Open Journal of Mathematical Sciences

PSR Press

Article

New Simpson's type inequalities via (α_1, m_1) -(α_2, m_2)-preinvexity on the coordinates in both the first and second sense

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Academic Editor: Waqas Nazeer
Received: 23 December 2022; Accepted: 25 March 2023; Published: 1 April 2023.

Abstract: In this paper, we established a new integral identity for twice partially differentiable functions. As a consequence, we established some new Simpson's type integral inequalities for functions of two independent variables whose mixed partial derivative is bounded and $(\alpha_1, m_1) - (\alpha_2, m_2)$-preinvex on the coordinates in both the first and second sense.

Keywords: Simpson's inequality; Hölder's inequality; Power mean inequality; Preinvexity.

MSC: 26D10; 26D15; 26A51.

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1. Introduction and Preliminaries

The inequality below is known in the literature as the Simpson's inequality:

$$\left| \int_a^b f(t)dt - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \right| \leq \frac{(b-a)^4}{2880}\left\|f^{(4)}\right\|_\infty,$$

where $f : [a,b] \to \mathbb{R}$ is a four times continuously differentiable function on (a,b) and $\left\|f^{(4)}\right\|_\infty = \sup_{t\in(a,b)} \left|f^{(4)}(t)\right| < \infty$.

The Simpson's inequality which is also know as Newton's type inequality has been studied and generalized by many authors in recent years due to its numerous applications in mathematical analysis as well as the applied sciences. For more information on recent results about the Simpson's inequality, we refer the interested reader to the papers [1–14]. In [15], Özdemir *et al.*, established the following generalizations of the Simpson's inequality for functions of two independent variables.

Theorem 1. *Let $f : \Delta \subset \mathbb{R}^2$ be a partially differentiable mapping on $\Delta = [a,b] \times [c,d]$. If $\frac{\partial^2 f}{\partial t \partial s}$ is bounded, that is, $M := \sup_{(x,y)\in[a,b]\times[c,d]} \left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right| < \infty$, then*

$$\begin{aligned}
&\Bigg|\frac{f\left(a, \frac{c+d}{2}\right) + f\left(b, \frac{c+d}{2}\right) + 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right)}{9} + \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{36} \\
&- \frac{1}{6(d-c)}\int_c^d \left[f(a,v) + 4f\left(\frac{a+b}{2}, v\right) + f(b,v)\right]dv - \frac{1}{6(b-a)}\int_a^b \left[f(u,c) + 4f\left(u, \frac{c+d}{2}\right) + f(u,d)\right]dt \\
&+ \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(u,v)dudv\Bigg| \leq \frac{25(b-a)(d-c)}{1296}M.
\end{aligned}$$

Theorem 2. *Let $f : \Delta \subset \mathbb{R}^2$ be a partially differentiable mapping on $\Delta = [a,b] \times [c,d]$. If $\frac{\partial^2 f}{\partial t \partial s}$ is convex on the co-ordinates on Δ, then the following inequality holds:*

$$\Bigg| \frac{f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)}{9}$$
$$+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{36}-\frac{1}{6(d-c)}\int_c^d\left[f(a,v)+4f\left(\frac{a+b}{2},v\right)+f(b,v)\right]dv$$
$$-\frac{1}{6(b-a)}\int_a^b\left[f(u,c)+4f\left(u,\frac{c+d}{2}\right)+f(u,d)\right]dt+\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(u,v)dudv\Bigg|$$
$$\le \frac{25(b-a)(d-c)}{72}\left[\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|}{72}\right].$$

Recently, several other generalizations of the Simpson's type inequality for functions of two independent variables have been established in the papers [16–20]. Motivated by the current research on convexity and integral inequalities, our goal in this paper is to established some generalizations of the Simpson's inequality for functions of two independent variables whose second-order mixed partial derivatives in absolute value is bounded and functions whose mixed partial derivative in absolute value to certain powers belongs to the class of $(\alpha_1, m_1) - (\alpha_2, m_2)$-preinvex functions on the coordinates in both the first and second sense. Theorems 1 and 2 are particular cases of some our results.

In what follows, we present the definitions of the key concepts related to the class of $(\alpha_1, m_1) - (\alpha_2, m_2)$-preinvex functions on the coordinates in both the first and second sense that would be important for our work and can be found in the papers [21–23].

Definition 1 ([21]). Let S be a nonempty subset of $\mathbb{R}^n$ and $\eta : S \times S \to \mathbb{R}^n$ be a bifunction. We say that S is invex at $x \in S$ with respect to η if

$$x + t\eta(y,x) \in S$$

holds for all $y \in S$ and $t \in [0,1]$.

S is said to be an invex set with respect to η if S is invex at each $x \in S$.

Remark 1. If we take the bifunction η to be given by $\eta(x,y) = x - y$ in Definition 1, then we have the concept of a convex set.

The following is the extension of the above concept to the cartesian product of two sets.

Definition 2 ([22]). Let S_1 and S_2 be two nonempty subsets of $\mathbb{R}^n$ and $\eta_i : S_i \times S_i \to \mathbb{R}^n$ for $i = 1,2$ be continuous function. We say that $S_1 \times S_2$ is invex at $(u,v) \in S_1 \times S_2$ if for each $(x,y) \in S_1 \times S_2$ and $t_1, t_2 \in [0,1]$,

$$(u + t_1\eta_1(x,u), v + t_2\eta_2(y,v)) \in S_1 \times S_2.$$

$S_1 \times S_2$ is said to be an invex set with respect to η_1 and η_2 if $S_1 \times S_2$ is invex at each $(u,v) \in S_1 \times S_2$.

Definition 3 ([23]). A function f on an invex set $S_1 \times S_2 \subset [0,b^*] \times [0,d^*]$ with $b^* > 0$ and $d^* > 0$ is said to be (α, m)-preinvex in the first sense on the co-ordinates with respect to η_1 and η_2 where $\alpha, m \in (0,1]$, if the partial mappings $f_y : S_1 \to \mathbb{R}, f_y(x) = f(x,y)$ and $f_x : S_2 \to \mathbb{R}, f_x(y) = f(x,y)$ are (α, m)-preinvex functions in the first sense with respect to η_1 and η_2 respectively for all $y \in S_2$ and $x \in S_1$.

Remark 2.
1. If $\alpha = m = 1$, $\eta_1(x,u) = x - u$ and $\eta_2(y,v) = y - v$ in Definition 3, then we have the concept of convex functions on the co-ordinates.
2. We deduce from Definition 3 that if f is co-ordinated (α, m)–preinvex function in the first sense, then we have

$$f(u + t\eta_1(x,u), v + s\eta_2(y,v))$$

$$\leq (1-t^{\alpha})(1-s^{\alpha})f(u,v)+m(1-t^{\alpha})s^{\alpha}f\left(u,\frac{y}{m}\right)+mt^{\alpha}(1-s^{\alpha})f\left(\frac{x}{m},v\right)+m^2t^{\alpha}s^{\alpha}f\left(\frac{x}{m},\frac{y}{m}\right).$$

Definition 4 ([23]). A function f on an invex set $S_1\times S_2\subset[0,b^*]\times[0,d^*]$ with $b^*>0$ and $d^*>0$ is said to be (α,m)-preinvex in the second sense on the co-ordinates with respect to η_1 and η_2 where $\alpha,m\in(0,1]$, if the partial mappings $f_y:S_1\to\mathbb{R}, f_y(x)=f(x,y)$ and $f_x:S_2\to\mathbb{R}, f_x(y)=f(x,y)$ are (α,m)-preinvex functions in the second sense with respect to η_1 and η_2 respectively for all $y\in S_2$ and $x\in S_1$.

