t^{\alpha}) + a_0 b_0 t^{\alpha} E_{\alpha,\alpha+1}(-t^{\alpha}) \\ &+ (a_0 b_1 + a_1 b_0) t^{2\alpha} E_{\alpha,2\alpha+1}(-t^{\alpha}) + \left(a_0 b_2 + \frac{a_1 b_1 \Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2} + a_2 b_0\right) t^{3\alpha} E_{\alpha,3\alpha+1}(-t^{\alpha}). \end{aligned} \tag{37}$$

Using (14) and equating coefficients of power t, yields

$$\begin{cases} a_0 = u_0 = 1.3, \qquad b_0 = v_0 = 0.6 \\ a_1 = u_0 - a_0 b_0 \Longrightarrow a_1 = 0.52, b_1 = -v_0 + a_0 b_0 \Longrightarrow b_1 = 0.18 \end{cases} \tag{38}$$

$$\begin{cases} a_2 = u_0 - a_0 b_0 - (a_0 b_1 + a_1 b_0) \Longrightarrow a_2 = -0.026, \\ b_2 = v_0 - a_0 b_0 + (a_0 b_1 + a_1 b_0) \Longrightarrow b_2 = 0.366, \end{cases} \tag{39}$$

$$\begin{cases} a_3 = u_0 - a_0b_0 - (a_0b_1 + a_1b_0) - \left(a_0b_2 + \frac{a_1b_1\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2} + a_2b_0\right) \\ a_3 = -0.4862 - \frac{0.0936\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}, \\ b_3 = -v_0 + a_0b_0 - (a_0b_1 + a_1b_0) + \left(a_0b_2 + \frac{a_1b_1\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2} + a_2b_0\right) \\ b_3 = 0.0942 + \frac{0.0936\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}, \end{cases} \cdots \tag{40}$$

Writing these terms into equation (18), we get the approximate solutions to the problem (4)-(5) as follows

$$u(t) = 1.3 + \frac{0.52}{\Gamma(\alpha+1)}t^{\alpha} - \frac{0.026}{\Gamma(2\alpha+1)}t^{2\alpha} - \frac{\left(0.4862 + \frac{0.0936\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\right)}{\Gamma(3\alpha+1)}t^{3\alpha} + \dots, \tag{41}$$

$$v(t) = 0.6 + \frac{0.18}{\Gamma(\alpha+1)}t^{\alpha} - \frac{0.366}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{\left(0.0942 + \frac{0.0936\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\right)}{\Gamma(3\alpha+1)}t^{3\alpha} + \dots, \tag{42}$$

which are the same as the ones we obtained in [35,36] if we set $\alpha = 1$.

5. Conclusion

In this study, the Laplace transform method has been successfully applied to two different versions of nonlinear problems. In applications, the method has good results with comparable exact solutions for the nonlinear fractional type of ordinary differential equations. The result obtained shows that the method is a powerful mathematical tool for solving fractional ordinary differential equations. In the future, we will explore the proposed method under other fractional derivatives. Moreover, we can generalize the proposed method by studying the Jafari transform which covers all classes of integral transform in the class of Laplace transform.

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