Remark 3. We deduce from Definition 4 that if f is co-ordinated (α,m)–preinvex function in the second sense, then we have

$$\begin{aligned}&f(u+t\eta_1(x,u),v+s\eta_2(y,v))\\&\quad\leq(1-t)^{\alpha}(1-s)^{\alpha}f(u,v)+m(1-t)^{\alpha}s^{\alpha}f\left(u,\frac{y}{m}\right)+mt^{\alpha}(1-s)^{\alpha}f\left(\frac{x}{m},v\right)+m^2t^{\alpha}s^{\alpha}f\left(\frac{x}{m},\frac{y}{m}\right).\end{aligned}$$

Definition 5 ([23]). A function f on an invex set $S_1\times S_2\subset[0,b^*]\times[0,d^*]$ with $b^*>0$ and $d^*>0$ is said to be $(\alpha_1,m_1)-(\alpha_2,m_2)$-preinvex in the first sense on the co-ordinates with respect to η_1 and η_2 where $\alpha,m\in(0,1]$, if the partial mappings $f_y:S_1\to\mathbb{R}, f_y(x)=f(x,y)$ and $f_x:S_2\to\mathbb{R}, f_x(y)=f(x,y)$ are (α_1,m_1)-preinvex functions in the first sense with respect to η_1 and (α_2,m_2)-preinvex functions in the first sense with respect to η_2 respectively for all $y\in S_2$ and $x\in S_1$.

Remark 4. We deduce from Definition 5 that if f is co-ordinated $(\alpha_1,m_1)-(\alpha_2,m_2)$–preinvex function in the first sense, then we have

$$\begin{aligned}f(u+t\eta_1(x,u),v+s\eta_2(y,v))\leq&(1-t^{\alpha_1})(1-s^{\alpha_2})f(u,v)+m_2(1-t^{\alpha_1})s^{\alpha_2}f\left(u,\frac{y}{m_2}\right)\\&+m_1t^{\alpha_1}(1-s^{\alpha_2})f\left(\frac{x}{m_1},v\right)+m_1m_2t^{\alpha_1}s^{\alpha_2}f\left(\frac{x}{m_1},\frac{y}{m_2}\right).\end{aligned}$$

Definition 6 ([23]). A function f on an invex set $S_1\times S_2\subset[0,b^*]\times[0,d^*]$ with $b^*>0$ and $d^*>0$ is said to be $(\alpha_1,m_1)-(\alpha_2,m_2)$-preinvex in the second sense on the co-ordinates with respect to η_1 and η_2 where $\alpha,m\in(0,1]$, if the partial mappings $f_y:S_1\to\mathbb{R}, f_y(x)=f(x,y)$ and $f_x:S_2\to\mathbb{R}, f_x(y)=f(x,y)$ are (α_1,m_1)-preinvex functions in the second sense with respect to η_1 and (α_2,m_2)-preinvex functions in the second sense with respect to η_2 respectively for all $y\in S_2$ and $x\in S_1$.

Remark 5. We deduce from Definition 6 that if f is co-ordinated $(\alpha_1,m_1)-(\alpha_2,m_2)$–preinvex function in the second sense, then we have

$$\begin{aligned}f(u+t\eta_1(x,u),v+s\eta_2(y,v))\leq&(1-t)^{\alpha_1}(1-s)^{\alpha_2}f(u,v)+m_2(1-t)^{\alpha_1}s^{\alpha_2}f\left(u,\frac{y}{m_2}\right)\\&+m_1t^{\alpha_1}(1-s)^{\alpha_2}f\left(\frac{x}{m_1},v\right)+m_1m_2t^{\alpha_1}s^{\alpha_2}f\left(\frac{x}{m_1},\frac{y}{m_2}\right).\end{aligned}$$

These generalized class of functions are the two dimensional versions of the functions introduced by Latif and Shoaib in [24]. For more information about these generalized class of functions and related results, we refer the interested to the paper [23].

2. Main results

To establish our main results, we need the following identity.

Lemma 1. *Let S_1 and S_2 be open invex subsets of $\mathbb{R}$ with respect to the bifunctions $\eta_1:S_1\times S_1\to\mathbb{R}$ and $\eta_2:S_2\times S_2\to\mathbb{R}$ and let $f:S_1\times S_2\to\mathbb{R}$ be a twice partially differentiable mapping. If $\frac{\partial f}{\partial t\partial s}\in L_1([a,a+\eta_1(a,b)]\times[c,c+\eta_2(c,d)])$ with $a,b\in S_1$ and $c,d\in S_2$ such that $\eta_1(a,b)\neq0$ and $\eta_2(c,d)\neq0$, then the following identity holds:*

$$\frac{1}{9}\left\{f\left(b+\eta_1(a,b),d+\frac{1}{2}\eta_2(c,d)\right)+f\left(b,d+\frac{1}{2}\eta_2(c,d)\right)+4f\left(b+\frac{1}{2}\eta_1(a,b),d+\frac{1}{2}\eta_2(c,d)\right)\right.$$

$$
\begin{aligned}
&+f\left(b+\frac{1}{2}\eta_1(a,b),d+\eta_2(c,d)\right)+f\left(b+\frac{1}{2}\eta_1(a,b),d\right)\Bigg\}\\
&+\frac{1}{36}\Big\{f(b+\eta_1(a,b),d+\eta_2(c,d))+f(b+\eta_1(a,b),d)+f(b,d+\eta_2(c,d))+f(b,d)\Big\}\\
&-\frac{1}{6\eta_2(c,d)}\int_d^{d+\eta_2(c,d)}\left[f(b+\eta_1(a,b),v)+4f\left(b+\frac{1}{2}\eta_1(a,b),v\right)+f(b,v)\right]dv\\
&-\frac{1}{6\eta_1(a,b)}\int_b^{b+\eta_1(a,b)}\left[f(u,d+\eta_2(c,d))+4f\left(u,d+\frac{1}{2}\eta_2(c,d)\right)+f(u,d)\right]dt\\
&+\frac{1}{\eta_1(a,b)\eta_2(c,d)}\int_d^{d+\eta_2(c,d)}\int_b^{b+\eta_1(a,b)}f(u,v)dudv\\
=\,&\eta_1(a,b)\eta_2(c,d)\int_0^1\int_0^1 p(t)p(s)\frac{\partial^2 f}{\partial t\partial s}(b+t\eta_1(a,b),d+s\eta_2(c,d))dtds
\end{aligned}
$$

where $p(w)=\begin{cases} w-\frac{1}{6}, & w\in\left[0,\frac{1}{2}\right]\\ w-\frac{5}{6} & w\in\left(\frac{1}{2},1\right]\end{cases}.$

Proof. By integrating by parts, we have

$$
\begin{aligned}
\int_0^1 p(t)\frac{\partial^2 f}{\partial t\partial s}(b+t\eta_1(a,b),v)dt &= \int_0^{\frac{1}{2}}\left(t-\frac{1}{6}\right)\frac{\partial^2 f}{\partial t\partial s}(b+t\eta_1(a,b),v)dt+\int_{\frac{1}{2}}^1\left(t-\frac{5}{6}\right)\frac{\partial^2 f}{\partial t\partial s}(b+t\eta_1(a,b),v)dt\\
&=\frac{1}{\eta_1(a,b)}\left[\frac{2}{3}\frac{\partial f}{\partial s}\left(b+\frac{1}{2}\eta_1(a,b),v\right)+\frac{1}{6}\frac{\partial f}{\partial s}(b,v)+\frac{1}{6}\frac{\partial f}{\partial s}(b+\eta_1(a,b),v)\right.\\
&\qquad\left.-\int_0^1\frac{\partial f}{\partial s}(b+t\eta_1(a,b),v)dt\right]. \qquad (1)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\int_0^1 p(s)\frac{\partial f}{\partial s}(u,d+s\eta_2(c,d))ds &= \frac{1}{\eta_2(c,d)}\left[\frac{2}{3}f\left(u,d+\frac{1}{2}\eta_2(c,d)\right)+\frac{1}{6}f(u,d)+\frac{1}{6}f(u,d+\eta_2(c,d))\right.\\
&\qquad\left.-\int_0^1 f(u,d+s\eta_2(c,d))ds\right]. \qquad (2)
\end{aligned}
$$

By using (1) and (2), we have

$$
\begin{aligned}
&\int_0^1\int_0^1 p(t)p(s)\frac{\partial^2 f}{\partial t\partial s}(b+t\eta_1(a,b),d+s\eta_2(c,d))dtds\\
&=\frac{1}{\eta_1(a,b)}\left[\frac{2}{3}\int_0^1 p(s)\frac{\partial f}{\partial s}\left(b+\frac{1}{2}\eta_1(a,b),d+s\eta_2(c,d)\right)ds+\frac{1}{6}\int_0^1 p(s)\frac{\partial f}{\partial s}(b,d+s\eta_2(c,d))ds\right.\\
&\quad\left.+\frac{1}{6}\int_0^1 p(s)\frac{\partial f}{\partial s}(b+\eta_1(a,b),d+s\eta_2(c,d))ds-\int_0^1\int_0^1 p(s)\frac{\partial f}{\partial s}(b+t\eta_1(a,b),d+s\eta_2(c,d))dsdt\right]\\
&=\frac{1}{\eta_1(a,b)}\left[\frac{2}{3}\left\{\frac{1}{\eta_2(c,d)}\left[\frac{2}{3}f\left(b+\frac{1}{2}\eta_1(a,b),d+\frac{1}{2}\eta_2(c,d)\right)+\frac{1}{6}f\left(b+\frac{1}{2}\eta_1(a,b),d\right)\right.\right.\right.\\
&\quad\left.\left.+\frac{1}{6}f\left(b+\frac{1}{2}\eta_1(a,b),d+\eta_2(c,d)\right)-\int_0^1 f\left(b+\frac{1}{2}\eta_1(a,b),d+s\eta_2(c,d)\right)ds\right]\right\}\\
&\quad+\frac{1}{6}\left\{\frac{1}{\eta_2(c,d)}\left[\frac{2}{3}f\left(b,d+\frac{1}{2}\eta_2(c,d)\right)+\frac{1}{6}f(b,d)+\frac{1}{6}f(b,d+\eta_2(c,d))-\int_0^1 f(b,d+s\eta_2(c,d))ds\right]\right\}\\
&\quad+\frac{1}{6}\left\{\frac{1}{\eta_2(c,d)}\left[\frac{2}{3}f\left(b+\eta_1(a,b),d+\frac{1}{2}\eta_2(c,d)\right)+\frac{1}{6}f(b+\eta_1(a,b),d)\right.\right.\\
&\quad\left.\left.+\frac{1}{6}f(b+\eta_1(a,b),d+\eta_2(c,d))-\int_0^1 f(b+\eta_1(a,b),d+s\eta_2(c,d))ds\right]\right\}\\
&\quad-\int_0^1\left\{\frac{1}{\eta_2(c,d)}\left[\frac{2}{3}f\left(b+t\eta_1(a,b),d+\frac{1}{2}\eta_2(c,d)\right)+\frac{1}{6}f(b+t\eta_1(a,b),d)\right.\right.\\
&\quad\left.\left.\left.+\frac{1}{6}f(b+t\eta_1(a,b),d+\eta_2(c,d))-\int_0^1 f(b+t\eta_1(a,b),d+s\eta_2(c,d))ds\right]\right\}dt\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\eta_1(a,b)\eta_2(c,d)}\left[\frac{4}{9}f\left(b+\frac{1}{2}\eta_1(a,b),d+\frac{1}{2}\eta_2(c,d)\right)+\frac{1}{9}f\left(b+\frac{1}{2}\eta_1(a,b),d\right)\right.\\
&\quad+\frac{1}{9}f\left(b+\frac{1}{2}\eta_1(a,b),d+\eta_2(c,d)\right)-\frac{2}{3}\int_0^1 f\left(b+\frac{1}{2}\eta_1(a,b),d+s\eta_2(c,d)\right)ds\\
&\quad+\frac{1}{9}f\left(b,d+\frac{1}{2}\eta_2(c,d)\right)+\frac{1}{36}f(b,d)+\frac{1}{36}f(b,d+\eta_2(c,d))-\frac{1}{6}\int_0^1 f(b,d+s\eta_2(c,d))ds\\
&\quad+\frac{1}{9}f\left(b+\eta_1(a,b),d+\frac{1}{2}\eta_2(c,d)\right)+\frac{1}{36}f(b+\eta_1(a,b),d)\\
&\quad+\frac{1}{36}f(b+\eta_1(a,b),d+\eta_2(c,d))-\frac{1}{6}\int_0^1 f(b+\eta_1(a,b),d+s\eta_2(c,d))ds\\
&\quad-\frac{2}{3}\int_0^1 f\left(b+t\eta_1(a,b),d+\frac{1}{2}\eta_2(c,d)\right)dt-\frac{1}{6}\int_0^1 f(b+t\eta_1(a,b),d)dt\\
&\quad\left.-\frac{1}{6}\int_0^1 f(b+t\eta_1(a,b),d+\eta_2(c,d))dt+\int_0^1\int_0^1 f(b+t\eta_1(a,b),d+s\eta_2(c,d))dsdt\right]\\
&= \frac{1}{\eta_1(a,b)\eta_2(c,d)}\left[\frac{1}{9}\left\{f\left(b+\eta_1(a,b),d+\frac{1}{2}\eta_2(c,d)\right)+f\left(b,d+\frac{1}{2}\eta_2(c,d)\right)\right.\right.\\
&\quad\left.+4f\left(b+\frac{1}{2}\eta_1(a,b),d+\frac{1}{2}\eta_2(c,d)\right)+f\left(b+\frac{1}{2}\eta_1(a,b),d+\eta_2(c,d)\right)+f\left(b+\frac{1}{2}\eta_1(a,b),d\right)\right\}\\
&\quad+\frac{1}{36}\left\{f(b+\eta_1(a,b),d+\eta_2(c,d))+f(b+\eta_1(a,b),d)+f(b,d+\eta_2(c,d))+f(b,d)\right\}\\
&\quad-\frac{1}{6}\int_0^1\left[f(b+\eta_1(a,b),d+s\eta_2(c,d))+4f\left(b+\frac{1}{2}\eta_1(a,b),d+s\eta_2(c,d)\right)+f(b,d+s\eta_2(c,d))\right]ds\\
&\quad-\frac{1}{6}\int_0^1\left[f(b+t\eta_1(a,b),d+\eta_2(c,d))+4f\left(b+t\eta_1(a,b),d+\frac{1}{2}\eta_2(c,d)\right)+f(b+t\eta_1(a,b),d)\right]dt\\
&\quad\left.+\int_0^1\int_0^1 f(b+t\eta_1(a,b),d+s\eta_2(c,d))dsdt\right]\\
&= \frac{1}{\eta_1(a,b)\eta_2(c,d)}\left[\frac{1}{9}\left\{f\left(b+\eta_1(a,b),d+\frac{1}{2}\eta_2(c,d)\right)+f\left(b,d+\frac{1}{2}\eta_2(c,d)\right)\right.\right.\\
&\quad\left.+4f\left(b+\frac{1}{2}\eta_1(a,b),d+\frac{1}{2}\eta_2(c,d)\right)+f\left(b+\frac{1}{2}\eta_1(a,b),d+\eta_2(c,d)\right)+f\left(b+\frac{1}{2}\eta_1(a,b),d\right)\right\}\\
&\quad+\frac{1}{36}\left\{f(b+\eta_1(a,b),d+\eta_2(c,d))+f(b+\eta_1(a,b),d)+f(b,d+\eta_2(c,d))+f(b,d)\right\}\\
&\quad-\frac{1}{6\eta_2(c,d)}\int_d^{d+\eta_2(c,d)}\left[f(b+\eta_1(a,b),v)+4f\left(b+\frac{1}{2}\eta_1(a,b),v\right)+f(b,v)\right]dv\\
&\quad-\frac{1}{6\eta_1(a,b)}\int_b^{b+\eta_1(a,b)}\left[f(u,d+\eta_2(c,d))+4f\left(u,d+\frac{1}{2}\eta_2(c,d)\right)+f(u,d)\right]dt\\
&\quad\left.+\frac{1}{\eta_1(a,b)\eta_2(c,d)}\int_d^{d+\eta_2(c,d)}\int_b^{b+\eta_1(a,b)} f(u,v)dudv\right].
\end{aligned}
$$

This completes the proof. □

Lemma 2. *For any $\alpha \geq 0$, we have*

$$
\begin{aligned}
\mathcal{B}(\alpha) &:= \int_0^1 |p(t)|t^{\alpha}dt = \int_0^1 |p(t)|(1-t)^{\alpha}dt\\
&=\frac{2^{-\alpha-1}}{3^{\alpha+2}(\alpha+1)}-\frac{2^{-\alpha-1}}{3^{\alpha+2}(\alpha+2)}+\frac{1}{2^{\alpha+2}(\alpha+2)}-\frac{1}{3\cdot 2^{\alpha+2}(\alpha+1)}\\
&\quad+\frac{5^{\alpha+2}\cdot 2^{-\alpha-1}}{3^{\alpha+2}(\alpha+1)}+\frac{1}{\alpha+2}-\frac{5}{6(\alpha+1)}+\frac{1}{2^{\alpha+2}(\alpha+2)}-\frac{5}{3\cdot 2^{\alpha+2}(\alpha+1)}-\frac{5^{\alpha+2}\cdot 2^{-\alpha-1}}{3^{\alpha+2}(\alpha+2)},
\end{aligned}
$$

$$
\mathcal{C}(\alpha) := \int_0^1 |p(t)|^{\alpha}dt = \frac{2}{6^{\alpha+1}(\alpha+1)}+\frac{2}{3^{\alpha+1}(\alpha+1)}
$$

and

$$\mathcal{D}(\alpha) := \int_0^1 |p(t)|(1-t^\alpha)dt = \mathcal{C}(1) - \mathcal{B}(\alpha)$$

where $p(t) = \begin{cases} t - \frac{1}{6}, & t \in \left[0, \frac{1}{2}\right], \\ t - \frac{5}{6}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$

Theorem 3. *Under the conditions of Lemma 1, suppose that* $\frac{\partial^2 f}{\partial t \partial s}$ *is bounded, i.e,* $M := \sup_{(x,y)\in[a,b]\times[c,d]} \left|\frac{\partial^2 f}{\partial t \partial s}(x,y)\right| < \infty$*, then*

$$\begin{aligned}
&\Bigg|\frac{1}{9}\left\{f\left(b+\eta_1(a,b), d+\frac{1}{2}\eta_2(c,d)\right) + f\left(b, d+\frac{1}{2}\eta_2(c,d)\right) + 4f\left(b+\frac{1}{2}\eta_1(a,b), d+\frac{1}{2}\eta_2(c,d)\right)\right.\\
&\quad \left. + f\left(b+\frac{1}{2}\eta_1(a,b), d+\eta_2(c,d)\right) + f\left(b+\frac{1}{2}\eta_1(a,b), d\right)\right\}\\
&\quad + \frac{1}{36}\left\{f(b+\eta_1(a,b), d+\eta_2(c,d)) + f(b+\eta_1(a,b), d) + f(b, d+\eta_2(c,d)) + f(b,d)\right\}\\
&\quad - \frac{1}{6\eta_2(c,d)}\int_d^{d+\eta_2(c,d)}\left[f(b+\eta_1(a,b), v) + 4f\left(b+\frac{1}{2}\eta_1(a,b), v\right) + f(b,v)\right]dv\\
&\quad - \frac{1}{6\eta_1(a,b)}\int_b^{b+\eta_1(a,b)}\left[f(u, d+\eta_2(c,d)) + 4f\left(u, d+\frac{1}{2}\eta_2(c,d)\right) + f(u,d)\right]dt\\
&\quad + \frac{1}{\eta_1(a,b)\eta_2(c,d)}\int_d^{d+\eta_2(c,d)}\int_b^{b+\eta_1(a,b)} f(u,v)dudv\Bigg|\\
&\leq \frac{25|\eta_1(a,b)||\eta_2(c,d)|}{1296}M.
\end{aligned}$$

Proof. By using the properties of the absolute value and the boundedness of $\frac{\partial^2 f}{\partial t \partial s}$, we have

$$\begin{aligned}
&\left|\int_0^1\int_0^1 p(t)p(s)\,\frac{\partial^2 f}{\partial t \partial s}(b+t\eta_1(a,b), d+s\eta_2(c,d))dtds\right|\\
&\qquad \leq \int_0^1\int_0^1 |p(t)p(s)|\left|\frac{\partial^2 f}{\partial t \partial s}(b+t\eta_1(a,b), d+s\eta_2(c,d))\right|dtds\\
&\qquad \leq M\int_0^1\int_0^1 |p(t)p(s)|\,dtds. \qquad (3)
\end{aligned}$$

By Lemma 2, we deduce that

$$\int_0^1\int_0^1 |p(t)p(s)|dtds = (\mathcal{C}(1))^2 = \frac{25}{1296}. \qquad (4)$$

The result follows directly from Lemma 1 by using (3) and (4).
□

Remark 6. If $\eta_1(x,u) = x-u$ and $\eta_2(y,v) = y-v$ in Theorem 3, then we obtain Theorem 1.

Theorem 4. *Under the conditions of Lemma 1, let* $S_1, S_2 \subseteq [0,\infty)$*,* $\eta(a,b) > 0$ *and* $\eta_2(c,d) > 0$*. If* $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$*,* $q \geq 1$ *is* $(\alpha_1, m_1) - (\alpha_2, m_2)$*-preinvex on the coordinates in the first sense, then*

$$\begin{aligned}
&\Bigg|\frac{1}{9}\left\{f\left(b+\eta_1(a,b), d+\frac{1}{2}\eta_2(c,d)\right) + f\left(b, d+\frac{1}{2}\eta_2(c,d)\right) + 4f\left(b+\frac{1}{2}\eta_1(a,b), d+\frac{1}{2}\eta_2(c,d)\right)\right.\\
&\quad \left. + f\left(b+\frac{1}{2}\eta_1(a,b), d+\eta_2(c,d)\right) + f\left(b+\frac{1}{2}\eta_1(a,b), d\right)\right\}\\
&\quad + \frac{1}{36}\left\{f(b+\eta_1(a,b), d+\eta_2(c,d)) + f(b+\eta_1(a,b), d) + f(b, d+\eta_2(c,d)) + f(b,d)\right\}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{6\eta_2(c,d)}\int_d^{d+\eta_2(c,d)}\left[f(b+\eta_1(a,b),v)+4f\left(b+\frac{1}{2}\eta_1(a,b),v\right)+f(b,v)\right]dv\\
&-\frac{1}{6\eta_1(a,b)}\int_b^{b+\eta_1(a,b)}\left[f(u,d+\eta_2(c,d))+4f\left(u,d+\frac{1}{2}\eta_2(c,d)\right)+f(u,d)\right]dt\\
&+\frac{1}{\eta_1(a,b)\eta_2(c,d)}\int_d^{d+\eta_2(c,d)}\int_b^{b+\eta_1(a,b)}f(u,v)dudv\Bigg|\\
\le\ &\eta_1(a,b)\eta_2(c,d)\left(\frac{25}{1296}\right)^{1-\frac{1}{q}}\left(\mathcal{D}(\alpha_1)\mathcal{D}(\alpha_2)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q+m_2\mathcal{D}(\alpha_1)\mathcal{B}(\alpha_2)\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c}{m_2}\right)\right|^q\right.\\
&\left.+m_1\mathcal{B}(\alpha_1)\mathcal{D}(\alpha_2)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a}{m_1},d\right)\right|^q+m_1m_2\mathcal{B}(\alpha_1)\mathcal{B}(\alpha_2)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a}{m_1},\frac{c}{m_2}\right)\right|^q\right)^{\frac{1}{q}}
\end{aligned}
$$

where $\mathcal{B}(\alpha)$ and $\mathcal{D}(\alpha)$ are as defined in Lemma 2.

Proof. By using the power mean inequality, we have

$$
\begin{aligned}
&\left|\int_0^1\int_0^1 p(t)p(s)\frac{\partial^2 f}{\partial t\partial s}(b+t\eta_1(a,b),d+s\eta_2(c,d))dtds\right|\\
&\le\left(\int_0^1\int_0^1|p(t)p(s)|\,dtds\right)^{1-\frac{1}{q}}\left(\int_0^1\int_0^1|p(t)p(s)|\left|\frac{\partial^2 f}{\partial t\partial s}(b+t\eta_1(a,b),d+s\eta_2(c,d))\right|^q dtds\right)^{\frac{1}{q}}. \quad (5)
\end{aligned}
$$

Using the $(\alpha_1,m_1)-(\alpha_2,m_2)$-preinvexity in the first sense on the coordinates of $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, we obtain

$$
\begin{aligned}
&\int_0^1\int_0^1|p(t)p(s)|\left|\frac{\partial^2 f}{\partial t\partial s}(b+t\eta_1(a,b),d+s\eta_2(c,d))\right|^q\\
&\le\int_0^1\int_0^1|p(t)p(s)|\left((1-t^{\alpha_1})(1-s^{\alpha_2})\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q dtds+m_2(1-t^{\alpha_1})s^{\alpha_2}\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c}{m_2}\right)\right|^q\right.\\
&\left.+m_1t^{\alpha_1}(1-s^{\alpha_2})\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a}{m_1},d\right)\right|^q+m_1m_2t^{\alpha_1}s^{\alpha_2}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a}{m_1},\frac{c}{m_2}\right)\right|^q\right)dtds\\
&=E_1\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|+E_2m_2\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c}{m_2}\right)\right|^q+E_3m_1\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a}{m_1},d\right)\right|^q+E_4m_1m_2\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a}{m_1},\frac{c}{m_2}\right)\right|^q,
\end{aligned}
$$

where

$$
\begin{aligned}
E_1&=\int_0^1\int_0^1|p(t)p(s)|\,(1-t^{\alpha_1})(1-s^{\alpha_2})dtds=\left(\int_0^1|p(t)|\,(1-t^{\alpha_1})dt\right)\left(\int_0^1|p(s)|\,(1-s^{\alpha_2})ds\right),\\
E_2&=\int_0^1\int_0^1|p(t)p(s)|\,(1-t^{\alpha_1})s^{\alpha_2}dtds=\left(\int_0^1|p(t)|\,(1-t^{\alpha_1})dt\right)\left(\int_0^1|p(s)|\,s^{\alpha_2}ds\right),\\
E_3&=\int_0^1\int_0^1|p(t)p(s)|\,t^{\alpha_1}(1-s^{\alpha_2})dtds=\left(\int_0^1|p(t)|\,t^{\alpha_1}dt\right)\left(\int_0^1|p(s)|\,(1-s^{\alpha_2})ds\right),\\
E_4&=\int_0^1\int_0^1|p(t)p(s)|\,t^{\alpha_1}s^{\alpha_2}dtds=\left(\int_0^1|p(t)|\,t^{\alpha_1}dt\right)\left(\int_0^1|p(s)|\,s^{\alpha_2}ds\right).
\end{aligned}
$$

By Lemma 2, we deduce that

$$E_1=\mathcal{D}(\alpha_1)\mathcal{D}(\alpha_2),\ \ E_2=\mathcal{D}(\alpha_1)\mathcal{B}(\alpha_2),\ \ E_3=\mathcal{B}(\alpha_1)\mathcal{D}(\alpha_2),\ \text{ and } E_4=\mathcal{B}(\alpha_1)\mathcal{B}(\alpha_2).$$

Thus,

$$
\begin{aligned}
&\int_0^1\int_0^1|p(t)p(s)|\left|\frac{\partial^2 f}{\partial t\partial s}(b+t\eta_1(a,b),d+s\eta_2(c,d))\right|^q\\
&\le\mathcal{D}(\alpha_1)\mathcal{D}(\alpha_2)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q+m_2\mathcal{D}(\alpha_1)\mathcal{B}(\alpha_2)\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c}{m_2}\right)\right|^q
\end{aligned}
$$

$$+ m_1\mathcal{B}(\alpha_1)\mathcal{D}(\alpha_2)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a}{m_1},d\right)\right|^q + m_1m_2\mathcal{B}(\alpha_1)\mathcal{B}(\alpha_2)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a}{m_1},\frac{c}{m_2}\right)\right|^q. \quad (6)$$

Substituting (6) and (12) in (5), we have

$$\begin{aligned}
&\left|\int_0^1\int_0^1 p(t)p(s)\frac{\partial^2 f}{\partial t\partial s}(b+t\eta_1(a,b),d+s\eta_2(c,d))dtds\right| \\
&\leq \int_0^1\int_0^1 |p(t)p(s)|\left|\frac{\partial^2 f}{\partial t\partial s}(b+t\eta_1(a,b),d+s\eta_2(c,d))\right|dtds \\
&\leq \left(\frac{25}{1296}\right)^{1-\frac{1}{q}}\left(\mathcal{D}(\alpha_1)\mathcal{D}(\alpha_2)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q + m_2\mathcal{D}(\alpha_1)\mathcal{B}(\alpha_2)\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c}{m_2}\right)\right|^q\right. \\
&\left.+ m_1\mathcal{B}(\alpha_1)\mathcal{D}(\alpha_2)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a}{m_1},d\right)\right|^q + m_1m_2\mathcal{B}(\alpha_1)\mathcal{B}(\alpha_2)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a}{m_1},\frac{c}{m_2}\right)\right|^q\right)^{\frac{1}{q}}. \quad (7)
\end{aligned}$$

By using the identity in Lemma 1 and (7) we obtain the desired result.
□

Remark 7. If $q = \alpha_1 = \alpha_2 = m_1 = m_2 = 1$, $\eta_1(x,u) = x - u$ and $\eta_2(y,v) = y - v$ in Theorem 4, then we obtain Theorem 1.

Theorem 5. *Under the conditions of Lemma 1, let $S_1, S_2 \subseteq [0,\infty)$, $\eta(a,b) > 0$ and $\eta_2(c,d) > 0$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q > 1$ is $(\alpha_1,m_1)-(\alpha_2,m_2)$-preinvex on the coordinates in the first sense, then*

$$\begin{aligned}
&\left|\frac{1}{9}\left\{f\left(b+\eta_1(a,b),d+\frac{1}{2}\eta_2(c,d)\right)+f\left(b,d+\frac{1}{2}\eta_2(c,d)\right)+4f\left(b+\frac{1}{2}\eta_1(a,b),d+\frac{1}{2}\eta_2(c,d)\right)\right.\right. \\
&\left.+f\left(b+\frac{1}{2}\eta_1(a,b),d+\eta_2(c,d)\right)+f\left(b+\frac{1}{2}\eta_1(a,b),d\right)\right\} \\
&+\frac{1}{36}\Big\{f(b+\eta_1(a,b),d+\eta_2(c,d))+f(b+\eta_1(a,b),d)+f(b,d+\eta_2(c,d))+f(b,d)\Big\} \\
&-\frac{1}{6\eta_2(c,d)}\int_d^{d+\eta_2(c,d)}\left[f(b+\eta_1(a,b),v)+4f\left(b+\frac{1}{2}\eta_1(a,b),v\right)+f(b,v)\right]dv \\
&-\frac{1}{6\eta_1(a,b)}\int_b^{b+\eta_1(a,b)}\left[f(u,d+\eta_2(c,d))+4f\left(u,d+\frac{1}{2}\eta_2(c,d)\right)+f(u,d)\right]dt \\
&\left.+\frac{1}{\eta_1(a,b)\eta_2(c,d)}\int_d^{d+\eta_2(c,d)}\int_b^{b+\eta_1(a,b)}f(u,v)dudv\right| \\
&\leq \eta_1(a,b)\eta_2(c,d)(\mathcal{C}(r))^{\frac{2}{r}}\left(\frac{1}{(\alpha_1+1)(\alpha_2+1)}\right)^{\frac{1}{q}}\left(\alpha_1\alpha_2\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q+m_2\alpha_1\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c}{m_2}\right)\right|^q\right. \\
&\left.+m_1\alpha_2\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a}{m_1},d\right)\right|^q+m_1m_2\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a}{m_1},\frac{c}{m_2}\right)\right|^q\right)^{\frac{1}{q}},
\end{aligned}$$

where $\mathcal{C}(r)$ is as defined in Lemma 2.

Proof. By using the Höder's inequality, we have

$$\begin{aligned}
&\left|\int_0^1\int_0^1 p(t)p(s)\frac{\partial^2 f}{\partial t\partial s}(b+t\eta_1(a,b),d+s\eta_2(c,d))dtds\right| \\
&\leq \left(\int_0^1\int_0^1 |p(t)p(s)|^r dtds\right)^{\frac{1}{r}}\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}(b+t\eta_1(a,b),d+s\eta_2(c,d))\right|^q dtds\right)^{\frac{1}{q}}. \quad (8)
\end{aligned}$$

Using the $(\alpha_1, m_1) - (\alpha_2, m_2)$-preinvexity in the first sense on the coordinates of $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$, we obtain

$$\begin{aligned}
&\int_0^1 \int_0^1 \left|\frac{\partial^2 f}{\partial t \partial s}(b + t\eta_1(a,b), d + s\eta_2(c,d))\right|^q \\
&\leq \int_0^1 \int_0^1 \Bigg((1-t^{\alpha_1})(1-s^{\alpha_2}) \left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q dtds + m_2 (1-t^{\alpha_1}) s^{\alpha_2} \left|\frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c}{m_2}\right)\right|^q \\
&\quad + m_1 t^{\alpha_1}(1-s^{\alpha_2}) \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a}{m_1}, d\right)\right|^q + m_1 m_2 t^{\alpha_1} s^{\alpha_2} \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a}{m_1}, \frac{c}{m_2}\right)\right|^q \Bigg) dtds \\
&= \left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q \int_0^1 \int_0^1 (1-t^{\alpha_1})(1-s^{\alpha_2}) dtds + m_2 \left|\frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c}{m_2}\right)\right|^q \int_0^1 \int_0^1 (1-t^{\alpha_1}) s^{\alpha_2} dtds \\
&\quad + m_1 \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a}{m_1}, d\right)\right|^q \int_0^1 \int_0^1 t^{\alpha_1}(1-s^{\alpha_2}) dtds + m_1 m_2 \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a}{m_1}, \frac{c}{m_2}\right)\right|^q \int_0^1 \int_0^1 t^{\alpha_1} s^{\alpha_2} dtds \\
&= \frac{1}{(\alpha_1+1)(\alpha_2+1)} \Bigg(\alpha_1 \alpha_2 \left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q + m_2 \alpha_1 \left|\frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c}{m_2}\right)\right|^q \\
&\quad + m_1 \alpha_2 \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a}{m_1}, d\right)\right|^q + m_1 m_2 \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a}{m_1}, \frac{c}{m_2}\right)\right|^q \Bigg).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_0^1 \int_0^1 \left|\frac{\partial^2 f}{\partial t \partial s}(b + t\eta_1(a,b), d + s\eta_2(c,d))\right|^q \\
&\leq \frac{1}{(\alpha_1+1)(\alpha_2+1)} \Bigg(\alpha_1 \alpha_2 \left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q + m_2 \alpha_1 \left|\frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c}{m_2}\right)\right|^q \\
&\quad + m_1 \alpha_2 \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a}{m_1}, d\right)\right|^q + m_1 m_2 \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a}{m_1}, \frac{c}{m_2}\right)\right|^q \Bigg). \qquad (9)
\end{aligned}$$

By using the identity in Lemma 1, (8), (9) and Lemma 2, we obtain the desired result.
□

Theorem 6. *Under the conditions of Lemma 1, let $S_1, S_2 \subseteq [0, \infty)$, $\eta(a,b) > 0$ and $\eta_2(c,d) > 0$. If $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$, $q \geq 1$ is $(\alpha_1, m_1) - (\alpha_2, m_2)$-preinvex on the coordinates in the second sense, then*

$$\begin{aligned}
&\Bigg| \frac{1}{9} \Bigg\{ f\left(b + \eta_1(a,b), d + \frac{1}{2}\eta_2(c,d)\right) + f\left(b, d + \frac{1}{2}\eta_2(c,d)\right) + 4f\left(b + \frac{1}{2}\eta_1(a,b), d + \frac{1}{2}\eta_2(c,d)\right) \\
&\quad + f\left(b + \frac{1}{2}\eta_1(a,b), d + \eta_2(c,d)\right) + f\left(b + \frac{1}{2}\eta_1(a,b), d\right) \Bigg\} \\
&\quad + \frac{1}{36} \Big\{ f(b + \eta_1(a,b), d + \eta_2(c,d)) + f(b + \eta_1(a,b), d) + f(b, d + \eta_2(c,d)) + f(b,d) \Big\} \\
&\quad - \frac{1}{6\eta_2(c,d)} \int_d^{d+\eta_2(c,d)} \left[f(b + \eta_1(a,b), v) + 4f\left(b + \frac{1}{2}\eta_1(a,b), v\right) + f(b,v) \right] dv \\
&\quad - \frac{1}{6\eta_1(a,b)} \int_b^{b+\eta_1(a,b)} \left[f(u, d + \eta_2(c,d)) + 4f\left(u, d + \frac{1}{2}\eta_2(c,d)\right) + f(u,d) \right] dt \\
&\quad + \frac{1}{\eta_1(a,b)\eta_2(c,d)} \int_d^{d+\eta_2(c,d)} \int_b^{b+\eta_1(a,b)} f(u,v) du dv \Bigg| \\
&\leq \eta_1(a,b)\eta_2(c,d) \left(\frac{25}{1296}\right)^{1-\frac{1}{q}} (\mathcal{B}(\alpha_1)\mathcal{B}(\alpha_2))^{\frac{1}{q}} \\
&\quad \times \left(\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q + m_2 \left|\frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c}{m_2}\right)\right|^q + m_1 \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a}{m_1}, d\right)\right|^q + m_1 m_2 \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a}{m_1}, \frac{c}{m_2}\right)\right|^q \right)^{\frac{1}{q}},
\end{aligned}$$

where $\mathcal{B}(\alpha)$ is as defined in Lemma 2.

Proof. By using the power mean inequality, we have

$$\left|\int_0^1\int_0^1 p(t)p(s)\frac{\partial^2 f}{\partial t\partial s}(b+t\eta_1(a,b),d+s\eta_2(c,d))dtds\right|$$
$$\leq\left(\int_0^1\int_0^1|p(t)p(s)|\,dtds\right)^{1-\frac{1}{q}}\left(\int_0^1\int_0^1|p(t)p(s)|\left|\frac{\partial^2 f}{\partial t\partial s}(b+t\eta_1(a,b),d+s\eta_2(c,d))\right|^q dtds\right)^{\frac{1}{q}}.$$

Using the $(\alpha_1,m_1)-(\alpha_2,m_2)$-preinvexity in the second sense on the coordinates of $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, we obtain

$$\begin{aligned}&\int_0^1\int_0^1|p(t)p(s)|\left|\frac{\partial^2 f}{\partial t\partial s}(b+t\eta_1(a,b),d+s\eta_2(c,d))\right|^q\\&\leq\int_0^1\int_0^1|p(t)p(s)|\left((1-t)^{\alpha_1}(1-s)^{\alpha_2}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q dtds+m_2(1-t)^{\alpha_1}s^{\alpha_2}\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c}{m_2}\right)\right|^q\right.\\&\quad\left.+m_1t^{\alpha_1}(1-s)^{\alpha_2}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a}{m_1},d\right)\right|^q+m_1m_2t^{\alpha_1}s^{\alpha_2}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a}{m_1},\frac{c}{m_2}\right)\right|^q\right)dtds\\&=A_1\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|+A_2m_2\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c}{m_2}\right)\right|^q+A_3m_1\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a}{m_1},d\right)\right|^q+A_4m_1m_2\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a}{m_1},\frac{c}{m_2}\right)\right|^q,\end{aligned}$$

where

$$\begin{aligned}A_1&=\int_0^1\int_0^1|p(t)p(s)|\,(1-t)^{\alpha_1}(1-s)^{\alpha_2}dtds=\left(\int_0^1|p(t)|\,(1-t)^{\alpha_1}dt\right)\left(\int_0^1|p(s)|\,(1-s)^{\alpha_2}ds\right),\\A_2&=\int_0^1\int_0^1|p(t)p(s)|\,(1-t)^{\alpha_1}s^{\alpha_2}dtds=\left(\int_0^1|p(t)|\,(1-t)^{\alpha_1}dt\right)\left(\int_0^1|p(s)|\,s^{\alpha_2}ds\right),\\A_3&=\int_0^1\int_0^1|p(t)p(s)|\,t^{\alpha_1}(1-s)^{\alpha_2}dtds=\left(\int_0^1|p(t)|\,t^{\alpha_1}dt\right)\left(\int_0^1|p(s)|\,(1-s)^{\alpha_2}ds\right),\\A_4&=\int_0^1\int_0^1|p(t)p(s)|\,t^{\alpha_1}s^{\alpha_2}dtds=\left(\int_0^1|p(t)|\,t^{\alpha_1}dt\right)\left(\int_0^1|p(s)|\,s^{\alpha_2}ds\right).\end{aligned}$$

By Lemma 2, we deduce that

$$A_1=A_2=A_3=A_4=\mathcal{B}(\alpha_1)\mathcal{B}(\alpha_2). \tag{10}$$

Thus,

$$\begin{aligned}&\int_0^1\int_0^1|p(t)p(s)|\left|\frac{\partial^2 f}{\partial t\partial s}(b+t\eta_1(a,b),d+s\eta_2(c,d))\right|^q\\&\leq\mathcal{B}(\alpha_1)\mathcal{B}(\alpha_2)\left(\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q+m_2\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c}{m_2}\right)\right|^q+m_1\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a}{m_1},d\right)\right|^q+m_1m_2\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a}{m_1},\frac{c}{m_2}\right)\right|^q\right).\end{aligned} \tag{11}$$

Also, by Lemma 2, we deduce that

$$\int_0^1\int_0^1|p(t)p(s)|\,dtds=(\mathcal{C}(1))^2=\frac{25}{1296}. \tag{12}$$

Substituting (15) and (12) in (10), we have

$$\begin{aligned}&\left|\int_0^1\int_0^1 p(t)p(s)\frac{\partial^2 f}{\partial t\partial s}(b+t\eta_1(a,b),d+s\eta_2(c,d))dtds\right|\\&\leq\int_0^1\int_0^1|p(t)p(s)|\left|\frac{\partial^2 f}{\partial t\partial s}(b+t\eta_1(a,b),d+s\eta_2(c,d))\right|dtds\\&\leq\left(\frac{25}{1296}\right)^{1-\frac{1}{q}}(\mathcal{B}(\alpha_1)\mathcal{B}(\alpha_2))^{\frac{1}{q}}\left(\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q+m_2\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c}{m_2}\right)\right|^q+m_1\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a}{m_1},d\right)\right|^q\right.\end{aligned}$$

$$+ m_1 m_2 \left| \frac{\partial^2 f}{\partial t \partial s} \left(\frac{a}{m_1}, \frac{c}{m_2} \right) \right|^q \Bigg)^{\frac{1}{q}}. \tag{13}$$

By using the identity in Lemma 1 and (13) we obtain the desired result.
□

Theorem 7. *Under the conditions of Lemma 1, let $S_1, S_2 \subseteq [0, \infty), \eta(a,b) > 0$ and $\eta_2(c,d) > 0$. If $\left| \frac{\partial^2 f}{\partial t \partial s} \right|^q, q > 1$ is $(\alpha_1, m_1) - (\alpha_2, m_2)$-preinvex on the coordinates in the second sense, then*

$$\begin{aligned}
&\Bigg| \frac{1}{9} \Bigg\{ f\left(b + \eta_1(a,b), d + \frac{1}{2}\eta_2(c,d)\right) + f\left(b, d + \frac{1}{2}\eta_2(c,d)\right) + 4f\left(b + \frac{1}{2}\eta_1(a,b), d + \frac{1}{2}\eta_2(c,d)\right) \\
&\quad + f\left(b + \frac{1}{2}\eta_1(a,b), d + \eta_2(c,d)\right) + f\left(b + \frac{1}{2}\eta_1(a,b), d\right) \Bigg\} \\
&\quad + \frac{1}{36} \Big\{ f(b + \eta_1(a,b), d + \eta_2(c,d)) + f(b + \eta_1(a,b), d) + f(b, d + \eta_2(c,d)) + f(b,d) \Big\} \\
&\quad - \frac{1}{6\eta_2(c,d)} \int_d^{d+\eta_2(c,d)} \left[f(b + \eta_1(a,b), v) + 4f\left(b + \frac{1}{2}\eta_1(a,b), v\right) + f(b,v) \right] dv \\
&\quad - \frac{1}{6\eta_1(a,b)} \int_b^{b+\eta_1(a,b)} \left[f(u, d + \eta_2(c,d)) + 4f\left(u, d + \frac{1}{2}\eta_2(c,d)\right) + f(u,d) \right] dt \\
&\quad + \frac{1}{\eta_1(a,b)\eta_2(c,d)} \int_d^{d+\eta_2(c,d)} \int_b^{b+\eta_1(a,b)} f(u,v) du dv \Bigg| \\
&\leq \eta_1(a,b)\eta_2(c,d)(C(r))^{\frac{2}{r}} \left(\frac{1}{(\alpha_1+1)(\alpha_2+1)} \right)^{\frac{1}{q}} \Bigg(\left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right|^q + m_2 \left| \frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c}{m_2}\right) \right|^q \\
&\quad + m_1 \left| \frac{\partial^2 f}{\partial t \partial s}\left(\frac{a}{m_1}, d\right) \right|^q + m_1 m_2 \left| \frac{\partial^2 f}{\partial t \partial s}\left(\frac{a}{m_1}, \frac{c}{m_2}\right) \right|^q \Bigg)^{\frac{1}{q}},
\end{aligned}$$

where $C(r)$ is as defined in Lemma 2.

Proof. By using the Höder's inequality, we have

$$\begin{aligned}
&\left| \int_0^1 \int_0^1 p(t)p(s) \frac{\partial^2 f}{\partial t \partial s}(b + t\eta_1(a,b), d + s\eta_2(c,d)) dt ds \right| \\
&\qquad \leq \left(\int_0^1 \int_0^1 |p(t)p(s)|^r dt ds \right)^{\frac{1}{r}} \left(\int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial s}(b + t\eta_1(a,b), d + s\eta_2(c,d)) \right|^q dt ds \right)^{\frac{1}{q}}.
\end{aligned} \tag{14}$$

Using the $(\alpha_1, m_1) - (\alpha_2, m_2)$-preinvexity in the second sense on the coordinates of $\left| \frac{\partial^2 f}{\partial t \partial s} \right|^q$, we obtain

$$\begin{aligned}
&\int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial s}(b + t\eta_1(a,b), d + s\eta_2(c,d)) \right|^q \\
&\quad \leq \int_0^1 \int_0^1 \Bigg((1-t)^{\alpha_1}(1-s)^{\alpha_2} \left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right|^q dt ds + m_2 (1-t)^{\alpha_1} s^{\alpha_2} \left| \frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c}{m_2}\right) \right|^q \\
&\qquad + m_1 t^{\alpha_1}(1-s)^{\alpha_2} \left| \frac{\partial^2 f}{\partial t \partial s}\left(\frac{a}{m_1}, d\right) \right|^q + m_1 m_2 t^{\alpha_1} s^{\alpha_2} \left| \frac{\partial^2 f}{\partial t \partial s}\left(\frac{a}{m_1}, \frac{c}{m_2}\right) \right|^q \Bigg) dt ds \\
&\quad = \left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right| \int_0^1 \int_0^1 (1-t)^{\alpha_1}(1-s)^{\alpha_2} dt ds + m_2 \left| \frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c}{m_2}\right) \right|^q \int_0^1 \int_0^1 (1-t)^{\alpha_1} s^{\alpha_2} dt ds \\
&\qquad + m_1 \left| \frac{\partial^2 f}{\partial t \partial s}\left(\frac{a}{m_1}, d\right) \right|^q \int_0^1 \int_0^1 t^{\alpha_1}(1-s)^{\alpha_2} dt ds + m_1 m_2 \left| \frac{\partial^2 f}{\partial t \partial s}\left(\frac{a}{m_1}, \frac{c}{m_2}\right) \right|^q \int_0^1 \int_0^1 t^{\alpha_1} s^{\alpha_2} dt ds \\
&\quad = \frac{1}{(\alpha_1+1)(\alpha_2+1)} \Bigg(\left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right|^q + m_2 \left| \frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c}{m_2}\right) \right|^q \\
&\qquad + m_1 \left| \frac{\partial^2 f}{\partial t \partial s}\left(\frac{a}{m_1}, d\right) \right|^q + m_1 m_2 \left| \frac{\partial^2 f}{\partial t \partial s}\left(\frac{a}{m_1}, \frac{c}{m_2}\right) \right|^q \Bigg).
\end{aligned}$$

Thus,

$$\int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial s}(b + t\eta_1(a,b), d + s\eta_2(c,d)) \right|^q$$
$$\leq \frac{1}{(\alpha_1+1)(\alpha_2+1)} \left(\left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right| + m_2 \left| \frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c}{m_2}\right) \right|^q \right.$$
$$\left. + m_1 \left| \frac{\partial^2 f}{\partial t \partial s}\left(\frac{a}{m_1}, d\right) \right|^q + m_1 m_2 \left| \frac{\partial^2 f}{\partial t \partial s}\left(\frac{a}{m_1}, \frac{c}{m_2}\right) \right|^q \right). \quad (15)$$

By using the identity in Lemma 1, (14), (15) and Lemma 2, we obtain the desired result.
□

3. Conclusion

We established a new integral identity involving the second-order mixed partial derivatives of functions of two independent variables on open invex sets. Utilizing this new identity, we established a new generalization of the Simpson's inequality for functions of two variables whose second-order mixed partial derivatives are bounded. We also established new Simpson's type integral inequalities for functions of two variables whose second-order mixed partial derivatives in absolute value to certain powers are $(\alpha_1, m_1) - (\alpha_2, m_2)$–preinvex on the coordinates in thee first and second sense.

Conflicts of Interest: The authors declare no conflict of interest.

Data Availability: No data is required for this research.

Funding Information: No funding is available for this research.

Acknowledgments: The author is grateful to the anonymous referees for their useful comments and suggestions on the manuscript.

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